Abstract

With the growing level of uncertainties in today’s power systems, the vulnerability analysis of a power system with uncertain parameters becomes a must. This paper proposes a two-stage adaptive robust optimization (ARO) model for the vulnerability analysis of power systems. The main goal is to immunize the solutions against all possible realizations of the modeled uncertainty. In doing so, the uncertainties are defined by some pre-determined intervals defined around the expected values of uncertain parameters. In our model, there are a set of first-stage decisions made before the uncertainty is revealed (attacker decision) and a set of second-stage decisions made after the realization of uncertainties (defender decision). This setup is formulated as a mixed-integer trilevel nonlinear program (MITNLP). Then, we recast the proposed trilevel program to a single-level mixed-integer linear program (MILP), applying the strong duality theorem (SDT) and appropriate linearization approaches. The efficient off-the-shelf solvers can guarantee the global optimum of our final MILP model. We also prove a lemma which makes our model [...]
Adaptive robust vulnerability analysis of power systems under uncertainty: A multilevel OPF-based optimization approach

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ABSTRACT

With the growing level of uncertainties in today’s power systems, the vulnerability analysis of a power system with uncertain parameters becomes a must. This paper proposes a two-stage adaptive robust optimization (ARO) model for the vulnerability analysis of power systems. The main goal is to immunize the solutions against all possible realizations of the modeled uncertainty. In doing so, the uncertainties are defined by some predetermined intervals defined around the expected values of uncertain parameters. In our model, there are a set of first-stage decisions made before the uncertainty is revealed (attacker decision) and a set of second-stage decisions made after the realization of uncertainties (defender decision). This setup is formulated as a mixed-integer trilevel nonlinear program (MITNLP). Then, we recast the proposed trilevel program to a single-level mixed-integer linear program (MILP), applying the strong duality theorem (SDT) and appropriate linearization approaches. The efficient off-the-shelf solvers can guarantee the global optimum of our final MILP model. We also prove a lemma which makes our model much easier to solve. The results carried out on the IEEE RTS and modified Iran’s power system show the performance of our model to assess the power system vulnerability under uncertainty.

1. Introduction

Nowadays, more than ever, electricity has become a key commodity for any growing society. Any failure or destruction of its infrastructure has a significant impact on the safety, security, economy, health, and well-being of a community [1]. The major blackouts that have hit many developed and developing countries in recent years are the most striking illustrations. For instance, in Iran, North America, and Italy, a total of 128 million people are affected in 2003. More recently, in 2012 and 2015, 670 million Indian people and 70 million Turkish people were temporarily deprived of electricity, respectively [2-5]. An estimate of the cost of major weather-related outages to the U.S. economy ranges from $20 to $55 billion annually [6]. Hence, the power-system vulnerabilities should be reduced for coping with several disruptive incidents such as natural hazards, deliberate outages, and random failures [7].

Recently, vulnerability analysis i.e., to identify critical assets whose outages lead to the most immense damage, has gained more attention in order to proactively and adequately, protect the system and mitigate such vulnerabilities [8]. A survey of the literature reveals that the proposed approaches are from analytical approaches to Monte Carlo (MC) simulations [7]. Complex network analysis (CNA) [9] has been developed recently for vulnerability analysis of several human-made infrastructures such as power systems and natural gas networks. In the pure CNA for a power system, each node represents a power-system bus, and each edge represents a transmission line. The pure CNA neglects weight and direction, and all the nodes and edges are identical [10]. In this context, several centralities are introduced, such as flow-betweenness centralities [11], and combined degree–betweennesscentrality [12]. The pure CNA also ignores the physical properties, electrical characteristics, and operational limits of power systems [10,13,14].

Researchers have updated the “pure” centralities to the “extended” centralities in which some of the power system characteristics are taken into account, such as the resistance and impedance of lines and cables [15,16], the reliability characteristics of transmission networks [17] and the active power flow and the capacity of the generators [1,10,18]. The CNA originally ignores the physics of the power system operation, an issue that was partially overcome with extended CNA. Moreover, different power flow-based methods are developed for vulnerability analysis, which can intrinsically and thoroughly consider physical
features of the power system [19,20]. A detailed comparison between power-flow based methods and other novel approaches are recently carried out in [7,21] (and the references therein).

Among the state-of-the-art approaches for vulnerability analysis, the optimization-based approaches lead to promising results without the necessity of ranking the critical assets. With the advent of high-performance computing infrastructures, the application of optimization-based approaches to power-system operation problems is considerably increasing [22]. The interdiction model as a multilevel optimization problem includes an upper-level problem (ULP) whose aim is to identify critical components so as to maximize the damage (LS) in the system and a lower-level problem (LLP) whose aim is mitigating the impacts of outages [23,24]. The interdiction model is developed based on bilevel and trilevel programs. For instance, Karush-Kuhn-Tucker (KKT) optimality conditions [25] and SDT [26] are used to convert a bilevel optimization problem to an equivalent single-level optimization problem [27]. Brown et al. [28] augmented the classical interdiction model by establishing a trilevel defender-attacker-defender model using limited (defensive) resources. Recently, this approach is used to identify the vulnerabilities of power systems exposed to natural disasters [8,29] and the hidden N-k contingencies [30]. Also, an ACOFP-based bilevel optimization approach for vulnerability analysis of a power system and the multi-period vulnerability analysis of power systems are conducted in [31] and [32], respectively.

The above-surveyed literature ignored the uncertainty of parameters. Due to the increasing uncertainty caused by the dramatic increase of intermittent renewable energy sources (RESs) such as wind power [33], together with the load forecast errors, and price-responsive demands [34], the traditional security assessment may not provide a holistic and optimal solution for the power system operation under uncertainty [35]. However, such uncertainties may jeopardize the operational security of power systems. In order to guarantee the operational security of power systems with such uncertainties, developing security models and tools for immunizing the system against worst uncertainty realizations has attracted growing attention in recent years [36].

In the literature, there are essentially two approaches to tackle uncertainty in an optimization problem, namely, stochastic programming (SP) and robust optimization (RO) [37,38] approaches. By employing SP and scenario-based approaches, we explore a few representative scenarios using the probability distributions [39,40]. So, to obtain a high-quality solution, a large set of discrete scenarios is needed, which may cause computational intractability if large systems are considered [41,42]. In contrast, the RO approach only defines the uncertainty in terms of bounded intervals using an uncertainty set, rather than a hard-to-obtain probability distribution of the uncertain data, and hence, the problem maintains at a moderate size [35,43]. The RO approach immunizes the solution against all realizations of the uncertain data within a deterministic uncertainty set and hence that might be conservative in comparing with the SP approach. As the main aim of vulnerability analysis is to guarantee the supply of demands in all situations, a conservative solution is suitable for this type of problem [44].

The RO-based analysis has been attracting considerable attention for various applications in the power-system operation area. For instance, the RO-based analysis is used for expansion planning problems such as transmission network expansion planning under uncertainties of renewable generation and load [38,45], coordinated investment in transmission and storage systems [43], the generation and transmission expansion planning [37]. It is also employed in finding the optimal unit commitment decision taking into account uncertainty (see these references as good examples: contingency-constrained unit commitment in [46], robust unit commitment with wind power and pumped hydro storage proposed in [47], an adaptive robust optimization (ARO) approach for unit commitment with recourse developed in [35], and finally an adaptive robust AC-based unit commitment model in [48]). Recently, a defender–attacker–defender model is proposed to deal with the power grid protection problem under uncertain attacks using the analytic hierarchy process [49]. Moreover, authors in [50] introduce several application areas where the ARO concepts are used.

Accordingly, in the current paper, we propose a two-stage ARO model for the vulnerability analysis of power systems, where first-stage, or the here-and-now decision, is the leader decision subject to a number
of plausible outages (NPOs) and second-stage, or the wait-and-see decision, is the dispatch decision which is robust against the worst case of all possible uncertainty realizations [51]. In doing so, the DC optimal power flow (DCOPF) is used in the LLP. The uncertainty realization and the attacker model are modeled as the middle-level problem (MLP) and the ULP, respectively. The proposed trilevel model is a MITNLNLP that is non-convex and NP-hard [52]. To solve the proposed MITNLNLP model, we first use the SDT to replace the LLP with its dual problem. Then, the MITNLNLP is reformulated to a single-level mixed-integer nonlinear program (MINLP) using the SDT and appropriate linearization approaches. The main contributions of the current paper are summarized below:

1. In contrast to previous literature, vulnerability of power system under different uncertainties is assessed using an adaptive robust trilevel optimization model. In this model, the uncertainties act as an attacker trying to maximize the damage. The proposed optimization model is robust against all possible realizations of uncertain power generations and load demands. Moreover, the level of robustness is controlled using physically-based budget constraints.

2. Since the proposed model is a MITNLNLP that is non-convex and NP-hard, an alternative MILP model is developed using the Big-M technique. We also prove a lemma which can be used to improve the computational tractability of our proposed final MILP model. The proposed final MILP model can be solved efficiently using off-the-shelf solvers such as Cplex. Numerical results using the IEEE reliability test system (RTS) and the modified model of Iran’s power system show the promising performance of our proposed MILP model.

The rest of this paper is structured as follows. Section 2 introduces assumptions, the two-stage adaptive robust vulnerability analysis, and the uncertainty set. The formulation of the proposed trilevel optimization model will be presented in Section 3. Section 4 proposes the mathematical techniques to recast the original MITNLNLP model to an equivalent MILP model. We also prove a lemma which improves the computational tractability of our proposed final MILP model. Sections 5 presents the test cases and the numerical results. Some important remarks are finally concluded in Section 6.

2. Problem description

2.1. Assumptions

For the sake of clarity, the main modeling assumptions are summarized below [25-28,48,53,54]:

1. We assume that a rational attacker trying to disable multiple assets simultaneously in order to maximize the damage, and the system damage is measured by the magnitude of LS.
2. The transmission lines and transformers are the targeting assets. However, in our framework to consider the substations, we can add a virtual bus and a virtual transmission line with zero impedance between the substations and their connected lines. Hence, by targeting the virtual lines, we can mathematically assess the vulnerability of the substations as well. Furthermore, two parallel transmission lines on the same tower are modeled with a single transmission line with double capacity.
3. The uncertain parameters are the maximum power of generation units and bus loads.
4. We employ the DCOPF in the LLP as the steady-state security constraints.

2.2. Uncertainty characterization

A pivotal component of an ARO model is the definition of uncertainty set, which determines how much uncertainty is considered in the model [55]. In this paper, we employ the most commonly used static uncertainty sets, i.e., the budget-based uncertainty set [35]. It is a polyhedral uncertainty set which assigns an interval for each uncertain parameter. Such uncertainty set is introduced by the following constraint [35,48]:

$$\sum_{k}^{K} \left| \bar{d}_k - \bar{d}_k \right| \leq \Delta; \quad \bar{d}_k \in [\bar{d}_k - \bar{d}_k, \bar{d}_k + \bar{d}_k]$$

(1)

where $\bar{d}_k$ is the $k^{th}$ component of the uncertain-parameter vectors (the maximum power of generation units and bus loads), and $K$ is the total number of buses that have uncertain power generation or load. The $\Delta$ is
the expected value of the uncertain parameter, $\tilde{d}$ is the variation from the expected value, and $\Delta$ is the “budget of uncertainty”. This inequality restricts the total variation of the uncertainty realization from the expected value [55]. When $\Delta = 0$, for all nodes $\tilde{d} = \nabla$, which means no uncertainty is considered. With increasing the budget of uncertainty, the size of the uncertainty set enlarges. This means that the resulting robust solutions are more conservative considering a larger total deviation from the expected values. Accordingly, the system will be immunized against a higher degree of uncertainty [35]. When $\Delta = K$, the uncertainty set will be the entire hypercube defined by the intervals for each $d_{\tilde{d}}$. In this paper, to model the uncertainty of both loads and power generations, two independent uncertainty sets are employed.

2.3. The two-stage adaptive robust vulnerability analysis

This paper aims to analyze the vulnerability of power systems capturing uncertainty in (renewable) power availability at generation units and in power consumption at load buses. From the safety point of view, the results of a model must be reliable in all circumstances, especially, with respect to uncertainties [51]. To ensure a robust and reliable result, an approach based on robust optimization is proposed for vulnerability analysis of power systems. The robust optimization approach determines a feasible solution to an optimization problem, which is optimal for the worse-case realization of the uncertain parameters within the uncertainty set [41,51]. Fig. 1 shows the two-stage nature of the decisions in our proposed model. In this model, a set of first-stage decisions are made before the realization of uncertainty (attacker’s decisions) and a set of second-stage decisions are made once the uncertain parameters are revealed (defender’s decision). Accordingly, our proposed model is fully adaptive to the specific realization of uncertainties. This two-stage model comprises three levels:

1. The ULP models the attacker prior to the uncertainty realization. The ULP maximizes the LS subject to the limited NPOs.
2. The LLP represents the uncertainty realization in the worst possible manner within an uncertainty set, and thus it seeks to maximize the LS.
3. The LLP models the defender trying to reduce the adverse effect of the outages considering the worse-case realization of uncertain parameters from the MLP.

The proposed ARO-based vulnerability analysis for power systems is illustrated in Fig. 1. Section 3 presents the mathematical formulation of the proposed model as a MITNLp. Then, this MITNLp is recast into a single-level MILP in Section 4, applying a series of proposed mathematical techniques [56].

3. The adaptive robust attacker-defender problem

This section presents the mathematical formulation of the proposed adaptive robust attacker-defender problem. This attacker-defender interaction considering the worse-case realization of uncertain parameters is modeled as the trilevel optimization problem (2)-(16). The Lagrange multipliers for the LLP are presented in the parentheses.

$$\begin{align*}
\text{Maximize} & \quad \sum_{i \in N} Ls_i^+ \\
\text{subject to:} & \quad 0.5 \times \sum_{i,j \in N} (1 - z_{ij}) = \text{NPO}; \quad \forall i, j \in N \\
z_{ij} = z_{ji}; \quad \forall i, j \in N
\end{align*}$$

subject to:

$$\begin{align*}
\sum_{i,j \in N} (1 - z_{ij}) = \text{NPO}; \quad \forall i, j \in N \\
z_{ij} = z_{ji}; \quad \forall i, j \in N
\end{align*}$$

where

$$\begin{align*}
\sum_{i \in N} Ls_i^+ \in \arg\left\{ \text{Maximize} \sum_{i \in N} Ls_i^+ \right\} \\
\text{subject to:} & \quad \forall i \in G \\
\forall i \in G: \quad & \quad \left| \frac{\tilde{P}_i - \bar{P}_i}{\bar{P}_i} \right| \leq DR_i; \\
\forall i \in D: & \quad \left| \frac{\tilde{P}_i - \bar{P}_i}{\bar{P}_i} \right| \leq DR_i;
\end{align*}$$

where

$$\begin{align*}
\sum_{i \in N} Ls_i^+ \in \arg\left\{ \text{Minimize} \sum_{i \in N} Ls_i \right\} \\
\text{subject to:} & \quad \forall \tilde{P}_i \in \mathbb{P} \subseteq \mathbb{P}_G; \quad \forall i \in N: (\lambda_i) \\
\forall \tilde{P}_i \in \mathbb{P} \subseteq \mathbb{P}_G; \quad \forall i \in N: (\mu_{ij}) \\
\forall \tilde{P}_i \in \mathbb{P} \subseteq \mathbb{P}_G; \quad \forall i \in N: (\nu_{ij})
\end{align*}$$

The optimization problem (2)-(16) comprises three optimization levels: (i) the ULP, i.e. (2)-(4), which is associated with the attacker; (ii) the MLP, i.e. (5)-(9), which characterizes the worse-case realization of the uncertainties to maximize the damage; and (iii) the LLP, i.e. (10)-(16), which models the system operator (i.e., defender) to mitigate the damage consequences. Equations (3)-(4) models the limited number of plausible outages. As $z_{ij} = z_{ji}$, to avoid double consideration, the total number of transmission-line outages in constraint (3) is multiplied by a factor of 0.5. When $z_{ij}$ is zero, the transmission line $ij$ is disconnected, otherwise, it is connected.

As previously mentioned, the uncertainties considered include:

(i) The available generation capacity is modelled as $\tilde{P}_G$: The level of availability of power generation varies based on several conditions such as equipment failures or the weather conditions for renewable units [48]. Constraint (6) ensures that this level will be in an interval. The total deviations of the available capacities with respect to the expected ones are bounded in constraint (7) by the uncertainty budget for generation units $DR_i$;

(ii) The level of loads for the load buses is modelled as $\tilde{P}_d$: Constraints (8) and (9) introduce an interval and the total deviations of the available loads, respectively. To adjust the budget of uncertainty, we employ integer values which are the number of generation units and load buses for $DR_i$ and $DR_d$, respectively.

In the LLP (10)-(16), the DCOPF is used whose aim is mitigating the impacts of outages and minimizing the damage consequences. Equation (10) is the objective function of the LLP. The asterisk in (2), (5), and (10)
means that $L_S$ is decided in the LLP. Equation (11) is the nodal power-balance equation for active power. Equation (12) represents the transmission-line flow calculation. Constraints (13)–(14) limit the active-power generation and the capacity of the transmission line, respectively. The angles of voltage phasors are limited using (15). Furthermore, active LS for each bus is limited to its maximum available load in (16).

4. Methodology

The derived MITNLP model is reformulated into a single-level MILP model using the “dualize-and-combine” technique in two steps: First, the LLP is recast to a max problem by the SBD [57]; then the derived model is recast to a single-level MILP with several proposed linearization techniques. We also prove a lemma which improves the computational tractability of our proposed final MILP model.

4.1. The lower-level optimization problem (LLP) and the SBD

The dual optimization problem of the linear programming (LP) problem (10)–(16) is derived below. The primal variables are presented in the parentheses.

$$
\text{Maximize} \quad \sum_{i \in N} (\lambda_i + \pi_i) \hat{P}_{d_i} + \sum_{i \in G} \tau_i \hat{P}_{d_i} + \sum_{i \in N} (\varphi_i - \varphi_i^0) S_{\text{max}} + \sum_{i \in N} (\vartheta_i - \omega_i ) P_{\text{max}}
$$

subject to:

$$
0.5 \times \sum_{j \in N} (1 - z_{ij}) = \text{NPO}; \quad \forall i, j \in N \quad (24)
$$

$$
z_{ij} = z_{ji}; \quad \forall i, j \in N \quad (25)
$$

$$
\bar{T}_{G_i} - \bar{P}_{G_i} - \bar{P}_{G_i} + \bar{P}_{G_i}; \quad \forall i \in G \quad (26)
$$

$$
\sum_{i \in G} \left| \frac{\bar{P}_{G_i} - \bar{T}_{G_i}}{P_{G_i}} \right| \leq \Delta R_i^G; \quad \forall i \in G \quad (27)
$$

$$
\bar{T}_{d_i} - \bar{P}_{d_i} - \bar{P}_{d_i} + \bar{P}_{d_i}; \quad \forall i \in D \quad (28)
$$

$$
\sum_{i \in D} \left| \frac{\bar{P}_{d_i} - \bar{T}_{d_i}}{P_{d_i}} \right| \leq \Delta R_i^D; \quad \forall i \in D \quad (29)
$$

$$
- \lambda_i + \mu_i + \varphi_i^0 = 0; \quad \forall i, j \in N : (P_g) \quad (30)
$$

$$
\lambda_i + \varpi_i = 0; \quad \forall i \in G : (P_d) \quad (31)
$$

$$\begin{align}
\text{Subject to:} & \\
- \lambda_i + \mu_i + \varphi_i^0 + \varpi_i = 0; & \forall i, j \in N : (P_g) \quad (18) \\
\lambda_i + \varpi_i = 0; & \forall i \in G : (P_d) \quad (19) \\
\sum_{j \in N} \varphi_j B_{ij} - \sum_{j \in N} \varphi_j B_{ji} + \vartheta_i = 0; & \forall i, j \in N : (\theta) \quad (20) \\
\varpi_i + \lambda_i \leq 1; & \forall i \in D : (L_S) \quad (21) \\
\lambda_i, \mu_i, \varphi_i^0, \varpi_i \leq 0; & \forall i, j \in N \quad (22)
\end{align}$$

4.2. Transforming the MITNLP to a single-level MILP

We first replace the LP problem (10)–(16) with its dual optimization problem derived in (17)–(22). Then, the optimization problem (2)–(16) is converted to a trilevel max-max-max problem. This trilevel max-max-max problem can be written equivalently as the single-level MINLP model in (23)–(34).

$$\begin{align}
\text{Maximize} \quad & \sum_{i \in N} (\lambda_i + \pi_i) \hat{P}_{d_i} + \sum_{i \in G} \tau_i \hat{P}_{d_i} + \sum_{i \in N} (\varphi_i - \varphi_i^0) S_{\text{max}} + \sum_{i \in N} (\vartheta_i - \omega_i ) P_{\text{max}} \\
\text{subject to:} & \\
0.5 \times \sum_{j \in N} (1 - z_{ij}) = \text{NPO}; & \forall i, j \in N \quad (24) \\
z_{ij} = z_{ji}; & \forall i, j \in N \quad (25) \\
\bar{T}_{G_i} - \bar{P}_{G_i} - \bar{P}_{G_i} + \bar{P}_{G_i}; & \forall i \in G \quad (26) \\
\sum_{i \in G} \left| \frac{\bar{P}_{G_i} - \bar{T}_{G_i}}{P_{G_i}} \right| \leq \Delta R_i^G; & \forall i \in G \quad (27) \\
\bar{T}_{d_i} - \bar{P}_{d_i} - \bar{P}_{d_i} + \bar{P}_{d_i}; & \forall i \in D \quad (28) \\
\sum_{i \in D} \left| \frac{\bar{P}_{d_i} - \bar{T}_{d_i}}{P_{d_i}} \right| \leq \Delta R_i^D; & \forall i \in D \quad (29) \\
- \lambda_i + \mu_i + \varphi_i^0 = 0; & \forall i, j \in N : (P_g) \quad (30) \\
\lambda_i + \varpi_i = 0; & \forall i \in G : (P_d) \quad (31) \\
\sum_{i \in N} \varphi_j B_{ij} - \sum_{i \in N} \varphi_j B_{ji} + \vartheta_i = 0; & \forall i, j \in N : (\theta) \quad (32) \\
\varpi_i + \lambda_i \leq 1; & \forall i \in D : (L_S) \quad (33) \\
\lambda_i, \mu_i, \varphi_i^0, \varpi_i \leq 0; & \forall i, j \in N \\
\lambda_i, \mu_i, \varphi_i^0, \varpi_i \geq 0; & \forall i, j \in N
\end{align}$$

The nonlinear terms are $\lambda_i \hat{P}_{d_i}, \pi_i \hat{P}_{d_i}$, and $\tau_i \hat{P}_{d_i}$ (products of two continuous variables in (23)), $\mu_i \varphi_i^0$ (products of integer and continuous variables in (32)) and absolute values in (27) and (29). Below, we propose different linearization techniques to linearize these nonlinear terms.

4.2.1. The nonlinear terms $\lambda_i \hat{P}_{d_i}, \pi_i \hat{P}_{d_i}$, and $\tau_i \hat{P}_{d_i}$

For linearizing the bilinear terms in the objective function, we observe the following properties of the LP problem (17)–(22).

**Property 1.** The feasibility set of the LP problem (17)–(22) is independent from the $P_g$ and $P_d$.

**Property 2.** The LP problem (10)–(16) is an always-feasible optimization.
problem.

Now we define the polyhedron \( U = \{l_i, \mu_i, \alpha_i, \omega_i, \beta_i, \gamma_i, \tau_i \} \) \((18)-(22))\). Using Property 1 and Property 2, no matter what the values of \( \hat{P}_g \) and \( \hat{P}_d \), the maximum over \( U \) of the objective function occurs at the extreme points of polyhedron \( U \). On the other hand, we know that \( U \) has a finite number of extreme points. If we denote the extreme points of \( U \) as \( u^p (p = 1,...,n_p) \), the value function of LP problem - can be written as the following piecewise linear function:

\[
V F(\hat{P}_g, \hat{P}_d) = \max \left\{ \sum_{i=1}^{n_p} \left( \lambda_i \alpha_i + \mu_i \beta_i \right) \hat{P}_d_{i,0} + \sum_{i=0}^{D} \left( \sigma_i^{0} - \sigma_i^{p} \right) S_{\mu,i}^{\text{max}} + \sum_{i=0}^{D} \left( \sigma_i^{0} - \sigma_i^{p} \right) \sigma_{\mu,i,p}^{\text{max}}, p = 1, \ldots, n_p \right\} \tag{35}
\]

This means that the \( VF(\hat{P}_g, \hat{P}_d) \) is the pointwise maximum of a set of affine functions as shown in (35). Now, in Lemma 1 below, we show that the \( VF(\hat{P}_g, \hat{P}_d) \) is a convex function in \( \hat{P}_g \) and \( \hat{P}_d \).

**Lemma 1.** The \( VF(\hat{P}_g, \hat{P}_d) \) is convex and its maximum with respect to \( \hat{P}_g \) and \( \hat{P}_d \) occurs at the extreme points of \( \hat{P}_g \) and \( \hat{P}_d \) variables.

**Proof.** The \( VF \) is the pointwise maximum of a set of convex (or affine) functions. And it is straightforward to show that the pointwise maximum of a set of convex functions is a convex function. Since, \( VF(\hat{P}_g, \hat{P}_d) \) is a convex function in \( \hat{P}_g \) and \( \hat{P}_d \), its maximum with respect to \( \hat{P}_g \) and \( \hat{P}_d \) occurs at the extreme points of \( \hat{P}_g \) and \( \hat{P}_d \) variables \([58]\).

**Fig. 2** below shows a simple illustration of function \( VF(\hat{P}_g, \hat{P}_d) \) in two-dimensional space. The \( x \) represents a general one-dimensional decision variable. The actual \( VF(\hat{P}_g, \hat{P}_d) \) function is an extension of the function in Fig. 2 in the multi-dimensional space. The black lines represent the affine functions in (35), and the red line represents the final convex piecewise function.

By substituting (35) in our trilevel max-max-problem, we have:

\[
\max_{\hat{P}_g, \hat{P}_d} \max_{\hat{P}_g, \hat{P}_d} VF(\hat{P}_d, \hat{P}_g) \tag{36}
\]

As we can see in the LLP (36), we maximize a convex function \( VF (\hat{P}_g, \hat{P}_d) \) over the box constraints \( \hat{P}_g - \hat{P}_d \leq \hat{P}_g \leq \hat{P}_g + \hat{P}_d \) and \( \hat{P}_d - \hat{P}_d \leq \hat{P}_d \leq \hat{P}_d + \hat{P}_d \). This means that the optimal solutions happen at the upper or lower bounds of these box constraints as discussed above. Accordingly, an optimal solution of the MLP is when the uncertain parameters reach either upper or lower bounds. Using Lemma 1, we write the \( \hat{P}_g \) and \( \hat{P}_d \) variables using binary variables as follows \([48]\):

\[
\hat{P}_g = \hat{P}_g + \hat{P}_g \beta_g - \hat{P}_g \beta_g^- \tag{37}
\]

\[
\hat{P}_d = \hat{P}_d + \hat{P}_d \beta_d - \hat{P}_d \beta_d^- \tag{38}
\]

where \( \beta_g^+, \beta_g^-, \beta_d^+, \beta_d^- \in \{0,1\} \). Based on these equations, the uncertain parameters reach their upper bounds with \( \beta_g^+ = 1 \), their lower bounds with \( \beta_g^- = 1 \) or meets its forecasted value when \( \beta_g^+ = \beta_g^- = 0 \). By substituting (37) and (38) in the nonlinear terms \( \hat{P}_g, \hat{P}_d \) variables in Subsection 4.2.1 and the term \( \hat{P}_g \mu_g \) in constraint (32). As an example, \( \hat{P}_g \mu_g \) can be linearized as follows:

\[
\hat{P}_g \mu_g - H_g \tag{39}
\]

\[
- B_1 \gamma_g \leq \hat{P}_g \mu_g \leq B_1 \gamma_g \tag{40}
\]

\[
\hat{P}_d \mu_g - B_1 (1 - \gamma_g) \leq \hat{P}_d \mu_g \leq \hat{P}_d \mu_g \tag{41}
\]

where \( B_1 \) is a suitably large constant.

**4.2.2. The bilinear terms**

Two auxiliary variables \( T \) and \( H \) \([53, 56, 39]\) can be used to linearize the bilinear terms \( \beta_g \beta_d \), \( \beta_g \mu_g \), \( \beta_d \mu_d \), \( \beta_g \beta_d \), \( \beta_g \beta_d \), \( \beta_d \beta_d \) and \( \gamma_g \beta_g \) in Subsection 4.2.1 and the term \( \gamma_g \beta_g \) in constraint (32). As an example, \( T_g \) can be linearized as follows:

\[
T_g = \mu_g - H_g \tag{39}
\]

\[
- B_1 \gamma_g \leq T_g \leq B_1 \gamma_g \tag{40}
\]

\[
- B_1 (1 - \gamma_g) \leq T_g \leq B_1 (1 - \gamma_g) \tag{41}
\]

where \( B_1 \) is a suitably large constant.

**4.2.3. Absolute values in (27) and (29)**

To linearize the general formulation of the budget-based uncertainty set in (1), the definitions in (37), and (38) are substituted in (1). So we will have:

\[
\sum_{i \in G} |\beta_i^+ - \beta_i^-| \leq DR_i \tag{42}
\]

\[
\sum_{i \in D} |\beta_i^+ - \beta_i^-| \leq DR_i \tag{43}
\]

Then, to remove the absolute value terms in (42) and (43), we can replace (42) and (43) with (44)-(45) and (46)-(47), respectively.

\[
\sum_{i \in G} (\beta_i^+ + \beta_i^-) \leq DR_i \tag{44}
\]

\[
\beta_i^+ + \beta_i^- \leq 1 \tag{45}
\]

\[
\sum_{i \in D} (\beta_i^+ + \beta_i^-) \leq DR_i \tag{46}
\]

\[
\beta_i^+ + \beta_i^- \leq 1 \tag{47}
\]

where \( \beta_i^+, \beta_i^-, \beta_i^+, \beta_i^- \in \{0,1\} \).

**4.3. Our final proposed MILP model**

Our final MILP model for the vulnerability analysis of power systems
Maximize \[ Z = \sum_{i=1}^{N} \beta_i \sum_{j=1}^{N} z_{ij} \] subject to:

\[
\begin{align*}
\sum_{i=1}^{N} (\beta_i^{+} + \beta_i^{-}) &\leq DR_t; \quad \forall i \in D \\
\beta_i^{+} + \beta_i^{-} &\leq 1; \quad \forall i \in G \\
\beta_i^{+} &\leq \lambda_i; \quad \forall i \in D \\
\beta_i^{-} &\leq \lambda_i; \quad \forall i \in D \\
\gamma_i &\leq 0; \quad \forall i \in G \\
\lambda_i + \gamma_i &\leq 0; \quad \forall i \in \mathcal{G} \\
\sum_{i=1}^{N} B_i T_0 - \sum_{(i,j) \in \mathcal{D}} B_i T_j + \omega_j + \varpi_i &= 0; \quad \forall i,j \in N \\
\lambda_i + \omega_j &\leq 1; \quad \forall i \in D \quad (53)
\end{align*}
\]

\[
\begin{align*}
T_0 &= \mu_j - H_j - B_i z_{ij} T_i \leq T_i \leq B_i z_{ij}; \\
-T_i (1 - z_{ij}) &\leq H_i \leq B_i (1 - z_{ij}); \quad \forall i,j \in N \quad (54)
\end{align*}
\]

\[
\begin{align*}
T_i &= \lambda_i - H_i - B_j z_{ij} T_j \leq T_i \leq B_j z_{ij} + B_j \beta_j^{+}; \\
-T_i (1 - \beta_j^{+}) &\leq H_i \leq B_j (1 - \beta_j^{+}); \quad \forall i \in D \quad (55)
\end{align*}
\]

\[
\begin{align*}
T_0 &= \pi_i - H_i, B_j z_{ij} T_j \leq T_i \leq B_j z_{ij} ; \\
-T_i &\leq H_i \leq B_j (1 - \beta_j^{+}); \quad \forall i \in D \\
T_j &= \gamma_i - H_j - B_i z_{ij} T_i \leq T_j \leq B_i z_{ij} ; \\
-T_j (1 - \beta_i^{+}) &\leq H_i \leq B_i (1 - \beta_i^{+}); \quad \forall j \in G \quad (56)
\end{align*}
\]

Our final MILP model (48)-(65) can be efficiently solved by a standard mixed integer programming solvers such as CPLEX [59] to the acceptable level of precision. These solvers can also certify the optimality of the solutions. This is while there is no guarantee that the

<table>
<thead>
<tr>
<th>Model statistics</th>
<th>Reference\textsuperscript{a} [31]</th>
<th>This work\textsuperscript{**}</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of equation blocks</td>
<td>12</td>
<td>46</td>
</tr>
<tr>
<td>Number of variable blocks</td>
<td>12</td>
<td>28</td>
</tr>
<tr>
<td>Number of nonzero elements</td>
<td>1993</td>
<td>2979</td>
</tr>
<tr>
<td>Number of single equations</td>
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<td>1046</td>
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<td>Number of single variables</td>
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<td>766</td>
</tr>
<tr>
<td>Number of binary variables</td>
<td>68</td>
<td>130</td>
</tr>
</tbody>
</table>

\textsuperscript{a} This is based on the DCOPP-based bilevel optimization approach without considering the uncertainties.

\textsuperscript{**} Our model has more variables and elements because it considers uncertainties in its formulation.
5. Numerical results

5.1. The IEEE RTS

The IEEE RTS consisting of 24 buses, 32 generator units, and 38 branches (lines and transformers) is used in this paper. Fig. 3 shows the single-line diagram of a single area IEEE RTS. Table 1 presents the generating unit characteristics. Other detailed data of this test system can be found in [60].

All the simulations have been performed using a processor at 2.4 GHz with a total RAM of 32 GB. We also implement our model using the GAMS modeling language environment and the solver CPLEX [61]. In all simulations, the CPLEX relative optimality criterion was set at 0.001. Table 2 shows the size and complexity of our model for the IEEE RTS comparing with the previous literature.

![Fig. 3](image1.jpg)

MITNLP model which is an NP-hard problem converges to the global optimum [22].

### Table 3

Results for the IEEE RTS without uncertainty in comparison with previous literature.

<table>
<thead>
<tr>
<th>NPO</th>
<th>Critical lines</th>
<th>load shedding (MW)</th>
<th>NPO</th>
<th>Critical lines</th>
<th>load shedding (MW)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>16-19,20-23</td>
<td>0</td>
<td>1</td>
<td>16-19,20-23</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>7-8,15-21,16-17</td>
<td>309</td>
<td>2</td>
<td>7-8,15-21,16-17</td>
<td>309</td>
</tr>
<tr>
<td>3</td>
<td>3-24,12-23,13-23,14-16</td>
<td>516</td>
<td>3</td>
<td>3-24,12-23,13-23,14-16</td>
<td>516</td>
</tr>
<tr>
<td>4</td>
<td>12-23,13-23,15-21,16-23</td>
<td>872</td>
<td>4</td>
<td>12-23,13-23,15-21,16-23</td>
<td>872</td>
</tr>
<tr>
<td>5</td>
<td>11-13,12-13,12-23,15-21,16-23,20-23</td>
<td>1198</td>
<td>5</td>
<td>11-13,12-13,12-23,15-21,16-23,20-23</td>
<td>1198</td>
</tr>
<tr>
<td>6</td>
<td></td>
<td></td>
<td>6</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

* Two parallel lines on the same tower are considered as two independent lines in references [25,27].

![Fig. 4](image2.jpg)

![Fig. 5](image3.jpg)

Fig. 4. (a) Comparison of the optimal solutions (LS) with different levels of load uncertainty (DR\(d\) = 0 (no uncertainty) to DR\(d\) = 17 (the most conservative case)), (b) Maximum difference of LS in comparison with “no uncertainty” case when DR\(d\) = 17.

Fig. 5. (a) Comparison of the optimal solutions (LS) with different levels of generation uncertainty (DR\(g\) = 0 (no uncertainty) to DR\(g\) = 14 (the most conservative case)), (b) Maximum difference of LS in comparison with “no uncertainty” i.e. when DR\(g\) = 14.
Vulnerability analysis with load uncertainty

In this subsection, the aim is to simulate the power system considering load uncertainty so as to assess vulnerability under load uncertainty. In order to consider the load uncertainty, the uncertainty budget for generation buses i.e., \( DR^d \) in our proposed model, should be set to zero in all simulations. Furthermore, we set the range of load variation to be \( P_d = \alpha^d P_d^\text{avg} \), \( \forall \in D \). In particular, in this case study, we consider that \( \alpha^d \) is always fixed at 0.05 (scenario I) and 0.1 (scenario II) for all load buses. Moreover, the different uncertainty levels of the loads are tuned up by varying the uncertainty budget \( DR^d \) for load buses. It takes values in the range of zero (no uncertainty) to the total number of load buses (N (D) = 17).

Fig. 4(a) shows the LS as a function of NPO under different load uncertainties. Moreover, this figure indicates that total LS increases when \( DR^d \) increases and the maximum total LS occurs when \( DR^d \) is at its maximum value 17. Furthermore, we have compared the maximum difference of LS in comparison with the “no uncertainty” case when \( DR^d \) = 17. Fig. 4(b) shows that the maximum differences are 30% and 60% in scenarios I and II when NPO = 3, respectively.

However, the load uncertainty is small, which leads to proposing different critical lines in some NPOs. For example, our model and the literature in [25, 27, 31] find similar critical lines (which are lines 16–19 and 20–23) under no uncertainty, when NPO is 2 (see Subsection 6.1). This is while our approach proposes different lines (lines 15–21 and 16–17), in a higher level of uncertainty (for example when \( DR^d \) > 6 in scenario I).

### 5.1.3. Vulnerability analysis with generation uncertainty

Similar to the previous subsection, the aim is to simulate the power system considering the uncertainty but only for the generation units. In order to consider the generation uncertainty, the uncertainty budget for load buses \( (DR^g) \) in our proposed model should be set to zero in all simulations. Furthermore, we set the range of generation variation to be \( P_{g_i} = \alpha^g P_{g_i}^\text{avg}, \forall \in G \). In contrast to peak load variations, the peak generation capacity can deviate much more due to the failure of generation units or connected RES units. So, in this case study we consider that \( \alpha^g \) is fixed at 0.2 (scenario III) and 0.5 (scenario IV) for all generation units. Moreover, the generation uncertainty levels are modeled by varying the uncertainty budget for generation units \( (DR^g) \). It takes values in the range of zero (no uncertainty) to the total number of generation units (N (D) = 14).

Fig. 5(a) shows the LS as a function of NPO under different generation uncertainties. Similarly, this figure shows that the total LS increases when \( DR^g \) increases and the minimum total LS occurs in the no-uncertainty case and the maximum total LS occurs when \( DR^g \) is at its maximum value of 14. Furthermore, Fig. 5(b) shows that the maximum differences of LS between uncertainty case and no-uncertainty case reach approximately 118% and 342% in scenarios III and IV, respectively. With NPO = 1, the IEEE RTS is “N-1” secure when there is no uncertainty. But, the IEEE RTS is not “N-1” secure when uncertainty increases in scenario III \( (DR^g \geq 4) \) and scenario IV \( (DR^g \geq 1) \), respectively. Note that when we set the NPO at zero the model considers the
uncertainty of generation units. Fig. 6 shows the LS as a function of DR^8 when NPO = 0. As can be seen in this figure, the system has LS when \( a_f^2 \) is larger than 17% (DR^8 ≤ 11). Similar to the load uncertainty, the generation uncertainty leads to proposing different critical lines for some NPOs.

### 5.1.4. Vulnerability analysis with both uncertainties

Both load and generation uncertainties are investigated in this subsection. The proposed model is employed for different levels of uncertainty, which is modeled by varying the uncertainty budgets DR^a and DR^b. They take values in the range of zero (no uncertainty) to the total number of generation/load units, which are 17 and 14 in this case study for load and generation units, respectively. Furthermore, we set the range of generation and load variations to be \( \tilde{P}_g = a_f^1 \tilde{P}_g, \forall i \in G \) and \( \tilde{P}_l = a_d^1 \tilde{P}_l, \forall i \in D \), respectively. We assume two scenarios in this subsection. In scenario V, \( a_f^1 \) and \( a_f^2 \) are fixed at 0.2 and 0.05, respectively, while for scenario VI, \( a_d^1 \) and \( a_d^2 \) are fixed at 0.5 and 0.1, respectively for all generation and load units.

Similarly, the proposed model is applied to the IEEE RTS. Fig. 7 and Fig. 8 show the LS as a function of DR^8 and DR^b in scenarios V and VI, respectively. If we categorize the uncertain parameters involved in power system studies into two main groups [62], namely topological parameters (e.g., failure and forced outages) and operational parameters (e.g., demand and generation values), our model can address both uncertain parameters simultaneously or independently.

Fig. 7(a) and Fig. 8(a) show that the IEEE RTS is unreliable and unsafe even when we do not have any outage of the transmission line (NPO = 0) for some of DR^a and DR^b. These figures present the worst-case scenarios that might be occurred due to solely uncertain operational parameters. As expected, the situation will be worse when we have transmission-line outages. Fig. 7(b) and Fig. 8(b) show the LS when we have 13 simultaneous outages. Moreover, Fig. 7(b) and Fig. 8(b) show that the minimum LS (1607 MW) is when we do not have uncertainty in generations and loads (when NPO = 13) and maximum differences of LS when we have the maximum level of the uncertainties in comparison with “no uncertainty” case are 9% and 32% in scenarios V and VI, respectively.

The critical lines and components are presented in Table 4 and Table 5, respectively, for scenarios V and VI. These tables show that increasing the level of uncertainty leads to higher LS levels and different critical lines in comparison with no uncertainty case (see Table 3). Moreover, based on the generation capacity and generation location in the network, some generation units and load buses with uncertainty play major roles in the worst-case scenarios. For instance, for scenario V and when DR^8 = DR^b = NPO = 1, the uncertainty in generation unit G6 and

---

### Table 4

<table>
<thead>
<tr>
<th>DR^a = DR^b = NPO</th>
<th>Critical lines</th>
<th>Generation unit</th>
<th>Load bus number</th>
<th>Load shedding (MW)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>15-21</td>
<td>G6, G15</td>
<td>15</td>
<td>39</td>
</tr>
<tr>
<td>2</td>
<td>15-21-16-17</td>
<td>G6, G13, G14</td>
<td>13, 15</td>
<td>741</td>
</tr>
<tr>
<td>3</td>
<td>7-8-15-21-16-17</td>
<td>G6, G10, G13, G14</td>
<td>10, 13, 15</td>
<td>1090</td>
</tr>
<tr>
<td>4</td>
<td>7-8-15-21-17-22-18-21</td>
<td>G4, G5, G6, G8, G9</td>
<td>10, 13, 14, 15, 19</td>
<td>1257</td>
</tr>
<tr>
<td>5</td>
<td>12-23-13-21-15-21-16-17-20-23</td>
<td>G2, G4, G5, G7, G8, G9</td>
<td>3, 9, 10, 14, 15, 19</td>
<td>1455</td>
</tr>
</tbody>
</table>

*In these components, when the uncertain parameters reach the boundary of their intervals, the worst-case scenario occurs.

---

### Table 5

<table>
<thead>
<tr>
<th>DR^a = DR^b = NPO</th>
<th>Critical lines</th>
<th>Generation unit</th>
<th>Load bus number</th>
<th>Load shedding (MW)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>15-21</td>
<td>G6, G15</td>
<td>15</td>
<td>39</td>
</tr>
<tr>
<td>2</td>
<td>15-21-16-17</td>
<td>G6, G13, G14</td>
<td>13, 15</td>
<td>741</td>
</tr>
<tr>
<td>3</td>
<td>7-8-15-21-16-17</td>
<td>G6, G10, G13, G14</td>
<td>10, 13, 15</td>
<td>1090</td>
</tr>
<tr>
<td>4</td>
<td>7-8-15-21-17-22-18-21</td>
<td>G4, G5, G6, G8, G9</td>
<td>10, 13, 14, 15, 19</td>
<td>1257</td>
</tr>
<tr>
<td>5</td>
<td>12-23-13-21-15-21-16-17-20-23</td>
<td>G2, G4, G5, G7, G8, G9</td>
<td>3, 9, 10, 14, 15, 19</td>
<td>1455</td>
</tr>
</tbody>
</table>

*In these components, when the uncertain parameters reach the boundary of their intervals, the worst-case scenario occurs.
load bus number 15 leads to a system which is not “N-1” secure.

5.2. Iran’s 400-kV power system

Iran’s 400-kV power system as a realistic test system is used in this subsection. This system is comprised of 28 generators, 52 buses, and 99 transmission lines as shown in Fig. 9. The solid assets are existing assets and the dashed assets are candidate assets that are planned for the expanded network [63]. The detailed data associated with this power system can be found in [63,64].

In this subsection, DR_d and DR_g take values in the range of zero (no uncertainty) to the total number of generation units and load buses which are 28 and 48, respectively. Furthermore, we set the range of generation and load variations to be \( \hat{P}_{G_i} = \alpha_g P_{G_i}, \forall i \in G \) and \( \hat{P}_{D_i} = \alpha_d P_{D_i}, \forall i \in D \), respectively. We assume \( \alpha_g \) and \( \alpha_d \) are fixed at 0.2 and 0.05 for all generation and load units, respectively. Moreover, the impact of the uncertainties on the vulnerability analysis is investigated for both existing and expanded networks.

Fig. 10 and Fig. 11 show the LS as a function of NPO in the existing and expanded networks, respectively, considering the uncertain parameters. As expected, the expanded network operates more reliable and robust than the existing network. For instance, when there is no uncertainty, the expanded network is N-1 secure, and the total possible LS i.e., 10390 MW occurs with more transmission line outages as compared to the existing network. Moreover, for a given NPO, the LS in the expanded network is lower than the one for the existing one.

Fig. 11 also highlights that the expanded network might be no longer N-1 secure when there are uncertain load and/or generation units. Finally, the proposed set of critical lines is also different. For example, when NPO = 1, line 1–2 is a critical transmission line for the existing network. However, considering the uncertainty and the most conservative case, the critical transmission line will change to line 30–31.

Fig. 9. Modified Iran’s 400-kV power system from [63], the solid assets are existing assets and the dashed assets are planned to be added.

Fig. 10. Comparison of the optimal solutions with different levels of uncertainty in the existing network (DR_d = DR_g = 0 (no uncertainty) to DR_g = 28 and DR_d = 48 (the most conservative case)).

Fig. 11. Comparison of the optimal solutions with different levels of uncertainty in the expanded network (DR_d = DR_g = 0 (no uncertainty) to DR_g = 28 and DR_d = 48 (the most conservative case)).
6. Conclusions

This study aims to propose a three-level optimization problem to analyze the power system vulnerability in the context of system uncertainty. A robust optimization approach has been proposed. In our proposed model, the ULP represents the attacker, the MLP models the worst-case uncertainty level, and finally, the LLP represents the defender. The proposed model is a MITNLP, which is hard to solve. Applying the SDT of the linear programs, the LLP is replaced by its dual program. Then the original MITNLP problem is recast to a max-max-max problem, which can be presented as a single-level MINLP. The nonlinear terms of the single-level MINLP model are linearized using appropriate linearization approaches. We also observe two properties of our MITNLP model and prove a lemma that improves the computational performance of our proposed final MILP model. Our final MILP model has been applied to the IEEE RTS and modified Iran’s power system, and the simulation results are carefully studied. Our simulation results show that the vulnerability analysis without considering uncertainties leads to optimistic results. Moreover, increasing the level of uncertainty in our case studies leads to higher levels of LS and different critical lines in comparison with no uncertainty case. An interesting future research project can be dedicated to investigating how our final model can be adjusted to model the asymmetric uncertainties accurately.

Declarations of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

References

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