Dynamic stochastic general equilibrium models with heterogeneous agents: theory, computation and application

PROEHL, Elisabeth Rita

Abstract
Dynamic stochastic general equilibrium models with ex-post heterogeneity due to idiosyncratic risk pose numerous challenges stemming from the cross-sectional distribution of endogenous variables which changes stochastically over time due to aggregate risk. In this thesis, I tackle various open questions. My first contribution is of a theoretical nature as I establish existence and uniqueness of the Aiyagari-Bewley growth model. The second challenge I address has a more practical concern. I propose a new numerical method to compute solutions to heterogeneous agent models. With the derived approximation error bounds, I ensure convergence to the rational expectations equilibrium. Equipped with this novel theoretically founded method, I show that even two standard economic models like the Aiyagari-Bewley growth model and the Huggett economy yield intriguing results. When agents progressively share idiosyncratic risk, heterogeneity increasingly amplifies aggregate risk. Furthermore, the volatility of the expected stationary cross-sectional distribution and of the stationary price distribution rises.

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DYNAMIC STOCHASTIC GENERAL EQUILIBRIUM MODELS WITH HETEROGENEOUS AGENTS: THEORY, COMPUTATION AND APPLICATION

by

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A thesis submitted to the
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Abstract

Dynamic stochastic general equilibrium models with ex-post heterogeneity due to idiosyncratic risk pose numerous challenges stemming from the cross-sectional distribution of endogenous variables. This distribution changes stochastically over time due to aggregate risk and thus, it becomes an infinite-dimensional element of the state space.

The first open question I tackle in this thesis is of a theoretical nature. Existence and uniqueness of solutions to models with ex-post heterogeneity have not been established yet in the case of a continuum of agents and unbounded utility. I close this gap for the Aiyagari-Bewley growth model à la Krusell and Smith (1998). To do so, I exploit the monotonicity properties of the model and use arguments from convex analysis. This methodology stands in contrast to the current literature on existence relying on compactness arguments.

Another open question related to this type of models is how to compute solutions of the rational expectations equilibrium. It is clear that models with ex-post heterogeneity have to be solved numerically in a way such that the recursive equilibrium price and policy functions depend on the cross-sectional distribution. To tackle this problem, I derive an iterative procedure to find the optimal policy functions. Convergence of this procedure for the growth model is ensured as a result of my theoretical analysis. Furthermore, I propose a rather unknown discretization technique to approximate the cross-sectional distribution. Most importantly, I derive the corresponding approximation error bounds. With this theoretically founded numerical method, I then compute the solution of the growth model and show that approximate aggregation, i.e., the fact that idiosyncratic risk averages out in aggregate variables, does not hold when agents share their idiosyncratic risk to a high degree.

Finally, in a more practical exploration, I compute the solution to an asset pricing model with ex-post heterogeneity which features bond trading. A proper investigation of this model has still been lacking due to the absence of a suitable computational method. I establish two novel economic results which also hold for the growth model. First, I find that ex-post heterogeneity does indeed amplify the effect of aggregate risk in models where agents share the idiosyncratic risk. This is in contrast to the existing literature, which often concludes that heterogeneity does not matter for aggregates. Second, I find that higher risk sharing of idiosyncratic risk among the heterogeneous agents leads to fatter tails in the price and the cross-sectional distribution, respectively, which means that more risk sharing can produce higher systemic risk.
Résumé

En raison du risque idiosyncratique, les modèles d’équilibre général dynamique stochastique avec ex-post hétérogénéité posent de nombreuses difficultés quant à la distribution transversale des variables endogènes. Cette distribution change de manière stochastique à travers le temps à cause du risque agrégé et, ainsi, elle devient un élément infiniment dimensionnel de l’espace d’état.


Une autre question ouverte liée à ce type de modèles concerne le calcul des solutions des équilibres à attentes rationnelles. Il est clair que les modèles avec ex-post hétérogénéité doivent être résolus numériquement de telle manière à ce que le prix d’équilibre récursif et la fonction de politique stratégique dépendent de la distribution transversale. Pour traiter ce problème, je dérive une procédure itérative pour obtenir les fonctions de politique stratégique optimales. La convergence de cette procédure pour le modèle de croissance est garantie par les résultats de mon analyse théorique. De plus, je propose une technique de discrétisation encore peu connue pour approximer la distribution transversale. Surtout, je dérive les bornes d’erreurs d’approximation correspondantes. Grâce à cette méthode numérique fondée sur des bases théoriques, je calcule ensuite la solution du modèle de croissance et montre que l’agrégation approximée, i.e. le fait que les risques idiosynchratiques se compensent sur les variables agrégées, ne tient pas lorsque les agents partagent leur risque idiosyncratique dans une large mesure.

Finalement, dans une exploration davantage pratique, je calcule la solution d’un modèle d’évaluation d’actifs avec ex-post hétérogénéité traitant du négoce d’obligations. Une enquête véritable de ce modèle a jusqu’ici manqué à cause de l’absence d’une méthode de calcul appropriée. J’établis deux nouveaux résultats économiques qui tiennent également pour le modèle de croissance. Premièrement, je trouve que l’hétérogénéité ex-post amplifie l’effet du risque agrégé dans les modèles où les agents partagent le risque idiosyncratique. Ce résultat contraste avec la littérature existante qui conclut souvent que l’hétérogénéité n’affecte pas les variables agrégées. Deuxièmement, je trouve qu’un partage plus élevé de risque idiosyncratique parmi les agents hétérogènes implique des queues plus épaisses dans le prix et la distribution transversale respectivement, ce qui signifie qu’un partage de risque plus prononcé peut aboutir à davantage de risque systémique.
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Introduction

Economies consist of heterogeneous agents who are exposed to various idiosyncratic risks, the most prominent example of which is labor income risk for households. This was first modeled in a dynamic stochastic general equilibrium (DSGE) model by Bewley (1977) where agents face idiosyncratic income shocks affecting their wealth, and, extended by Aiyagari (1994) to include a production technology. They show that individual precautionary savings contribute to aggregate savings because idiosyncratic risk cannot be fully insured. Other examples of idiosyncratic risks are firm-specific productivity shocks in models of firm exit and entry as in Hopenhayn (1992) or uninsurable persistent income shocks as in Constantinides and Duffie (1996), who show that these shock have a strong impact on the equity premium. Lately, there has been a renewed interest in heterogeneous agent models. Many recent studies reevaluate the impact of heterogeneity in the economy and find strong implications. Let me mention just a few examples. Khan and Thomas (2008) model heterogeneity to investigate the effect of non-convex adjustment costs and find that they are important to produce a realistic investment rate distribution. Apart from achieving a realistic wealth distribution using heterogeneous households trading in two assets, Kaplan et al. (2018) investigate the effect of monetary policy on the consumption of households. They find that the indirect effects significantly outweigh the direct ones. In finance, Storesletten et al. (2007) find a moderate effect of idiosyncratic risk on the Sharpe ratio, but a significant negative impact on inter-generational risk sharing. Overall, there is plenty of evidence that idiosyncratic risks have a sizable impact on the economy.

Such models often do not feature aggregate risk, however, to avoid the practical as well as theoretical difficulties related to it in solving the model. The challenge lies in handling the cross-sectional distribution of the agents’ idiosyncratic variables, which becomes an infinite-dimensional element of the state space. Moreover, this distribution changes stochastically over time depending on the realization of the aggregate shocks. The aggregate variables evolve, in turn, depending on how the cross-sectional distribution changes.

Chapter 1

Existence of solutions to heterogeneous agent models, in particular, the Aiyagari-Bewley growth model with aggregate risk has long been an open research question. It has first been examined by Miao (2006) who builds the existence argument on a fixed point of the value function which directly depends on the cross-sectional distributions. Cao (2016) improves the argument by treating the case of zero aggregate capital more carefully. However, as pointed out in Cheridito and Sagredo (2016b), these two approaches are subject to a flaw in the theoretical argument, namely that weak convergence of measures does not imply convergence of moments. Cheridito and Sagredo (2016a) provide an alternative proof. They derive the existence of a sequential equilibrium in first proving existence
of a value function which depends on a series of fixed aggregates and in a second step, proving existence of a fixed point in the aggregates consistent with the agent’s optimal choices. All of these papers mostly focus on sequential equilibria and build on fixed-point theory relying on compactness. Miao (2006) extends his results to recursive equilibria depending on the current state of exogenous and endogenous variables, the distribution and the current value function, but this extension is also prone to the Cheridito and Sagredo (2016b) critique. A recent work which does establish existence of simple recursive equilibria is Brumm et al. (2017). They present an existence result for recursive equilibria of a very generic model. To keep the generality, they focus on bounded utility and finitely many agents. Hence, the existence of recursive equilibria for the model of Krusell and Smith (1998), which features unbounded utility and a continuum of agents, is still an open question.

This chapter contributes to closing that gap by taking a different approach to the existing literature. I consider a generic equilibrium model with aggregate risk and a continuum of heterogeneous agents who maximize their CRRA utility when trading in securities. I make the restrictive assumption that the securities’ returns are explicitly defined in terms of the cross-sectional distribution and the exogenous shocks. This assumption limits the class of models to production economies like the Aiyagari-Bewley model or questions in partial equilibrium. For this reason, this chapter does not span the same general set of models as in Brumm et al. (2017). However, for the restricted set of models, I am able to establish uniqueness which is otherwise far-fetched. My methodology differs from the existing literature in two aspects.

First, a recursive equilibrium in a heterogeneous agent model with a continuum of agents is defined as a set of functions which depend on the cross-sectional distribution. I develop an alternative representation by showing that there is an equivalent set of equilibrium functions which have a random variable instead of a distribution as an argument. Sets of random variables are typically well behaved, especially the set of square-integrable random variables. The advantage of this approach fully enfolds when considering the Euler equation of the equilibrium problem. As I work with the random variable of security holdings instead of their distribution, I can substitute this random variable into the Euler equations of the individual agents. This transforms the continuum of individual Euler equations which are linked by the definition of the rate of returns into one Euler equation on random variables. It significantly simplifies the problem.

The second aspect in which I depart from the existing literature lies in the type of fixed point argument I use to establish existence. In contrast to the existing literature which relies on the compactness of the state space requiring bounded utility functions, this chapter exploits the monotonicity properties of the model and can thus handle unbounded utility functions. This approach is inspired by a series of papers by Rockafellar (1969, 1970, 1976a,b). Rather than using fixed-point theory on compact spaces, it relies on results from convex analysis and monotone operator theory. I show that the generalized Euler equation on random variables is a maximal monotone operator which implies that there exists a convex Lagrangian which has the generalized Euler equation as its first-order condition. Furthermore, there exists a root of such an operator if one can find a candidate policy at which the generalized Euler equation has a negative value and another policy at which it has a positive value. Since this equilibrium problem can be solved using arguments from convex analysis, uniqueness of the solution can be examined in a straightforward manner. When using fixed-point theory relying on compactness instead, it is typically much more difficult to investigate the uniqueness of a solution. A nice side-effect of exploiting the
monotonicity properties of the equilibrium model is that there exists a straightforward iterative procedure which is guaranteed to converge to the equilibrium solution.

Chapter 2

This chapter tackles the more practical challenges which models with heterogeneous agents pose. As these models typically do not have closed-form solutions, they have to be solved numerically. However, the cross-sectional distribution is an element of an infinite-dimensional space which results in a dimensionality problem. It is an open question how one should approximate this space.

In their seminal paper, Krusell and Smith (1998) were the first to propose a global solution algorithm for the Aiyagari growth model with aggregate risk. They handle the dimensionality problem in assuming bounded rationality, which means that agents are not required to observe the whole cross-sectional distribution to predict the movement of aggregate variables. They rather use a parametric law of motion for the aggregate variables depending on a finite number of moments. Given that assumption, they then solve the model by iterating on the following two steps: Solving for the optimal policies given a guess of parameters for the aggregate variables’ law of motion, and second, estimating new parameters for the law of motion given a set of simulated data from the new optimal policy. The main economic result from this seminal work is that, given the bounded rationality assumption, adding moments higher than the mean to the parametric law of motion does not change the equilibrium solution. Hence, the idiosyncratic risk does not matter for aggregation. Various more recent papers improve the original algorithm mainly by eliminating the agent dimension in the simulation step, and, by varying the parametric form of the law of motion. However, these works still rely on the bounded rationality assumption and a two-step iterative procedure with a simulation.

The existing methodology of global solution methods for heterogeneous agent models with aggregate risk has several drawbacks. First, it is not clear whether the assumed parametric law of motion for the aggregates in the bounded rational expectations equilibrium is indeed close to its equivalent in the fully rational expectations equilibrium. Generally, it is unknown whether the bounded rational solution is at all close to the fully rational solution since there is no theory on measuring their distance. Second, it is not clear a priori how many moments are necessary for the bounded rational equilibrium to exist. In fact, Kubler and Schmedders (2002) show that there are models, for which recursive equilibria depending only on aggregate wealth, i.e., the first moment of the cross-sectional distribution, do not exist. Third, it is unclear whether the existing algorithms converge to the bounded rational equilibrium for every model setup as theoretical convergence results are lacking.

The contribution of this chapter is twofold. To begin with, I construct a global solution algorithm for DSGE models with heterogeneous agents and aggregate risk, which does not assume bounded rationality and for which conditions for convergence exist. More specifically, I apply the proximal point algorithm from Chapter 1 to compute the optimal policy functions of the example model.

My main contribution in this chapter, however, is that I show how to discretize the space of cross-sectional distributions rather than assuming a parametric law of motion for the aggregate variables. I use a projection method which extends the polynomial projection of real functions to a projection of square-integrable random variables and hence, can be interpreted as a probabilistic polynomial projection. This technique is called generalized polynomial chaos. It has several advantages over standard polynomial projection
using, for instance, Chebyshev polynomials. First of all, polynomial chaos does not require smooth functions and can, therefore, handle distributions with mass points. This is very relevant for our example of the Aiyagari-Bewley growth model which features mass points in the cross-sectional distribution due to the hard borrowing constraint. A further complication is that the location of these mass points is endogenous such that one cannot simply treat them separately. Another advantage is that this probabilistic polynomial projection converges very fast. For a standard calibration, I find that a projection on polynomials up to the first order is enough to obtain a satisfactory precision of the solution. Hence, I only need two dimensions to sufficiently approximate the cross-sectional distribution in the growth model. Furthermore, by approximating the full distribution, the aggregate variables emerge automatically in a nonparametric fashion. Therefore, I do not require a separate step in the solution algorithm to estimate their law of motion. No simulation is necessary.

When comparing the results of our algorithm to existing methods for the benchmark Aiyagari-Bewley growth model with aggregate risk, I find a significant improvement in precision for individual policies in terms of Euler equation errors as well as a significant improvement in the prediction of the law of motion of aggregate variables. Furthermore, there is a significant improvement in precision when truncating the polynomial chaos expansion in our algorithm at order one, rather than at order zero. Note that the order zero approximation leads to optimal policies, which depend solely on the mean of the distribution, whereas, order one and higher lead to policies, which depend on an approximation of the whole cross-sectional distribution. This implies that idiosyncratic risk matters in this fully rational equilibrium. I do find a reason though, why Krusell and Smith (1998) obtain their approximate aggregation result. When computing the expected ergodic cross-sectional distributions, I see that they do not change significantly when the order of truncation increases. In another calibration of the model with high unemployment benefit, approximate aggregation does not persist. The Krusell-Smith algorithm produces an expected ergodic cross-sectional distribution with much fatter tails than our theoretically founded algorithm in this case. This suggests that bounded rationality can, in fact, contribute to inequality.

Equipped with this novel numerical method which has a theoretical foundation, I then investigate the effects of heterogeneity in the growth model. I find two interesting economic results. First, I show that heterogeneity does indeed amplify aggregate risk when agents share a significant amount of idiosyncratic risk though an unemployment insurance. The model with aggregate risk produces an expected stationary capital distribution with has fatter tails than in the case where aggregate risk is absent. Even though the magnitude of this effect is small, it reinforces the result that approximate aggregation fails for the Aiyagari-Bewley model when the level of idiosyncratic risk sharing is high. The second economic result is of a larger magnitude. Comparing model configurations with high and low levels of unemployment insurance yields the insight that more risk sharing produces fatter tails in the expected stationary capital distribution. This result does, in fact, resemble the volatility paradox in Brunnermeier and Sannikov (2014) which states that risk sharing can lead to more systemic risk.

This chapter relates to several strands of literature. First of all, it is related to the literature on numerical algorithms. In general, there are two types of algorithms: Local solution methods are based on perturbation techniques, whereas, global solution methods are based on projection techniques or a mixture of projection and simulation techniques. My algorithm and the aforementioned seminal algorithm by Krusell and Smith (1998)
belong to the latter group. The algorithm by Krusell and Smith (1998) was also the subject of a special issue of the Journal of Economic Dynamics and Control in January 2010. This special issue presents various alternative algorithms which are compared in den Haan (2010). They have in common that they assume bounded rationality, and hence, use a small finite number of moments instead of the full cross-sectional distribution to approximate the policy function and the law of motion of aggregate variables. One problem, which is addressed by Algan et al. (2008); Young (2010); Rios-Rull (1997); den Haan (1997) and summarized in Algan et al. (2010), is the cross-sectional variation due to the simulation of a finite number of agents in Krusell and Smith (1998) when estimating the law of motion parameters. They use parametric and nonparametric procedures to get around this issue. However, the variation due to simulating over aggregate exogenous shocks remains. In contrast to the simulation approach, den Haan and Rendahl (2010) use direct aggregation to obtain the law of motion. Interestingly, Algan et al. (2008) and Reiter (2010a) parameterize the cross-sectional distribution itself to obtain a better prediction of the law of motion, but their parametric functional forms are somewhat ad hoc. They do not span the space of square-integrable random variables. I use the algorithm by Reiter (2010a) in my numerical comparison and find that it performs significantly worse than the algorithm proposed herein.

Local solution methods based on perturbations do not assume bounded rationality. To reduce dimensionality, they first solve for the optimal policy and stationary distribution of the model without aggregate shocks using projection methods, and then, perturb this solution to accommodate aggregate shocks. The most prominent perturbation algorithm goes back to Reiter (2009, 2010b). Childers (2015) investigates the theoretical underpinning of these perturbations. Mertens and Judd (2013) use perturbations for the law of motion. Winberry (2018) combines the law of motion approach of Algan et al. (2008) with the perturbation of Reiter (2009). He also presents a model where the aggregation result by Krusell and Smith (1998) does not hold. There are two major drawbacks for perturbation methods: First, the perturbation in aggregate shocks is often only linear, or at most quadratic. Therefore, any higher-order nonlinear effects of aggregate shocks like risk are not accounted for. Second, as for all perturbation methods, the solutions are only accurate for small aggregate shocks. Crises scenarios consisting of a large aggregate shock or a long series of aggregate shocks in one direction cannot be analyzed with confidence.

It is also worth pointing out the relation to the literature on mean field games and their numerical solutions because they are essentially continuous-time versions of DSGE models with ex-post heterogeneity. Achdou et al. (2017) show how to use partial differential equations to solve heterogeneous agent models. Kaplan et al. (2018) put forward a very interesting application of this methodology to monetary policy questions. However, their models incorporate only idiosyncratic shocks without aggregate risk. Applying generalized polynomial chaos, as in the algorithm presented herein, to extend their framework to aggregate risk could yield interesting results.

Chapter 3

This chapter is an extension of the first two chapters where I restrict the set of models to production economies and, thus, exclude models which feature zero or unit-net supply conditions. These models are, however, very important as one can explore the reasons why people trade in a centralized market. Additionally, the cross-sectional distribution plays a much stronger role in the price discovery than in production economies.

Extending the theory from Chapter 1 to these models lies outside the scope of this
thesis which is why I take a more practical approach. I devise a computational algorithm by extending the proximal point algorithm from Chapter 1. To accommodate zero or unit-net supply conditions, I add a second step in which I solve for the equilibrium prices given the policies produced by the proximal point algorithm. In both steps, I use polynomial chaos to discretize the space of cross-sectional distributions. For this type of models, this discretization technique enfold its full power as the equilibrium prices are much more sensitive to changes in the cross-sectional distribution.

I illustrate this computational method with the Huggett (1993) model which has been extended in Krusell et al. (2011) to accommodate aggregate risk in the restrictive case of maximally tight borrowing constraints. This model features an endowment economy with aggregate risk and heterogeneous agents who trade in a bond in zero net supply. Adding aggregate risk without the maximally tight borrowing constraint to this model leads to significantly different prices compared to the model without aggregate risk. They are lower in recessions and higher in booms. This shows that heterogeneity amplifies aggregate risk as the agents share their idiosyncratic risk by trading the bond. The risk sharing occurs as agents with a low endowment sell the bond to transform future endowment into current consumption, whereas, agents with high endowment buy the bond to build up precautionary savings. It follows that tighter borrowing constraints restrict the level of risk sharing. When considering the stationary distribution of equilibrium prices, I show that the increase in risk sharing by relaxing the constraint results in higher price volatility. Thus, I recover the volatility paradox in the Huggett model as well.
Chapter 1

Theory

Abstract
In this paper, I study the existence and uniqueness of recursive equilibria in production economies with aggregate risk. The economy features a continuum of agents who face idiosyncratic shocks and borrowing constraints. In particular, I establish existence for the Aiyagari-Bewley growth model à la Krusell and Smith (1998). In contrast to the existing literature, I do not rely on compactness to establish a fixed point. I instead exploit the monotonicity property of the equilibrium model and rely on arguments from convex analysis.

The paper proceeds as follows. I first introduce a generic model framework. Second, I characterize the recursive equilibrium by functions depending on random variables which results in a generalized Euler equation substituting the continuum of individual Euler equations. In Section 1.3, I establish the monotonicity properties which are necessary for the existence of equilibria. The following section applies this general framework to the Aiyagari-Bewley economy with aggregate risk. Lastly, I introduce the corresponding convergent iterative procedure which can be used to compute the equilibrium numerically. Appendix A.1 contains all proofs.

1.1 A Generic Model

Consider a discrete-time infinite-horizon model with a continuum of agents of measure one. There are two kinds of exogenous shocks, an aggregate shock and an idiosyncratic shock. The aggregate shock characterizes the state of the economy with outcomes in $Z^{ag} \subset \mathbb{R}$. It follows a first-order Markov process with transition probability $\mathbb{P}(\cdot|z^{ag}) : Z^{ag} \times \sigma(Z^{ag}) \to [0,1]$ defined on the generating Borel-$\sigma$-algebra. The idiosyncratic shock with outcomes in $Z^{id} \subset \mathbb{R}$ represents the agent-specific risk. It is a first-order Markov process which is i.i.d. across agents and whose transition probability at any point in time $t$ is conditional on the aggregate shocks $\mathbb{P}(\cdot|z^{id}_t, z^{ag}_{t-1}, z^{ag}_t) : Z^{id} \times \sigma(Z^{id} \times Z^{ag} \times Z^{ag}) \to [0,1]$. I denote the compound exogenous process $(z^{ag}_t, z^{id}_t)_{t \geq 0}$ by $(z_t)_{t \geq 0} \in \mathcal{Z}$ with $\mathcal{Z} = Z^{ag} \times Z^{id}$. The only requirement I impose on the exogenous stochastic processes is square integrability.

Assumption 1 (Square integrability). The aggregate and idiosyncratic exogenous pro-
cesses \((z^{ag}_t)_{t \geq 0}\) and \((z^{id}_t)_{t \geq 0}\) are square integrable, i.e., \(\mathbb{E}[(z^{ag}_t)^2] < \infty\) and \(\mathbb{E}[(z^{id}_t)^2] < \infty\) at any time point \(t \in \mathbb{N}\).

This specification of the aggregate and idiosyncratic shock is fairly flexible. It does include finite state Markov chains as well as continuous Markov processes in discrete time. Linear growth ensures square integrability in the latter case.

**Example.** Examples for both exogenous processes include the following.

(i) Finite Markov chain: Define a finite state space \(S = \{s_1, \ldots, s_N\}\). Then, \(z_t \in S\) with the transition probabilities being given by \(\pi_{ij} = \mathbb{P}(z_t = s_i|z_{t-1} = s_j)\) if \(z\) is an aggregate process or \(\pi_{ij} = \mathbb{P}(z_t = s_i|z_{t-1} = s_j, z^{ag}_t)\) if \(z\) is idiosyncratic.

(ii) AR(1) process: Assume a normally distributed innovation \(\eta \sim N(0, \sigma^2)\) and define \(z_{t+1} = c + a z_t + \eta\) with \(c\) constant and \(a \in [0, 1]\). The dependency of the idiosyncratic shock on the aggregate shock can be achieved by letting the mean and/or volatility of \(\eta\) vary depending on the current aggregate outcome.

Agents can invest in \(n\) one-period securities. An agent’s share of security \(j \in \{1, \ldots, n\}\) is denoted by \((x^j_t)_{t \geq 0}\). The security \(j \in \{1, \ldots, n\}\) pays a rate of return \((r^j_t)_{t \geq 0}\) after one holding period.\(^1\) Each agent chooses her share of the securities and consumption such that they satisfy certain constraints. First, individual consumption must be positive at all times \(c_t > 0\), \(t \geq 0\), and security holdings are subject to a borrowing constraints \(x^j_t \geq \bar{x}^j\), \(t \geq 0\), where \(\bar{x}^j \leq 0\) for \(j \in \{1, \ldots, n\}\). Second, given the initial holdings \(x^j_0 \geq \bar{x}^j\), each agent adheres to a budget constraint, which equates individual consumption and current security holdings to current endowment and the return on previous holdings

\[
\sum_{j=1}^{n} x^j_t + c_t = e(z_t) + \sum_{j=1}^{n} (1 + r^j_t)x^j_{t-1} \forall t \geq 0. \tag{1.1}
\]

The endowment process \(e\) is given exogenously. The returns are aggregate endogenous variables. They are defined through the equilibrium condition which aggregates over the security holdings to equalize demand and supply. There are two possibilities how returns and the equilibrium condition can be connected. In production economies, the returns are explicitly set by an optimizing firm and, thus, depend on the firm’s aggregate capital demand. In asset markets, returns are implicitly defined by a zero or unit-net supply equilibrium condition. I restrict the analysis of this chapter to the former case.

**Assumption 2.** Suppose that the equilibrium returns \((r^j_t)_{t \geq 0}\) for security \(j\) are of the form

\[
r^j_t = f^j (z_t, S_t),
\]

where \(S_t\) is the exogenously given process describing the vector of the securities’ aggregate supply. The function \(f^j : \mathbb{Z} \times \mathbb{R} \rightarrow \mathbb{R}\) is once continuously differentiable in its second argument and bounded below by \(f(z, S) > -1\) for any \((z, S) \in \mathbb{Z} \times \mathbb{R}\).

\(^1\)Note that I choose to work solely with assets which pay a variable return for a fixed price rather than with assets which have a fixed return and can be bought for a variable price as these two types of assets are equivalent. To convert an asset with variable return into an asset with fixed return, we can simply substitute the asset holding at the transaction date with \(x^j_t = y^j_t p^j_t\), where the price is given by \(p^j_t = 1/\mathbb{E}_t[1 + r^j_{t+1}]\). This leads to a fixed payout \(y^j_t\).
Remark. This is a strong assumption which restricts the set of models to production economies or partial equilibria which define returns explicitly in terms of the aggregate security supply. It does not cover models with zero or unit-net supply conditions where returns are implicitly defined which requires solving an inverse problem. I leave the extension of the methodology presented herein to this latter case to future research.

Agents optimize their utility. I assume that all agents have a time-separable CRRA utility with a risk aversion coefficient \( \gamma > 1 \) or logarithmic utility when \( \gamma = 1 \). Then, given an agent’s initial security holdings \( x^j_{t-1} \geq \bar{x}^j \), the individual optimization problem reads

\[
\max_{\{c_t,x^j_t\} \in \mathbb{R}^{n+1}} E \left[ \sum_{t=0}^{\infty} \beta^t c_t^{1-\gamma} - 1 \right] \quad (1.2)
\]

s.t. \( \sum_{j=1}^{n} x^j_t + c_t = e(z_t) + \sum_{j=1}^{n} (1 + r^j_t)x^j_{t-1} \forall t \geq 0 \)

\[ c_t > 0, \ x^j_t \geq \bar{x}^j \forall j \in \{1, \ldots, n\}, \ t \geq 0, \]

where \( \beta \in (0, 1) \) is the time preference parameter.

Let me now introduce the cross-sectional distribution of the model. I use the methodology of Fubini extension by Sun (2006) to ensure the validity of the law of large numbers when aggregating over the continuum of agents with measure one. In particular, denote the atomless measure space of agents by \( (I, \mathcal{I}, \lambda) \) with \( \lambda(I) = 1 \) and the sample probability space at time \( t \) by \( (\mathcal{Z}^{id}, \sigma(\mathcal{Z}^{id}), P^{id}) \) with \( P^{id} = \mathbb{P}(\cdot | z_{t-1}, z^{id}_t) \). Let \( f \) be a measurable function mapping the Fubini extension \( (I \times \mathcal{Z}^{id}, \mathcal{I} \times \sigma(\mathcal{Z}^{id}), \lambda \times P^{id}) \) into \( \mathbb{R} \). If the random variables \( f(i,.) \) are essentially pairwise independent, then \( f(i,.) \) have a common distribution \( \mu \) for \( \lambda \)-almost all \( i \in I \). The same holds for the samples \( f(.,z^{id}) \).

When \( f \) represents individual security holdings, we have that \( x^j_t = f^j(i, z^{id}_t) \) for agent \( i \) and, thus, \( x^j_t \) is distributed according to \( \mu^j_t: \mathcal{Z}^{id} \times [\bar{x}^j, \infty) \rightarrow [0,1] \). Hence, I denote the cross-sectional distribution of agent-specific variables at the beginning of period \( t \) by \( \mu_t: \mathcal{Z}^{id} \times \prod_{j=1}^{n} [\bar{x}^j, \infty) \rightarrow [0,1] \). Note that the aggregate shocks cause the cross-sectional distribution to vary over time, which is indicated by the time subscript of \( \mu_t \).

The equilibrium conditions of the model aggregate over the cross-sectional distribution to equate the securities’ demand and supply. Let \( E \) denote a linear aggregation operator on the cross-sectional distribution \( E: \mathcal{P}(\mathcal{Z}^{id} \times \prod_{j=1}^{n} [\bar{x}^j, \infty)) \rightarrow \mathbb{R}^n \) which computes the vector of aggregate security holdings. Then, the equilibrium condition reads

\[
E[\mu_t] = S_t, \quad (1.3)
\]

where \( S \) denotes the exogenous process of aggregate supply of the securities. The equilibrium condition implies that the equilibrium returns depend on the aggregate security holdings.

Before I define the equilibrium for this model, let me clarify the time line with Figure 1.1. Note that I specify the time line slightly differently from existing papers. Often, \( x^j_t \) is substituted with \( x^j_{t+1} \) in the budget constraint (1.1) because this is the security holding with a payout at \( t + 1 \). In contrast to that notation, however, I want to emphasize the time period, at which the agent optimally chooses the magnitude of her security holdings. Taking this view, the optimal consumption and security holdings choices have the same time subscript. My time line, therefore, indicates which filtration the endogenous variables are adapted to.
Figure 1.1: **Time line of events.** Before period $t$, the agent observes how much securities everybody decided to hold in the previous period. At period $t$, the agent observes the exogenous shocks $z_t$, and therefore, knows the beginning-of-period cross-sectional distribution $\mu_t$ and the aggregation quantity $\mathbb{E}[\mu_t]$. The agent then decides how much to consume $c_t$ and how much to invest $x^j_t$ in security $j$.

In a competitive equilibrium, the individual problems are solved subject to the equilibrium condition (1.3). In this thesis, I consider a particular competitive equilibrium of recursive form. To define a recursive equilibrium, I will switch to prime-notation for convenience, where a prime denotes variables in the current period and variables with no prime refer to the previous period.

**Definition 3 (Recursive equilibrium).** Consider the measurable functions $^\text{2}$

\[
\begin{align*}
g_c & : \mathbb{Z} \times \mathbb{R}^n \times \mathcal{P} \left( \mathbb{Z}^{id} \times \prod_{j=1}^n [\bar{x}^j, \infty) \right) \to \mathbb{R} \\
g_x & : \mathbb{Z} \times \mathbb{R}^n \times \mathcal{P} \left( \mathbb{Z}^{id} \times \prod_{j=1}^n [\bar{x}^j, \infty) \right) \to \mathbb{R}^n \\
g_r & : \mathbb{Z} \times \mathcal{P} \left( \mathbb{Z}^{id} \times \prod_{j=1}^n [\bar{x}^j, \infty) \right) \to \mathbb{R}^n, g(z, \mu) = f(z, \mathbb{E}[\mu]),
\end{align*}
\]

where the equilibrium returns are given by Assumption (2) inserting the equilibrium condition (1.3). A solution to the agents’ individual optimization problems (1.2) subject to the equilibrium condition (1.3) given an initial cross-sectional distribution of individual security holdings $\mu_0$ is called recursive if for any point in time, the current equilibrium rates of return and the optimal consumption and security holdings choices for $j \in \{1, \ldots, n\}$ are given by

\[
\begin{align*}
r^j' &= g^j_r (z', \mu') \\
c' &= g^c_c (z', x^1, \ldots, x^n, \mu') \\
x^j' &= g^x_x (z', x^1, \ldots, x^n, \mu')
\end{align*}
\]

$^2$ I assume that the equilibrium functions $g(\ldots, \mu')$ are measurable w.r.t. the probability triple \( \left( \mathbb{Z}^{id} \times \prod_{j=1}^n [\bar{x}^j, \infty), \sigma \left( \mathbb{Z}^{id} \times \prod_{j=1}^n [\bar{x}^j, \infty) \right) \right), \mu' \).
for any agent with previous-period security holdings \((x^1,\ldots,x^n)\) who observes the current-period exogenous shock \(z' = (z'^a, z'^d)\) and the beginning-of-current period cross-sectional distribution \(\mu'\).

1.2 Characterizing the Incomplete Markets Equilibrium

Now that the model and its equilibrium are defined in a general manner, I explain how the recursive equilibrium induces the law of motion of the cross-sectional distribution and how that leads to an operator which characterizes the equilibrium.

1.2.1 A Consistent Law of Motion

In a recursive equilibrium, we can easily define a law of motion of \(\mu\) to \(\mu'\) which is consistent with the agents’ optimal choices. Given a fixed distribution \(\mu\) over the cross-section of individual security holdings at the beginning of the previous period, the distribution changes in two steps \(\mu \rightarrow \tilde{\mu} \rightarrow \mu'\). In the first step, the agents implement their optimal previous-period security holdings according to the recursive equilibrium from Definition 3, which leads to the end-of-previous period distribution

\[
\tilde{\mu}\left(z^{id}, x\right) = \int_{\zeta \in Z_{id} \cap \{\zeta \leq z^{id}\}} \int_{\bar{x}_1}^{\infty} \cdots \int_{\bar{x}_n}^{\infty} 1_{\{g_x(z^{ag}, \zeta, \chi, \mu) \leq x\}} d\mu(\zeta, \chi).
\]

In the second step, the current-period shocks \(z'\) realize for all agents and shift the quantities of the agents with a specific idiosyncratic shock according to the exogenous transition probabilities conditional on the outcome of the aggregate shock. The beginning-of-current period distribution is hence, computed by integrating over the transition probabilities that the idiosyncratic state changes from \(z^{id}\) to \(z^{id'}\) given the observed trajectory of \(z^{ag}\) to \(z^{ag'}\). The distribution \(\mu'\) is given by

\[
\mu'\left(z^{id'}, x\right) = \int_{z^{id} \in Z^{id}} \tilde{\mu}\left(z^{id}, x\right) \mathbb{P}\left(z^{id'} \mid dz^{id}, z^{ag}, z^{ag'}\right)
\]

\[
= \int_{z^{id} \in Z^{id}} \int_{\zeta \in Z_{id} \cap \{\zeta \leq z^{id}\}} \int_{\bar{x}_1}^{\infty} \cdots \int_{\bar{x}_n}^{\infty} 1_{\{g_x(z^{ag}, \zeta, \chi, \mu) \leq x\}} d\mu(\zeta, \chi) \mathbb{P}\left(z^{id'} \mid dz^{id}, z^{ag}, z^{ag'}\right).
\]

for all \(z^{id'} \in Z^{id}\) and \(x \in \mathbb{R}^n\). Note that the rate of returns \(r'\) follow immediately from this definition of the current-period distribution according to Assumption (2).

1.2.2 Rewriting the Recursive Equilibrium

As we consider a heterogeneous agent model with a continuum of agents, the equilibrium defined in Definition 3 consists of policy functions which depend on the cross-sectional distribution. Since functions on measures are typically difficult to handle, I will show in this section that the equilibrium functions can be restated in a more tractable form. Inspired by the approach taken in a paper on mean field games by Carmona and Delarue (2015), I lift the functions on distributions into functions on the space of random variables. Carmona and Delarue (2015) trace the idea of converting distributions into random variables back to lectures on mean field games by Pierre-Louis Lions given at the Collège
This implies that at any time 

\[ \mu \]

The conditional random variable of security holdings which is distributed according to conditional distribution of the security holdings is given by to define a conditional random variable representing the security holdings. Note that the I use the corresponding probability space (section, I define it in terms of random variables. Since the initial distribution \( \mu_0 \) is given, I use the corresponding probability space \( (Z^{id} \times \prod_{j=1}^{n} [\bar{z}^j, \infty), \sigma(Z^{id} \times \prod_{j=1}^{n} [\bar{z}^j, \infty)), \mu_0) \) to define a conditional random variable representing the security holdings. Note that the conditional distribution of the security holdings is given by

\[
\mu^{id}_0 (x) = \frac{\mu_0 (z^{id}, x)}{P(z^{id} \mid z^{ag}_0)}. 
\]

The conditional random variable of security holdings which is distributed according to \( \mu^{id}_0 \) is denoted by \( \chi_0(z^{id}) \). Given a trajectory of aggregate shocks \( (z^{ag}_t)_{t \geq 0} \), I define the series of conditional security holdings \( (\chi_t(z^{id}))_{t \geq 0} \) by induction using (1.4)

\[
\chi_t(z^{id}) = \int_{Z^{ag}} g_x (z^{ag}_{t-1}, z^{id}, \chi_{t-1}(z^{id}), \mu_{t-1}) \, P(z^{id} \mid dz^{id}, z^{ag}_{t-1}, z^{ag}_t), \quad t \geq 1. \tag{1.5} 
\]

This implies that at any time \( t \), the random variable of security holdings \( \chi_t(z^{id}) \) conditional on an idiosyncratic state \( z^{id} \) is a function of the trajectory of aggregate shocks \( (z^{ag}_t, \ldots, z^{ag}_t) \) and the initial random variable of security holdings \( \chi_0(z^{id}) \). Therefore, the security holdings at any time point are measurable w.r.t. \( \mu_0 \), i.e., \( \chi_t(z^{id}) \in L_{\mu_0} \) where I use the short-hand notation

\[
L_{\mu_0} = L \left( Z^{id} \times \prod_{j=1}^{n} [\bar{z}^j, \infty), \sigma(Z^{id} \times \prod_{j=1}^{n} [\bar{z}^j, \infty)), \mu_0 \right). 
\]

Accordingly, we can write any cross-sectional distribution \( \mu_t \) as a function of \( (z^{ag}_t, \ldots, z^{ag}_t) \) and \( \chi_{t-1}(z^{id}) \)

\[
\mu_t(z^{id}, x) = \int_{\zeta \in Z^{id} \cap (\zeta \leq z^{id})} \chi_t(\zeta) \leq x \, P(d\zeta \mid z^{ag}_t), \quad t \geq 1. \tag{1.6} 
\]

which implies that the cross-sectional distribution at any time \( t \) is a measurable function w.r.t. the initial distribution \( \mu_t \in L_{\mu_0} \). Due to (1.6), we can also rewrite the recursive equilibrium in terms of the conditional random variable of the beginning-of-current period security holdings \( \chi(z^{id}) \).

**Proposition 4** (Recursive equilibrium with random variables). **Consider the recursive equilibrium in Definition 3. According to (1.6), we can rewrite the equilibrium functions**
with functions

\[ h_c : \mathbb{Z} \times \mathbb{R}^n \times L_{\mu-1} \to \mathbb{R} \]
\[ h_x : \mathbb{Z} \times \mathbb{R}^n \times L_{\mu-1} \to \mathbb{R}^n \]
\[ h_r : \mathbb{Z} \times L_{\mu-1} \to \mathbb{R}^n, \]
\[ g(z, \chi) = f(z, \mathbb{E}[\chi]) , \]

such that

\[ r^j' = g^j_r (z', \mu') = h^j_r \left( z', \chi \left( z^{id} \right) \right) \]
\[ c' = g_c (z', x^1, \ldots, x^n, \mu') = h_c \left( z', x^1, \ldots, x^n, \chi \left( z^{id} \right) \right) \]
\[ x^j' = g^j_x (z', x^1, \ldots, x^n, \mu') = h^j_x \left( z', x^1, \ldots, x^n, \chi \left( z^{id} \right) \right) \]

where \( \chi(z^{id}) \in L_{\mu_0} \) is the conditional random variable of the beginning-of-current period security holdings of the form (1.5).

1.2.3 The Euler Equation Operator

In this section, I derive an operator which characterizes the equilibrium. Note that the optimal policy functions of the recursive equilibrium solve the Euler equation, which, if a suitable transversality condition holds, is necessary and sufficient for optimality. The set of Euler equations corresponding to the model from Section 1.1 reads

\[ 0 = \left( e(z') + \sum_{j=1}^n \left[ (1 + r^j') x^j - x^j' \right] \right)^{-\gamma} \]
\[ - \mathbb{E}(z''|z') \left[ \beta \left( 1 + r^{j''} \right) \left( e(z'') + \sum_{j=1}^n \left[ (1 + r^{j''}) x^j' - x^j'' \right] \right)^{-\gamma} \right] \]
\[ + y^j', \forall j \in \{1, \ldots, n\}. \]

Note that I attach the borrowing constraints of the security holdings \( x' \) with Lagrange multipliers \( y' \). Inserting the recursive equilibrium functions according to Definition 3, yields

\[ 0 = \left( e(z') + \sum_{j=1}^n \left[ (1 + g^j_r (z', \mu')) x^j - g^j_x (z', x, \mu') \right] \right)^{-\gamma} \]
\[ - \mathbb{E}(z''|z') \left[ \beta \left( 1 + g^j_r (z'', \mu'') \right) \left( e(z'') \right) \right] \]
\[ + \sum_{j=1}^n \left\{ (1 + g^j_r (z'', \mu'')) g^j_x (z', x, \mu') - g^j_x (z'', g_x (z', x, \mu'), \mu'') \right\}^{-\gamma} \]
\[ + y^j (z', x, \mu'), \forall j \in \{1, \ldots, n\} , \]

where the Lagrange multipliers are functions which complement \( g_x \) such that \( y \perp g_x \geq 0 \).

The set of Euler equations has to hold at any exogenous state for any agent in the economy which means that it has to hold for a.e. \( z' \) and \( \mu' \)-a.e. \( x \). Recall from the previous section that the beginning-of-period security holdings \( x \) can be defined by a random variable \( \chi(z^{id}) \in L_{\mu_0} \) of the form (1.5). In terms of this random variable, the Euler equations
have to hold pointwise. By inserting this random variable into the equilibrium functions instead of $x$ and applying Proposition 4, the equilibrium functions are of the form $h_x(z', \chi(z^{id'}), \chi(z^{id''}))$. For notational convenience, I drop one of the two identical random variables in the following. Before defining the operator which characterizes the equilibrium, I make the following assumption on the equilibrium functions from Proposition 4.

**Assumption 5** (Square integrability).

(i) The initial conditional random variable of security holdings distributed according to the initial conditional cross-sectional distribution $\chi(z^{id}) \sim \mu_0^{z^{id}}$ is square integrable w.r.t. $\mu_0$.

(ii) The equilibrium functions $h_c, h_x$ and $h_r$ from Proposition 4 at a fixed $\chi \in L_{\mu_0}$ are square integrable w.r.t. their components $(z', x)$. The corresponding probability distribution is the product of the distribution of the exogenous shocks with the conditional distribution of security holdings $\mu'$. Due to (1.6), we can write $\mu'$ in terms of $\mu_0$, and therefore, I assume $h_c, h_x, h_r \in L_\mathbb{P}^2$ with

$$L_\mathbb{P}^2 = L^2 \left( \mathbb{Z} \times \prod_{j=1}^n [\bar{x}_j, \infty), \sigma \left( \mathbb{Z} \times \prod_{j=1}^n [\bar{x}_j, \infty) \right), P(\cdot | z) \mu_0^{z^{id'}} \right),$$

i.e., $P(z', x) = P(z'|z) \mu_0^{z^{id'}}(x)$.

(iii) The endowment function $e$ and the function $f$ defining the equilibrium rate of return are square integrable w.r.t. the exogenous shocks.

I now define the operator characterizing the equilibrium on the space of square-integrable functions $L_\mathbb{P}^2$.

**Definition 6** (Euler equation operator). Suppose that Assumptions 2 and 5 hold. Then, the Euler equation operator corresponding to the model from Section 1.1 is defined by $T : L_\mathbb{P}^2 \rightarrow L_\mathbb{P}^2$ with $h \mapsto [T^1[h], \ldots, T^n[h]]$ with

$$T^i[h_{z,i}] (z', \chi(z^{id'})) =$$

$$\left( e(z') + \sum_{j=1}^n \left\{ \left( 1 + f_j(z', E[\chi(z^{id'})]) \right) \chi^j(z^{id'}) - h^j_x (z', \chi(z^{id'})) \right\} \right)^{-\gamma}$$

$$- E[z''|z'] \left[ \beta \left( 1 + f^i(z'', E[\chi(z^{id''})]) \right) \left( e(z'') + \sum_{j=1}^n \left\{ \left( 1 + f^j(z'', E[\chi'(z^{id''})]) \right) h^j_x (z', \chi(z^{id''})) - h^j_x(z'', \chi'(z^{id''})) \right\} \right)^{-\gamma} \right],$$

where $i \in \{1, \ldots, n\}$ and

$$\chi'(z^{id''}) = \int_{z^{id'}} h_x(z', \chi(z^{id'})) P(z^{id''} | dz^{id'}, z^{ag'}, z^{ag''})$$

defines the law of motion of the random variable of security holdings in line with (1.5).
Remark. Note that the Euler equation operator directly incorporates the equilibrium conditions (1.3) due to Assumption 2. Furthermore, the Euler equation operator summarizes the Euler equations of all agents which is possible by switching to the random variables $\chi$.

To summarize, we obtain a candidate equilibrium solution by finding functions $h_x, y \in L^2_P$ which solve the following equation

$$T[h_x] (z', \chi(z'^d)) + y (z', \chi(z'^d)) = 0, \quad h_x \perp y \geq 0$$

(1.7) $P$-almost surely. Given such a solution, the original recursive equilibrium functions are recovered by

$$g^j (z', \mu') = h^j (z', x, \chi(z'^d))$$
$$g_c (z', x, \mu') = h_c (z', x, \chi(z'^d))$$
$$g^x (z', x, \mu') = h^x (z', x, \chi(z'^d)),$$

where $h_.$ is evaluated at the random variable $\chi(z'^d) \sim \mu'(z', x)/P(z'|z'ag)$. If this candidate solution additionally satisfies a suitable transversality condition, it is indeed an equilibrium solution. I explain in the next section how to ensure that solving (1.7) leads to an equilibrium solution.

1.3 Existence of an Equilibrium Solution

As is shown in Stokey et al. (1989), the extension of existence results with bounded utility functions to unbounded utility functions like the case of CRRA is typically done via constant returns to scale. However, due to the idiosyncratic shocks, there is a disjunction between individual security holdings and their rates of returns which aggregate over the individual holdings. Each agent in the continuum has zero weight and cannot influence aggregates. Therefore, it can happen that the individual security holdings grow substantially for an agent, but the rate of return does not change significantly to counteract this growth. In terms of Stokey et al. (1989) this model, thus, falls into the category of unbounded returns. To establish existence, I rely on arguments of monotonicity because compactness cannot be proven.

As I do not rely on a standard fixed-point theorem, let me first state the main mathematical result which I use to establish existence.

**Corollary 7 (Rockafellar (1969)).** Let $C$ be a Hilbert spaces over $\mathbb{R}$, and let $M : C \to C^*$ be a maximal monotone operator.\(^3\) Suppose that there exists a subset $B \subset C$ such that $0 \in \text{int} (\text{conv}(M(B)))$. Then, there exists a $c \in C$ such that $0 \in M(c)$.

Remark. This corollary essentially generalizes the result that there exists a root for a continuous real function $f : \mathbb{R} \to \mathbb{R}$ if there exist two points $a, b \in \mathbb{R}$ with $f(a) > 0$ and $f(b) < 0$ to higher-order spaces. Note that requiring continuity is not enough for mappings

\(^3\) Monotonicity (see e.g., Phelps, 1997; Bauschke and Combettes, 2017): Let $E$ be a Hilbert space. An operator $M : E \to E$ is called a monotone operator if for any two elements of its graph $(e, f), (\tilde{e}, \tilde{f}) \in G(M) = \{(e, f) \in E^2 | f \in M(e)\}$ it holds that $(e - \tilde{e}, f - \tilde{f}) \geq 0$. It is, additionally, called maximal monotone if any $(\tilde{e}, \tilde{f}) \in E^2$ with $(e - \tilde{e}, f - \tilde{f}) \geq 0 \forall (e, f) \in G(M)$ is necessarily also an element of the graph $(\tilde{e}, \tilde{f}) \in G(M)$.
on multidimensional spaces.\(^4\) Instead, the operator needs to be maximal monotone. If this property is satisfied, the corollary requires a subset \(B\) in the domain of the operator such that the interior of the convex hull of the subset’s image contains zero. If one finds two elements \(c_-\) and \(c_+\) such that the image \(T(c_-)\) is negative and the image \(T(c_+)\) is positive, it is possible in the general case to construct a set \(B\) such that the corollary holds.

The goal is to apply this corollary to the left-hand side of equation (1.7). Before I can do so, however, I have to establish that the operator

\[
M[h_x, y](z', \chi(z'')) = T[h_x](z', \chi(z'')) + y(z', \chi(z'')), \quad h_x \perp y \geq 0, \tag{1.8}
\]

is maximal monotone. I proceed in two steps. First, I consider the unconstrained case where by definition \(y = 0\). From the maximal monotonicity of \(T\), I then derive the same property for \(M\) in the constrained case.

1.3.1 Maximal Monotonicity in the Unconstrained Case

Let me first restrict \(T\) to the set of functions \(h_x \in L^2_P\) which are continuous in the security holdings variable \(\chi\) which is denoted by \(C(L^2_P)\). I now define an admissible set \(\mathcal{H}_\epsilon\) and show that the Euler equation operator \(T : \mathcal{H}_\epsilon \subset C(L^2_P) \rightarrow C(L^2_P)\) is maximal monotone. The proofs can be found in Appendix A.1.

**Proposition 8** (Admissible set). Consider the model from Section 1.1 and suppose that Assumptions 2 and 5 hold. For an arbitrary \(\epsilon > 0\), define the subspace \(\mathcal{H}_\epsilon \subset C(L^2_P)\) as the set of random variables for which the following inequalities are satisfied \(P\text{-}a.s.\) for any element \(h \in \mathcal{H}_\epsilon\) and \(\chi \in L^2_P\).

(i) **Limited bond holdings:**

\[
\sum_{j=1}^{n} h^j(z', \chi) \leq \epsilon(z') + \sum_{j=1}^{n} \left(1 + f^j(z', E[\chi])\right) \chi^j - \epsilon
\]

(ii) **Bounded Gâteaux derivative:**\(^5\)

\[
\left\langle \sum_{j=1}^{n} \delta h^j(z', \chi; \tilde{\chi}), \tilde{\chi} \right\rangle \leq \left\langle \sum_{j=1}^{n} \left(1 + f^j(z', E[\chi])\right) \tilde{\chi}, \tilde{\chi} \right\rangle + \left\langle \frac{\partial}{\partial x} f^j(z', E[\chi]) E[\chi] \chi, \tilde{\chi} \right\rangle
\]

for any \(\tilde{\chi} \in L^2_P\), where \(\langle \cdot, \cdot \rangle\) denotes the inner product in \(L^2_P\).

Then, \(\mathcal{H}_\epsilon\) is a Hilbert space.

**Remark.** The admissible set includes all functions of current security holdings \(h_x\), which are continuous and grow at most linearly in the previous security holdings, and have bounded slope in any direction. The first condition of the admissible set ensures that consumption is positive. Furthermore, due to the at most linear growth of \(h \in \mathcal{H}_\epsilon\), square integrability w.r.t. \(\mu_0\) is preserved under the composition \(h \circ h\).

\(^4\) A simple counterexample is \(f : \mathbb{R}^2 \rightarrow \mathbb{R}^2\) with \(f(x, y) = [\log(x + y), (x + y)^3]\).

\(^5\) Gâteaux derivative (see e.g., Zeidler, 1986b): Let \(E\) be a Hilbert space. The directional derivative of an operator \(M : E \rightarrow E\) at a point \(e \in E\) in the direction \(\tilde{e} \in E\) is defined by \(\delta M(e; \tilde{e}) = d/dt M(e + t\tilde{e})|_{t=0}\).
Using the fact that monotonicity is equivalent to \( \langle \delta T [h; \tilde{h}], \tilde{h} \rangle \geq 0 \) and that a continuous monotone operator is maximal monotone, I establish maximal monotonicity for the Euler equation operator in the following Lemma. However, I need to make another assumption.

**Assumption 9.** Aggregating over all securities, the change in the rate of return has a negative cross-sectional correlation with the change in security holdings

\[
\sum_{j=1}^{n} \langle \mathbb{E}(z''|z') \left[ \delta f^j \left( z'', \mathbb{E} [h]; \tilde{h} \right) \right], \tilde{h}^j \rangle \leq 0
\]

for any \( h, \tilde{h} \in \mathcal{H}_e \).

Assumption 9 describes the typical general equilibrium effect that current prices of a security rise when the aggregate demand, i.e., the cross-sectional average of current security holdings, increases. This means that the return on the current choice of security holdings decreases. This effect arises naturally in models where the returns are implicitly given, e.g., through a unit-net supply equilibrium condition as can be seen in the following illustrative example.

**Example.** Consider an asset pricing model with two securities. I specify this example according to the generic framework in Section 1.1. Each security pays a dividend after one period of holding the asset given by an exogenous process depending solely on the aggregate shock. The dividend yield is denoted by \( d'_i = d_i(z^{ag'}) \), \( i \in \{1, 2\} \) and the dividend payout is denominated in units of the consumption good. At any period of time the current asset value is given by the amount of shares held by an agent multiplied with the current asset price \( p'_i \), \( i \in \{1, 2\} \). Hence, the budget constraint reads

\[
e' + p'_1 x'_1 + p'_2 x'_2 = e (z') + \left( d_1 \left( z^{ag'} \right) + p'_1 \right) x_1 + \left( d_2 \left( z^{ag'} \right) + p'_2 \right) x_2.
\]

Note that the budget constraint in the individual optimization problem (1.2) is recovered by substituting \( x_i = \frac{\tilde{x}_i}{p_i'} \). The rate of return for the current security holdings choice is, hence, given by

\[
r''_i = \frac{d_i \left( z^{ag''} \right) + p''_i}{p'_i} - 1.
\]

The current price \( p'_i \) is defined by the equilibrium condition

\[
1 = \int_{\zeta \in \mathbb{Z}^d} \int_{x_1}^{\infty} \int_{x_2}^{\infty} x'_i d\mu'(\zeta, x'_1, x'_2)
\]

normalizing the total amount of shares of security \( i \) to one. Since equilibrium prices are aggregate variables, they are given by recursive equilibrium functions \( p'_i = g_{p'}(z^{ag'}, \mu') \). Therefore, current prices depend on the current choice of security holdings which yields that the rate of return for the current security holdings choice \( r''_i \), which is paid out in the next period, depends on the current price \( p'_i \). If the price increases, the rate of return decreases. In equilibrium, prices \( i \) are positively related to the aggregate security holdings \( i \) which follows from the first-order condition of the individual optimization problem

\[
p'_1 x'_1 + p'_2 x'_2 = e (z') + \left( d_1 \left( z^{ag'} \right) + p'_1 \right) x_1 + \left( d_2 \left( z^{ag'} \right) + p'_2 \right) x_2
\]

\[
- \left( \mathbb{E}(z''|z') \left[ \frac{d_i \left( z^{ag''} \right) + p''_i}{p'_i} (e'^{-\gamma})^{-\gamma} \right] \right)^{-\frac{1}{\gamma}}.
\]
Aggregating across agents yields
\[
p_i' = \frac{\int_{Z^{id}} \int_{x_1} \int_{x_2} \left( \left( d_i(z^{ag}''') + p'' \right) (c^{''})^{-\gamma} \right)^{-1} \mu'(z^{id'}, x_1', x_2') \right) \int_{Z^{id}} e(z') dP(z^{id'} | z^{ag'}) + d_1(z^{ag'}) + d_2(z^{ag'})}{\int_{Z^{id}} e(z') dP(z^{id'} | z^{ag'}) + d_1(z^{ag'}) + d_2(z^{ag'})}.
\]

The only variable on the right-hand side which depends on changes of \(x_i'\) is \(c''\). The admissible set for the consumption function in equilibrium requires that the aggregated Gâteaux derivative of consumption w.r.t. changes in the cross-sectional random variable is positive (see Assumption (ii) in Proposition 8). Therefore, prices increase with higher security holdings in the aggregate or in other words, the cross-sectional correlation of prices and security holdings is positive.

Contrary to this illustrative example, however, I restrict the type of models in this chapter to the case of explicitly defined returns by Assumption 2. Therefore, I have to add Assumption 9 to preserve the general equilibrium effect. In fact, Assumption 9 demands a slightly weaker general equilibrium effect than in the illustrative example as it aggregates over all assets, whereas, the example features a negative relation between returns and aggregate security holdings per asset. Under this assumption, maximal monotonicity holds.

**Lemma 10** (Maximal monotone Euler equation operator). Suppose that Assumption 9 and the assumptions of Proposition 8 hold. Then, the Euler equation operator \(T : \mathcal{H} \subset C(L_2^B) \rightarrow C(L_2^B)\) specified in Definition 6 is maximal monotone.

Note that the operator can be extended to the whole domain \(C(L_2^B)\) in a way which preserves maximal monotonicity (see e.g. Phelps, 1997). In our case, this means that one can define \(\overline{T}\) on \(C(L_2^B)\) such that \(\overline{T}[h] = T[h]\) for all \(h \in \mathcal{H}\).

### 1.3.2 Maximal Monotonicity in the Constrained Case

To show that the operator \(M\) is maximal monotone also in the constrained case where the Lagrange multiplier is not necessarily zero, I introduce a Lagrangian which has the Euler equation operator as the sole element of its subdifferential. Once, we have such an objective function, we can add the borrowing constraints using the Lagrange multiplier \(y\). The operator which ensures optimality of this newly introduced Lagrangian is then also maximal monotone.

It is important to understand that the first-order condition of the individual optimization problem (1.2) does not coincide with the Euler equation operator. The reason is that this optimization problem uses the sequential equilibrium policy, whereas, the Euler equation operator directly works with the recursive equilibrium functions and equilibrium returns. However, as \(T\) is maximal monotone, there exists a Lagrangian \(L_T : \mathcal{H} \times C(L_2^B) \rightarrow [-\infty, \infty]\) such that \(T\) maximizes \(L_T\) in its second argument (see Ghoussoub, 2008, Theorem 5.1). According to (Lemma 5.1, Ghoussoub, 2008), the Lagrangian associated with \(T\) is given by

\[
L_T(h, p) = \sup_{g \in \mathcal{H}} \{ \langle p, g \rangle + \langle T[g], h - g \rangle \}
\]

and maximizing the Lagrangian over \(p\) for a given \(h\) yields \(p^* = T[h]\) with the function value \(L_T(h, p^*) = \langle p^*, h \rangle = \langle T[h], h \rangle\). For notational convenience, I denote \(L_T(h, p^*)\) by \(L_T(h)\).
1.3. Existence of an Equilibrium Solution

Remark. The Lagrangian $L_T$ aggregates over first-order Taylor approximations of the agents’ utility over two time points. The aggregation denoted by the inner product happens w.r.t. the cross-sectional distribution. Therefore, we can interpret this Lagrangian as the objective function of a benevolent social planner. For each agent, the social planner uses a linearization of the agent’s utility at two time points. As we are looking for a recursive equilibrium, summing the utility over two time points suffices to optimize in the infinite horizon. The social planner weights each agent equally since aggregation over the cross-sectional distribution evaluates each state by the amount of agents which currently have that same state.

Now that a suitable objective function associated with $T$ is defined, I can attach the borrowing constraint $h \geq \bar{x}$. Therefore, I obtain a Lagrangian for the constrained problem $L : \mathcal{H}_c \times C(L^2_P) \to [-\infty, \infty]$ given by

$$L(h, y) = L_T(h) + \langle h - \bar{x}, y \rangle.$$  

I show in the following the first-order conditions of this Lagrangian form a maximal monotone operator which implies the same for (1.8).

**Lemma 11 (Maximal monotone $M$).** Consider the model from Section 1.1 and suppose that Assumptions 2, 5 and 9 hold. Then, the operator associated with the constrained problem $M : \mathcal{H}_c \subset C(L^2_P) \to C(L^2_P)$ in (1.8) where $\mathcal{H}_c$ as in Proposition 8 is maximal monotone.

1.3.3 Sufficiency and Uniqueness

Now that the property of maximal monotonicity is established for the left-hand side of the Euler equation (1.7), one can apply Corollary 7, which means that one has to find two points $(h_1, y_1), (h_2, y_2) \in \mathcal{H}_c \times C(L^2_P)$ such that $M[h_1, y_1] > 0$ and $M[h_1, y_1] < 0$ to obtain candidate solutions for the equilibrium problem. The Euler equation is normally only a necessary condition for the equilibrium. It needs to be verified that the candidate solution indeed maximizes individual utility.

**Lemma 12 (Sufficiency).** Consider the model from Section 1.1 and suppose that Assumptions 2, 5 and 9 hold. Then, the Euler equation in (1.7) where the Euler equation operator is defined on $\mathcal{H}_c$ as in Proposition 8 is necessary and sufficient for an equilibrium solution.

The sufficiency is mainly due to the fact that monotone operators are concepts from convex analysis. It is well known that the first-order condition is necessary and sufficient for a standard convex optimization problem. The problem, I consider in this chapter is not standard but this property continues to hold.

Another property from standard convex analysis carries over which is uniqueness. A strictly convex optimization problem has a unique solution. I obtain an equivalent result due to strict monotonicity of the Euler equation operator.

**Lemma 13 (Uniqueness).** Consider the model from Section 1.1 and suppose that Assumptions 2, 5 and 9 hold. Then, the Euler equation in (1.7) where the Euler equation operator is defined on $\mathcal{H}_c$ as in Proposition 8 has a unique solution.

The uniqueness result refers to recursive equilibrium solutions in the set of continuous square-integrable functions. Even though there might exist other forms of sequential
equilibria, I argue that recursive equilibria are the most important type of sequential equilibria for practical purposes. It is striking that the recursive equilibrium is unique for this fairly elaborate class of models with aggregate and idiosyncratic risk, especially given the wealth of literature on multiplicity of equilibria. It is well known that multiplicity can occur, for instance, in overlapping generations models, in the Arrow-Debreu setup or in bank run models. The main difference between these simpler setups and the one in this chapter lies in the specification of risk and the type of equilibrium solution considered. In these simpler models, one typically solves for a steady-state equilibrium where large populations have to coordinate on finitely many possible actions. The coordination problem, i.e., the requirement to know which exact action the other agents choose, results in multiplicity. Morris and Shin (2000) show that this coordination problem is resolved and uniqueness obtained by introducing even a small amount of uncertainty about the other agent’s behavior. The model investigated in this chapter features the exact same remedy in form of idiosyncratic risk. A similar mechanism is at work in game theory when moving from pure strategy Nash equilibria to mixed strategy Nash equilibria, although the result is different as the uncertainty in mixed strategies ensures existence when pure strategies might fail.

Now, all the ingredients to establish existence are ready, the only step missing is to find two concrete feasible functions \((h_1, y_1), (h_2, y_2) \in \mathcal{H}_\epsilon \times C(L^2_\mathcal{P})\) such that the images of the Euler equation’s left-hand side are negative and positive, respectively. To find specific points, we need more structure on the rate of returns. I will, therefore, illustrate the established results with the Aiyagari-Bewley growth model with aggregate risk. The strategy to identify these two points is also applicable to other models. One thing to keep in mind is that one has to choose \(\epsilon > 0\) small enough in \(\mathcal{H}_\epsilon\).

### 1.4 Example of the Aiyagari-Bewley Growth Model

I use the same growth model with aggregate shocks as in den Haan et al. (2010) and Krusell and Smith (1998). It is an Aiyagari-Bewley economy which fits the framework in this chapter. The aggregate shock characterizes the state of the economy with outcomes in \(Z^{ag} = \{0, 1\}\) standing for a bad and good state, respectively. The idiosyncratic shock with outcomes in \(Z^{id} = \{0, 1\}\) indicates that an agent is unemployed or employed, respectively. Hence, the transition probabilities of the compound process \(p^{z'\mid z}\) are exogenously given by a four-by-four matrix.

The security market consists of a claim to aggregate capital \((K_t)_{t \geq 0}\). An agent’s share of physical capital is denoted by \((k_t)_{t \geq 0}\). The aggregation operator \(E[\mu_t] = K_t\) is hence defined by

\[
K_t = \sum_{z^{id} = 0}^{1} \int_{-\infty}^{\infty} kd\mu_t\left(z^{id}, k\right) \forall t \geq 0,
\]

where \(\mu_t\) is the cross-sectional distribution of idiosyncratic exogenous and endogenous variables at the beginning of time \(t\), i.e., before the agents choose their optimal capital savings. Each agent chooses her share of physical capital and consumption such that they satisfy certain constraints. First, individual consumption must be positive at all times \(c_t > 0, t \geq 0\), and capital holdings are subject to a hard borrowing constraint \(k_t \geq \bar{x} = 0, t \geq 0\). Second, given an initial cross-sectional distribution \(\mu_0^6\) with non-negative support,

\(^6\) The initial cross-sectional distribution \(\mu_0\) does not only imply the initial aggregate capital \(K_0\), but also the initial aggregate economic state as it is pinned down by the employment rate \(P(z^{id} = 1|z^{ag}) =\)
1.4. Example of the Aiyagari-Bewley Growth Model

each agent adheres to a budget constraint, which equates individual consumption and current capital stock to productive income and saved capital stock\(^7\)

\[ k_t + c_t = I(z_t, k_{t-1}, K_t) + [1 - \delta] k_{t-1} \forall t \geq 0, \tag{1.10} \]

where \(k_{t-1}\) is distributed according to \(\mu_0/\mathcal{P}(z_0^id|z^ag)\). The parameters in this budget constraint are defined as follows. The capital stock brought forward from period \(t-1\) depreciates by a rate \(\delta \in (0, 1)\). The productive income is given by

\[ I(z_t, k_{t-1}, K_t) = R(z_t^ag, K_t) k_{t-1} \]

\[ + z_t^{id} \pi [1 - \tau_t] W(z_t^ag, K_t) + \left[1 - z_t^{id}\right] \nu W(z_t^ag, K_t). \tag{1.11} \]

It is composed of, first, the return on capital stock, and second, labor income, which equals the individual’s wage \(W\) when the agent is employed and a proportional unemployment benefit \(\nu W\) otherwise. The agent’s wage is subject to a tax rate \(\tau_t = \nu (1 - p_t^e)/\pi p_t^e\) whose sole purpose it is to redistribute money from the employed to the unemployed. The parameter \(\nu \in (0, 1)\) denotes the unemployment benefit rate, whereas, \(p_t^e = \mathcal{P}(z_t^id = 1|z_t^ag)\) is the employment rate at time \(t\) and \(\pi > 0\) is a time endowment factor. It is reasonable to assume \(\nu/\pi < 1 - \tau_t \iff \nu < \pi p_t^e\) for all \(t \geq 0\). The wage \(W\) and the rental rate \(R\) are derived from a Cobb-Douglas production function for the consumption good

\[ W(z_t^ag, K_t) = (1 - \alpha) (1 + z_t^ag a - (1 - z_t^ag) a) \left[ \frac{K_t}{\pi p_t^e} \right]^{\alpha} \]

\[ R(z_t^ag, K_t) = \alpha (1 + z_t^ag a - (1 - z_t^ag) a) \left[ \frac{K_t}{\pi p_t^e} \right]^{\alpha-1}, \]

where \(a \in (0, 1)\) is the absolute aggregate productivity rate and \(\alpha \in (0, 1)\) is the output elasticity parameter. Labor supply is defined by the employment rate \(p_t^e\) scaled by the time endowment factor \(\pi\).

The question is how this model fits the framework introduced earlier. There are two securities in this model, the share in physical capital and time line, therefore, indicates which filtration the endogenous variables are adapted to.

\[ \pi, \text{ time endowment factor} \]

\[ p, \text{ elasticity parameter}. \]

Labor supply is defined by the employment rate \(p_t^e\) scaled by the time endowment factor \(\pi\).

\[ \text{Proposition 14. Assumption 9 is satisfied in the Aiyagari-Bewley economy with aggregate risk given in Krusell and Smith (1998) and den Haan et al. (2010).} \]

\[ (1/K_0) \int_0^\infty \kappa d\mu_0 (1,k). \]

\(^7\)Note that I specify the time line slightly differently than den Haan et al. (2010) and Krusell and Smith (1998). These authors substitute \(k_t\) with \(k_{t+1}\) in the budget constraint (1.10) because this is the capital, which is put forward as start capital to period \(t+1\). In contrast to that notation, however, I want to emphasize the time period, at which the agent optimally chooses the magnitude of her capital savings. Taking this view, the optimal consumption and capital savings choice have the same time subscript. My time line, therefore, indicates which filtration the endogenous variables are adapted to.
Assume that all agents have time-separable CRRA utility with a risk aversion coefficient $\gamma > 1$ and time preference parameter $\beta \in (0, 1)$. Then, given the initial cross-sectional distribution $\mu_0$ with non-negative support, the individual optimization problem reads

$$
\max_{(c_t, k_t) \in \mathbb{R}^2} \mathbb{E} \left[ \sum_{t=0}^{\infty} \beta^t c_t^{1-\gamma} - 1 \right]_{1-\gamma}
$$

subject to

$$
k_t + c_t = I(z_t, k_{t-1}, K_t) + [1 - \delta] k_{t-1} \forall t \geq 0$$

$$
c_t > 0, k_t \geq 0 \forall t \geq 0
$$

where the productive income $I$ is defined as in (1.11). I make the following technical assumption on the model parameters.

**Assumption 15.** Suppose that $\beta (1 - \delta)^{1-\gamma} < 1$.

I can now apply the results from the previous section to establish existence and uniqueness of a solution. I show that the two points which result in the left-hand side of the Euler equation being greater and smaller than zero correspond to the save everything/consume nothing and the save nothing/consume everything strategies. From these two polar strategies, I construct a set which contains zero in its convex hull so that Corollary 7 can be applied.

**Theorem 16** (Existence of a unique recursive equilibrium). *Consider the growth model together with Assumption 15. Define the admissible set $\mathcal{H}_\epsilon$ as in Proposition 8. Then, there exists a unique continuous square-integrable function $h \in \mathcal{H}_\epsilon$, $h : \mathbb{Z} \times \mathbb{R} \times L^2_{\mathbb{P}} \rightarrow \mathbb{R}$, which maximizes the individual optimization problem (1.13) with rates of returns as in (1.12).*

### 1.5 An Iterative Solution Procedure

Due to the fact that I do not rely on compactness to establish existence for this type of model, the convergent iterative procedure of the contraction mapping theorem does not apply here. Hence, I cannot compute the equilibrium using value function iteration. However, the monotonicity approach leads to another convergent iterative procedure which is similar. This procedure is explained subsequently.

We can construct an iterative procedure $P$ where $h^{n+1} = P(h^n)$ with $h^n$ converging to a solution of (1.7) by exploiting the monotonicity of the Euler equation operator $T$. To illustrate the idea, I will first look at the simplified problem without borrowing constraint. We can rewrite the Euler equation by

$$
T[h] = 0 \Leftrightarrow T[h] + h = h \Leftrightarrow (T + \text{Id})[h] = h \Leftrightarrow h = (T + \text{Id})^{-1}[h],
$$

where $\text{Id}$ is the identity operator. The last equality contains the resolvent of the Euler equation $(T + \text{Id})^{-1}$. This operator has a very desirable property. Since the Euler equation operator is maximal monotone, its resolvent is firmly nonexpansive, a property slightly stronger than Lipschitz continuity with coefficient one and it can be proven that iterating on the resolvent as in

$$
h^{n+1} = (T + \text{Id})^{-1}[h^n],
$$
where \( n \) is the iteration count, converges to the optimal policy, i.e. the root of the Euler equation operator.\(^8\) This iterative procedure results in the proximal point algorithm. To understand how such a resolvent is constructed, let us look at a simplified example first.

**Example (Resolvent of a subdifferential).** Let \( \mathcal{E} \) be a Hilbert space. Consider a lower semicontinuous proper convex function \( F : \mathcal{E} \to [-\infty, \infty] \). It is well known that its subdifferential \( \partial F \) is maximal monotone (see e.g., Bauschke and Combettes, 2017, Theorem 20.48). We are looking for a fixed point \( e^* \in \mathcal{E} \) of the resolvent of \( F \), which can be computed by simple iteration with iteration count \( n \),

\[
e_n \xrightarrow{n \to \infty} e^* \text{ with } e_{n+1} = (\partial F + \text{Id})^{-1}(e_n).
\]

The resolvent \((\partial F + \text{Id})^{-1}\) can be represented by

\[
e_{n+1} = (\partial F + \text{Id})^{-1}(e_n) \iff e_n = (\partial F + \text{Id})(e_{n+1}) \iff 0 = (\partial F + \text{Id})(e_{n+1}) - \text{Id}(e_n) \iff e_{n+1} = \arg \min_{e \in \mathcal{E}} F(e) + \frac{1}{2\lambda} \|e - e_n\|^2.
\]

The latter is the update of the proximal point algorithm.\(^9\)

This example shows that the proximal point algorithm in our case translates into an algorithm on augmented Lagrangians. To ensure convergence, a regularization term containing the previous iterate has to be added to the Lagrangian \( L_T \) in (1.9) associated with the left-hand side of the Euler equation. I follow Rockafellar (1976b) for defining the proximal point algorithm’s update. The augmented Lagrangian is a function \( L^A : \mathcal{H}_T \times L^2(\mathbb{Z}^{id} \times \mathbb{R}, \mathcal{B}(\mathbb{Z}^{id} \times \mathbb{R}), \mu) \to [-\infty, \infty] \) given by

\[
L^A\left(h, y; z', \chi(z')^{\text{id}} \right), h^n) = L_T(h) + \frac{1}{2\lambda} \|h - h^n\|_{L^p}^2 + \begin{cases}
- y(h - \bar{x}) + \frac{1}{2\lambda} \|h - \bar{x}\|^2_{L^p} & , h - \bar{x} \leq \frac{y}{\lambda} \\
- \frac{1}{2\lambda} \|y\|^2_{L^p} & , h - \bar{x} > \frac{y}{\lambda}
\end{cases},
\]

where \( L_T \) as in (1.9) and \( \lambda > 0 \) is the step size parameter of the proximal point algorithm. The first line of the augmented Lagrangian features the Lagrangian corresponding to the Euler equation operator from Definition 6. The second line consists of the objective’s proximal point augmentation, which transforms the first-order condition into its resolvent. The last line corresponds to the inequality constraint. It also consists of the Lagrange term and the augmentation, but it is defined piecewise to account for the case of a binding constraint.

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\(^8\)As a root, the optimal policy of the Euler equation operator represents an eigenfunction of the Euler equation operator’s eigenvalue zero. This set of eigenfunctions is the same set which corresponds to the eigenvalue problem of the resolvent \( \lambda \text{Id} - (\text{Id} + T)^{-1} = 0 \) for the eigenvalue \( \lambda = 1 \). As the resolvent is Lipschitz continuous with coefficient one, which follows from the maximal monotonicity of the Euler equation operator, we can, in fact, characterize the resolvent’s spectrum. The spectrum for nonlinear operators is not uniquely defined as the corresponding spectral theory is much more complex than for linear operators (see e.g. Appell et al., 2004). However, due to the Lipschitz property, we can use the definition by Kachurovskij leading to a compact spectrum with spectral radius of one. Hence, the optimal policy represents the eigenfunction corresponding to the resolvent’s maximal eigenvalue.

\(^9\)The proximal point update presented here is a simplified version. Rockafellar (1976a) proves convergence for the generalized resolvent \( \lambda^n (\text{Id} + 1/\lambda^n T)^{-1} \), also called Yosida approximation, where \( \{\lambda^n\}_{n=1}^\infty \) is either constant and bounded away from zero or a series \( 0 < \lambda^n \not\to \lambda^\infty \leq \infty \).
Remark. The augmentation term in the Lagrangian (1.14) of the proximal point algorithm does, in fact, represent a Tikhonov regularization. This regularization is necessary because the equilibrium correspondence is not a contraction mapping which implies that the proximal point algorithm can be interpreted as the equivalent to value function iteration for heterogeneous agent models. Furthermore, it has been shown in (Bauschke and Combettes, 2017, Theorem 27.23) that regularizations other than Tikhonov are admissible as well as long as the regularization function is uniformly convex in the policy $h$. An avenue for future research might, therefore, be to explore alternatives like the Sobolev regularization. However, one should keep in mind that the policies will not be differentiable everywhere when there are borrowing constraints.

With the augmented Lagrangian as above, I now state the algorithm to approximate a recursive equilibrium of the growth model in Algorithm 1.1.

**Algorithm 1.1 Proximal point algorithm**

▷ A Initialization
1: Set $n = 0$. Initialize the agents’ choices individual capital and the Lagrange multiplier $H^n = (h^n, y^n)$.
2: Set the parameter $\lambda > 0$.
3: Set the termination criterion small $\tau > 0$ and the initial distance larger $d > \tau$.
▷ B Iterative procedure
4: while $d > \tau$ do
5: Update $H^{n+1}$ by
$$h^{n+1} \left(z', \chi(z^{id'})\right) \approx \arg \min_{h \in H,} L^A \left(h, y^n; z', \chi(z^{id'}), h^n\right)$$
$$y^{n+1} \left(z', \chi(z^{id'})\right) = \max \left\{0, y^n \left(z', \chi(z^{id'})\right) - \lambda h^{n+1} \left(z', \chi(z^{id'})\right)\right\}$$
where $L^A$ is defined as in (1.14).
6: Compute the distance $d = \|H^{n+1} - H^n\|_{L^2}$.
7: Set $n = n + 1$.
8: end while
Chapter 2

An Approach to Compute Solutions to a Production Economy

Abstract
Dynamic stochastic general equilibrium models with ex-post heterogeneity due to idiosyncratic risk have to be solved numerically. This is a nontrivial task as the cross-sectional distribution of endogenous variables becomes an element of the state space due to aggregate risk. Existing global solution methods often assume bounded rationality in terms of a parametric law of motion of aggregate variables in order to reduce dimensionality. In this chapter, I remove that assumption and compute a fully rational equilibrium dependent on the whole cross-sectional distribution. Dimensionality is tackled by polynomial chaos expansions, a projection technique for square-integrable random variables, resulting in a nonparametric law of motion. Moreover, I establish approximation error bounds. Economically, I find that bounded rationality can lead to more wealth inequality than the fully rational equilibrium in the Aiyagari-Bewley growth model. Furthermore, I find that more risk sharing amplifies aggregate risk and leads to higher systemic risk.

The chapter proceeds as follows. In the next section, I present the Aiyagari-Bewley growth model with aggregate risk, which serves as the benchmark model for the algorithm. In Section 2, I introduce the iterative procedure for which I establish conditions for convergence. Section 3 explains the technique I use to discretize the space of distributions, and thus, makes it possible to keep track of the cross-sectional distribution. Furthermore, I derive approximation error bounds. In Section 4, I compare the proximal point algorithm to three existing global solution methods. Lastly, I analyze which economic insights this novel computational method yields. Appendix B contains all proofs, information on the calibration used, an extension of the iterative solution procedure and robustness checks.
2.1 The Aiyagari-Bewley Economy with Aggregate Risk

For illustration, I use the same growth model with aggregate shocks as in den Haan et al. (2010), which is used for a comparison of Krusell-Smith-style algorithms in the special issue of the Journal of Economic Dynamics and Control of January 2010. Consider a discrete-time infinite-horizon model with a continuum of agents of measure one. There are two kinds of exogenous shocks, an aggregate shock and an idiosyncratic shock. The aggregate shock characterizes the state of the economy with outcomes in \( Z^{ag} = \{0, 1\} \) standing for a bad and good state, respectively. The idiosyncratic shock with outcomes in \( Z^{id} = \{0, 1\} \) indicates that an agent is unemployed or employed, respectively. It is i.i.d. across agents conditional on the aggregate shock. I denote the compound exogenous process \( (z_{t}, z_{t}^{id})_{t \geq 0} \) by \( (z_{t})_{t \geq 0} \in Z \) with \( Z = Z^{ag} \times Z^{id} \). The transition probabilities are exogenously given by a four-by-four matrix.

The security market consists of a claim to aggregate capital \((K_{t})_{t \geq 0}\). An agent’s share of physical capital is denoted by \((k_{t})_{t \geq 0}\). The aggregate endogenous variable \( K \) is hence defined by

\[
K_{t} = \frac{1}{\sum_{z_{t}^{id} = 0}^{\infty}} \int_{-\infty}^{\infty} kd\mu_{t}(z_{t}^{id}, k) \quad \forall t \geq 0, \tag{2.1}
\]

where \( \mu_{t} \) is the cross-sectional distribution of idiosyncratic exogenous and endogenous variables at the beginning of time \( t \), i.e. before the agents choose their optimal capital savings. It is simply the probability distribution of individual capital across the unemployed and the employed agents given the trajectory of aggregate shocks

\[
\mu_{t}(z_{t}^{id}, k) = \mathbb{P}\left(\left\{z_{t}^{id} = z_{t}^{id}\right\} \cap \{k_{t-1} \leq k\} \mid z_{t}^{ag}, \ldots, z_{0}^{ag}\right) \tag{2.2}
\]

for all \( t \geq 0, z_{t}^{id} \in Z^{id} \) and \( k \in \mathbb{R}. \)

The aggregate shocks cause the cross-sectional distribution to vary over time, which is indicated by the time subscript of \( \mu_{t} \).

Each agent chooses her share of physical capital and consumption such that they satisfy certain constraints. First, individual consumption must be positive at all times \( c_{t} > 0, t \geq 0 \), and capital holdings are subject to a hard borrowing constraint \( k_{t} \geq 0, t \geq 0 \). Second, given an initial capital endowment \( k_{-1} \geq 0 \) and an initial cross-sectional distribution \( \mu_{-1} \) with non-negative support, each agent adheres to a budget constraint, which equates individual consumption and current capital stock to productive income and

\[\text{Note that we can use the methodology of Fubini extension by Sun (2006) to ensure the validity of the law of large numbers when aggregating over the set of agents. In particular, let us denote the atomless measure space of agents by } (I, \mathcal{I}, \lambda) \text{ with } \lambda(I) = 1 \text{ and the sample probability space by } (Z^{id}, \sigma(Z^{id}), \mathbb{P}^{z^{id}|z^{ag}}). \]

Let \( f \) be a measurable function mapping the Fubini extension \((I \times Z^{id}, \mathcal{I} \otimes \sigma(Z^{id}), \lambda \otimes \mathbb{P}^{z^{id}|z^{ag}}) \) into \( \mathbb{R} \). If the random variables \( f(i,) \) are essentially pairwise independent, then \( f(i,) \) have a common distribution \( \mu \) for \( \lambda \)-almost all \( i \in I \). The same holds for the samples \( f(., z^{id}) \). When \( f \) represents individual capital, we get

\[
K = \int_{I} k(i) d\lambda(i) = \frac{1}{\sum_{z_{t}^{id} = 0}^{\infty}} \int_{-\infty}^{\infty} kd\mu \left( z_{t}^{id}, k \right).
\]

\[\text{The initial cross-sectional distribution } \mu_{-1} \text{ does not only imply the initial aggregate capital } K_{-1}, \text{ but also the initial aggregate economic state as it is pinned down by the employment rate } \mathbb{P}(z_{t}^{ag} = 1 | z_{t}^{ag}) = (1/K_{-1}) \int_{0}^{\infty} kd\mu_{-1}(1, k). \]
saved capital stock

\[ k_t + c_t = I(z_t, k_{t-1}, K_t) + [1 - \delta] k_{t-1} \quad \forall t \geq 0. \]  \hspace{1cm} (2.3)

The parameters in this budget constraint are defined as follows. The capital stock brought forward from period \( t - 1 \) depreciates by a rate \( \delta \in (0, 1) \). The productive income is given by

\[
I \left( z_t, k_{t-1}, K_t \right) = R \left( z_t^{ag}, K_t \right) k_{t-1} \\
+ z_t^{id} \pi [1 - \tau_t] W \left( z_t^{ag}, K_t \right) + \left[ 1 - z_t^{id} \right] \nu W \left( z_t^{ag}, K_t \right).
\]  \hspace{1cm} (2.4)

It is composed of, first, the return on capital stock, and second, labor income, which equals the individual’s wage \( W \) when the agent is employed and a proportional unemployment benefit \( \nu W \) otherwise. The agent’s wage is subject to a tax rate \( \tau_t = \nu (1 - p_t^e)/(\pi p_t^e) \) whose sole purpose is to redistribute money from the employed to the unemployed. The parameter \( \nu \in (0, 1) \) denotes the unemployment benefit rate, whereas, \( p_t^e = P(z_t^{id} = 1|z_t^{ag}) \) is the employment rate at time \( t \) and \( \pi > 0 \) is a time endowment factor. It is reasonable to assume \( \nu/\pi < 1 - \tau_t \Leftrightarrow \nu < \pi p_t^e \) for all \( t \geq 0 \). The wage \( W \) and the rental rate \( R \) are derived from a Cobb-Douglas production function for the consumption good

\[
W \left( z_t^{ag}, K_t \right) = (1 - \alpha) \left( 1 + z_t^{ag} a - (1 - z_t^{ag}) a \right) \left[ \frac{K_t}{\pi p_t^e} \right]^\alpha
\]

\[
R \left( z_t^{ag}, K_t \right) = \alpha \left( 1 + z_t^{ag} a - (1 - z_t^{ag}) a \right) \left[ \frac{K_t}{\pi p_t^e} \right]^{\alpha-1},
\]

where \( a \in (0, 1) \) is the absolute aggregate productivity rate and \( \alpha \in (0, 1) \) is the output elasticity parameter. Labor supply is defined by the employment rate \( p_t^e \) scaled by the time endowment factor \( \pi \).

Assume that all agents have time-separable CRRA utility with a risk aversion coefficient \( \gamma > 1 \) and time preference parameter \( \beta \in (0, 1) \). Then, given an agent’s initial capital endowment \( k_{-1} \geq 0 \) and the initial cross-sectional distribution \( \mu_{-1} \) with non-negative support, the individual optimization problem reads

\[
\max_{\{c_t, k_t\} \in \mathbb{R}^2} \mathbb{E} \left[ \sum_{t=0}^{\infty} \beta^t c_t^{1-\gamma} - 1 \right] \\
\text{s.t. } k_t + c_t = I \left( z_t, k_{t-1}, K_t \right) + [1 - \delta] k_{t-1} \quad \forall t \geq 0 \\
c_t > 0, k_t \geq 0 \forall t \geq 0
\]  \hspace{1cm} (2.5)

where the productive income \( I \) is defined as in (2.4). I make the following technical assumption on the model parameters.

**Assumption 17.** Suppose that \( \beta(1 - \delta)^{1-\gamma} < 1 \).

---

\(^3\)Note that I specify the time line slightly differently than den Haan et al. (2010) and Krusell and Smith (1998). These authors substitute \( k_t \) with \( k_{t+1} \) in the budget constraint (2.3) because this is the capital, which is put forward as start capital to period \( t+1 \). In contrast to that notation, however, I want to emphasize the time period, at which the agent optimally chooses the magnitude of her capital savings. Taking this view, the optimal consumption and capital savings choice have the same time subscript. My time line, therefore, indicates which filtration the endogenous variables are adapted to.
In a competitive equilibrium, the individual problems are solved subject to the equilibrium condition (2.1) that aggregate capital equals the expected optimal individual capital holdings. In this thesis, I consider a particular competitive equilibrium of recursive form. To define a recursive equilibrium, I will switch to prime-notation for convenience, where a prime denotes variables in the current period and variables with no prime refer to the previous period.

**Definition 18** (Recursive equilibrium). A solution to the agents’ individual optimization problems (2.5) subject to the equilibrium condition (2.1) given an initial cross-sectional distribution of individual capital \( \mu_{-1} \) with non-negative support is called recursive if there exist functions \( g_i : Z \times R \times \mathcal{P}(Z^id \times R) \rightarrow R, \ i \in \{c,k\} \), such that, for any point in time, the current optimal consumption and capital savings choices equal \( c' = g_c(z', k, \mu') \) and \( k' = g_k(z', k, \mu') \) for any agent with previous-period capital stock \( k \) who observes the current-period exogenous shock \( z' = (z^{ag'}, z^{id}) \) and the beginning-of-current-period cross-sectional distribution \( \mu' \).

**Remark.** Note that I will solely work with the savings policy \( k' = g_k(z', k, \mu') \) in the following as the consumption policy follows directly due to the budget constraint. The subscript \( k \) is dropped for convenience.

In order to obtain a full description of equilibrium, we need to define a consistent law of motion of \( \mu' \) to \( \mu'' \). Given a fixed distribution \( \mu' \) over the cross-section of individual capital at the beginning of the current period and a recursive equilibrium, the distribution changes in two steps \( \mu' \rightarrow \mu'' \). In the first step, the agents implement their optimal capital savings, which leads to the end-of-current-period distribution

\[
\hat{\mu}'(z^{id'}, k) = \mathbb{P}\left(\left\{ z' = z^{id'} \right\} \cap \left\{ g\left(z'^{ag'}, z', \kappa, \mu'\right) \leq k \right\} | z^{ag'} \right), \tag{2.6}
\]

where \( (z', \kappa) \sim \mu' \) is a random variable distributed according to the cross-sectional distribution. In the second step, the next-period shocks \( z'' \) for all agents realize and shift the quantities of employed and unemployed agents depending on the outcome of the aggregate shock. Formerly employed agents either stay employed or become unemployed, the same holds for the formerly unemployed. Therefore, the distribution at the beginning of the next period \( \mu'' \) is given by

\[
\mu''(z^{id''}, k) = \sum_{z^{id'} \in Z^{id}} \frac{p(z^{ag''}, z^{id''})|z^{ag''}}{p(z^{ag'}|z^{ag'})} \hat{\mu}'(z^{id'}, k) \tag{2.7}
\]

\[
= \sum_{z^{id'} \in Z^{id}} \frac{p(z^{ag''}, z^{id''})|z^{ag''}}{p(z^{ag'}|z^{ag'})} \cdot \mathbb{P}\left(\left\{ z' = z^{id'} \right\} \cap \left\{ g\left(z'^{ag'}, z', \kappa, \mu'\right) \leq k \right\} | z^{ag'} \right)
\]

for all \( z^{id''} \in Z^{id} \) and \( k \in R \). The multipliers in front of the end-of-current-period distribution are the probabilities that the employment status changes from \( z^{id'} \) to \( z^{id''} \) given the observed trajectory of \( z^{ag'} \) to \( z^{ag''} \). From this definition of the new distribution, the new aggregate capital \( K'' \) follows immediately due to (2.1).

Now that all model ingredients are defined, the next section lays out the methodology to solve for the recursive equilibrium in this growth model. The parameter values of the model are set according to the calibration of den Haan et al. (2010) and can be found in Appendix B.1.
2.2 The Proximal Point Algorithm for the Growth Model

I show the existence and uniqueness of the growth model in Chapter 1. Furthermore, I introduce an iterative procedure named the proximal point algorithm which converges to the solution of the equilibrium model. I use the proximal point algorithm in this chapter to compute the solution to the growth model. Let me recall some of its notions.

I show in Chapter 1 that the recursive equilibrium can be written in terms of random variables.

**Definition 19** (Recursive equilibrium with random variables). The recursive equilibrium in Definition 18 can be rewritten using a function $h : \mathbb{Z} \times \mathbb{R} \times L_{\mu-1} \to \mathbb{R}$ such that $k' = g(z', k, \mu') = h(z', k, \kappa(z'))$ for any agent with previous-period capital stock $k$ who observes the current-period exogenous shock $z' = (z''', z')$ and the beginning-of-current-period conditional cross-sectional random variable $\kappa(z') \in L_{\mu-1}$. The function for consumption follows from the budget constraint.

Using this functional form of the recursive equilibrium, I define the model’s Euler equation operator $T : \mathcal{H}_c \to C(L_P^\beta)$ such that

$$T(h) = -\frac{\partial}{\partial c} u \left( I \left( z', \kappa(z'), \mathbb{E} \left[ \mathbb{E} \left[ \kappa(z') | z'' \right] \right] \right) \right) + [1 - \delta] \kappa(z') - h \left( z', \kappa(z') \right)$$

$$+ \sum_{z'' \in \mathbb{Z}} p'' |z'' \beta \left( 1 - \delta + R \left( z'', \mathbb{E} \left[ h \left( z', \kappa(z') \right) \right] \right) \right)$$

$$\cdot \frac{\partial}{\partial c} u \left( I \left( z'', h \left( z', \kappa(z') \right), \mathbb{E} \left[ h \left( z', \kappa(z') \right) \right] \right) \right)$$

$$+ [1 - \delta] \kappa' \left( z'' \right) - h \left( z'', \kappa' \left( z'' \right) \right),$$

where $\kappa(z') \sim \mu' / P(z'' | z''')$ denotes the conditional random variable of individual capital and

$$\kappa' \left( z'' \right) = \mathbb{E}(z'' | z'') \left[ h \left( z', \kappa(z') \right) \right]$$

denotes the next-period conditional capital savings random variable which accounts for the agents’ changing idiosyncratic states. Recall that the individual capital in the recursive functional form is substituted by the corresponding random variable in the Euler equation operator which is why it depends on $h(z', \kappa(z'))$ instead of $h(z', k, \kappa(z'))$. This yields the model’s first-order condition

$$T(h) + y = 0 \text{ a.e. with } h \perp y \geq 0.$$  

(2.9)

There exists a Lagrangian $L : \mathcal{H}_c \times C(L_P^\beta) \to [\infty, \infty]$ such that $T$ maximizes $L$ in its second argument. According to (Lemma 5.1, Ghoussoub, 2008), the Lagrangian associated with $T$ is given by

$$L(h, p) = \sup_{g \in \mathcal{H}_c} \left\{ \langle p, g \rangle + \langle T(g), h - g \rangle \right\}$$

(2.10)

and maximizing the Lagrangian over $p$ for a given $h$ yields $p^* = T(h)$ with the function value $L(h, p^*) = \langle p^*, h \rangle = \langle T(h), h \rangle$. For notational convenience, I denote $L(h, p^*)$ by $L(h)$.

I follow Rockafellar (1976b) for defining the proximal point algorithm’s update. This algorithm iterates on the resolvent of the first-order condition associated with the Lagrangian (2.10). Hence, each iteration on the resolvent updates the agents’ optimal choices.
for individual capital $h$ as well as the Lagrange multiplier $y$ for the inequality constraint of (2.5). The $(n+1)$-th iterate of the agent’s optimal choices, i.e., $h^{n+1}$, is the minimizer of the Lagrangian augmented by terms featuring the $n$-th iterate. The augmented Lagrangian is a function $L^A : H \times L^2(\mathbb{Z}^d \times \mathbb{R}, B(\mathbb{Z}^d \times \mathbb{R}), \mu) \to [-\infty, \infty]$ given by

$$L^A (h, y; z', \kappa(z^{id'}), h^n) = L(h, 0) + \frac{1}{2\lambda} \|h - h^n\|^2_{L^2_P} + \begin{cases} -yh + \frac{\lambda}{2} \|h\|^2_{L^2_P}, & h \leq \frac{y}{\lambda} \\ -\frac{y}{2\lambda} \|y\|^2_{L^2_P}, & h > \frac{y}{\lambda} \end{cases}$$

where $L$ as in (2.10), $\lambda > 0$ is the step size parameter of the proximal point algorithm. The first line of the augmented Lagrangian features the Lagrangian corresponding to the Euler equation (2.9). The second line consists of the objective’s proximal point augmentation, which transforms the first-order condition into its resolvent. The last line corresponds to the inequality constraint. It also consists of the Lagrange term and the augmentation, but it is defined piecewise to account for the case of a binding constraint.

With the augmented Lagrangian as above, I now state the algorithm to approximate a recursive equilibrium of the growth model in Algorithm 2.1.

**Algorithm 2.1** Proximal point algorithm for the growth model

1. **Initialization**
   1. Set $n = 0$. Initialize the agents’ choices individual capital and the Lagrange multiplier $H^n = (h^n, y^n)$.
   2. Set the parameter $\lambda > 0$.
   3. Set the termination criterion small $\tau > 0$ and the initial distance larger $d > \tau$.

2. **Iterative procedure**

   4. While $d > \tau$ do
      5. Update $H^{n+1}$ by
         $$h^{n+1}(z', \kappa(z^{id'})) \approx \arg \min_{h \in H} L^A (h, y^n; z', \kappa(z^{id'}), h^n)$$
         $$y^{n+1}(z', \kappa(z^{id'})) = \max \{0, y^n(z', \kappa(z^{id'})) - \lambda h^{n+1}(z', \kappa(z^{id'}))\}$$
         where $L^A$ is defined as in (2.11) with $L$ as in (2.10) and $T$ as in (2.8).
   6. Compute the distance $d = \|H^{n+1} - H^n\|_{L^2_P}$.
   7. Set $n = n + 1$.
   8. End while

shows that the proximal point algorithm converges to an optimum of the Lagrangian even if the update of the optimal consumption and individual capital in line 5 is only approximate. This is important as the minimizer of the augmented Lagrangian is often not known in closed form, but it can be approximated with standard nonlinear solvers.

I make the form of approximation of the policy update precise to show convergence in the following theorem.

**Theorem 20** (Convergence of Algorithm 2.1). Consider the growth model from Section 2.1 together with Assumption 17. Define the admissible set of policies $H_\mu$ as in Proposition...
2.3. Discretizing the Space of Distributions

8. When computing the next iterate $h^{n+1}$ in Algorithm 2.1, line 5, as a solution to the formula

$$X(z', k, K') \| T(h) + (h - h^n) + (-y + \lambda h) 1_{\{h \leq y}\} \|_1 \leq \frac{\epsilon^2}{2\lambda}$$

(2.12)

for any $(z', k, K') \in \mathbb{Z} \times \mathbb{R}_{\geq 0}$, where $X$ is the cash-at-hand, i.e., productive income plus savings, of the agent with start capital $k$

$$X(z', k, K') = I(z', k, K') + [1 - \delta] k$$

with the income $I$ as in (2.4) and the Euler equation $T$ as in (2.8), then, Algorithm 2.1 converges to a solution of the growth model’s Euler equation.

Remark. (i) Equation (2.12) is easily implemented by any root solver using a tolerance level of $\epsilon^2/(2\lambda)$.

(ii) Note that the augmentations in the augmented Lagrangian (2.11) go to zero when the proximal point algorithm converges. Hence, the optimal policy to which the Algorithm 2.1 converges, is a root to the equilibrium’s Euler equation.

The mathematical details are explained in Appendix B.2.1, which also contains the proof. The convergence rate of Algorithm 2.1 is $O(n^{-1})$ as is shown by Güler (1991). The proximal point algorithm can, however, be accelerated, which goes back to Güler (1992). The convergence rate of the accelerated algorithm is $O(n^{-2})$, which was proven in Salzo and Villa (2012). I explain the acceleration in Appendix B.3.

We have to discretize the policy function in order to implement the algorithm. The difficulty herein lies in the cross-sectional distribution as an element of the state space because the space of distributions is infinite-dimensional. A solution to this problem is discussed in the next section.

2.3 Discretizing the Space of Distributions

The goal is to solve equation (2.9) in order to obtain the optimal policy $h$. Note that $h$ depends on the conditional cross-sectional random variable $\kappa(z^{id})$ defined by some square-integrable transformation of the initial cross-sectional distribution. Since we do not want to limit ourselves to a specific series of shocks to make this transformation of $\mu_0$ precise, I need to discretize the space of distributions to compute the optimal policy for any possible current-period conditional cross-sectional distribution of $\kappa(z^{id}) \sim \mu' / \mathbb{P}(z^{id} | z^{ag})$.

The existing literature often resorts to using a finite number of moments to characterize the cross-sectional distribution. However, even though there is a one-to-one correspondence between a distribution and its moment-generating function, this function does not exist for all distributions. Hence, any moment-based method cannot span the full space of square-integrable distributions and for some models, especially the ones producing fat-tailed cross-sectional distributions, such an approximation method is bound to fail. Another option would be a histogram representation or a spline interpolation of the distribution, but the discretized state space becomes very large very quickly in this case. Unfortunately, we have to rule out projection on orthogonal polynomials, which is widely used in computational economics, as well, because a prerequisite is a sufficiently smooth distribution. Due to the occasionally binding borrowing constraint, however, the cross-sectional distribution exhibits mass points at the constraint and elsewhere. This fact is illustrated in the following proposition.
Proposition 21 (A condition for mass points\textsuperscript{4}). Consider a recursive equilibrium as in Definition 18 with an explicit debt constraint $k \geq \delta$ with $\delta \in \mathbb{R}$. Suppose that there exists a $\hat{z} \in \mathbb{Z}$ with $p_{\hat{z} | \hat{z}} > 0$ and a $\hat{k} > \delta$ such that $g_k(\hat{z}, k, \mu) \leq k$ for all $k \in [\delta, \hat{k}]$. Furthermore, assume that the optimal capital savings function has a kink at $k^* := \max\{k \geq \delta : g_k(\hat{z}, k, \mu) = \delta\} > \delta$, i.e., the debt constraint is binding, and that $g$ is strictly increasing in $k \geq k^*$. Then, the cross-sectional distribution has a mass point at the constraint $\delta$. If, additionally, there exists a $\bar{z} \in \mathbb{Z}$ with $p_{\bar{z} | \bar{z}} > 0$ and $g_k(\bar{z}, \delta, \mu) > \delta$, then the cross-sectional distribution has multiple mass points.

This result implies jumps in the cross-sectional distribution $\mu'$. Hence, standard orthogonal polynomial projection methods do not work well here as one would need a very large number of projection points and hence, would be confronted with the curse of dimensionality.

There is an efficient way of approximating distributions circumventing the aforementioned problems. Similar to Chapter 1, it is important to view the cross-sectional distribution as a random variable, which implies that I solve for the recursive equilibrium in terms of random variables. Hence, instead of the polynomial projection of a c.d.f. on the real line, I will use polynomial projection in the space of square-integrable random variables. One can interpret this approach as a probabilistic rather than a deterministic polynomial projection. This technique is called polynomial chaos and is well known in the physics and engineering literature. It is a method, which projects a square-integrable distribution onto orthogonal polynomials which have random variables rather than the real line as arguments. The advantage of this approach is that it spans the whole space of square-integrable random variables and hence, one can be sure to approximate any cross-sectional distribution sufficiently well. This includes discrete distributions and mixtures of discrete and continuous distributions. Furthermore, when the basic random variables and their corresponding family of polynomials are chosen carefully, the speed of convergence easily outperforms standard polynomial projection. Hence, a low order of polynomials is enough to obtain a good approximation of the cross-sectional distribution. In the growth model, I find that order two to three is sufficient depending on the calibration of the model. Therefore, we can substitute $\kappa(z^{id})$ in the recursive equilibrium with two to three projection coefficients. In the following, I will summarize the method of polynomial chaos in general and how this technique is applied to the growth model. Subsequently, I derive approximation error bounds which yield the convergence of this method.

2.3.1 Polynomial Chaos

The standard polynomial chaos expansion is an approach to represent random variables by a series of polynomials mapping basic random variables into the space of square-integrable random variables $L^2$. Originally, this approach yields the so-called Wiener-Hermite expansion, i.e., a projection onto Hermite polynomials, which take Gaussians as basic random variables. The well known Cameron-Martin theorem (see e.g., Ernst et al., 2012, Theorem 2.1) shows that this construction spans all square-integrable random variables, which are measurable w.r.t. the basic random variables. Xiu and Karniadakis (2002) extend this concept to sets of orthogonal polynomials mapping more general basic random variables, e.g., uniform, gamma or binomial variables, into $L^2$. The $L^2$-convergence result for these generalized polynomial chaos expansions is proven in Ernst et al. (2012). The main

\textsuperscript{4}The proof can be found in Appendix B.2.2.
2.3. Discretizing the Space of Distributions

The purpose of this generalization is the gain in convergence speed when the basic random variables are chosen such that they are similar to the approximated random variable. To summarize, given a basic random variable $\xi \in L^2$ with distribution $\xi \sim F$, which has finite moments of all orders, and a set of orthogonal polynomials $\{\Phi_i\}_{i=0}^\infty$, where $i$ denotes the order of each polynomial, we can represent any random variable $\kappa \in L^2$ with distribution $\kappa \sim \mu$ by

$$\kappa = \sum_{i=0}^\infty \varphi_i \Phi_i (\xi),$$  \hspace{1cm} (2.13)

where $\varphi_i$ are constant projection coefficients.

It is important to note that there is an explicit connection between the basic random variable and the set of orthogonal polynomial used. The orthogonality condition of the polynomials reveals this relation. For polynomials of order $i, j \in \{0, 1, \ldots\}$, it reads

$$\langle \Phi_i, \Phi_j \rangle = \int_{-\infty}^\infty \Phi_i (\xi) \Phi_j (\xi) \ dF (\xi) = \frac{\delta_{ij}}{a_i^2},$$  \hspace{1cm} (2.14)

where $\delta_{ij}$ denotes the Kronecker symbol and $a_i \neq 0$ are constants. One can see that the weighting function, which defines the orthogonal polynomials, has to equal the distribution of the basic random variable. Once a basic random variable is fixed, we can generate the corresponding orthogonal polynomials by the three-term recurrence relation (see e.g., Gautschi, 1982; Zheng et al., 2015)

$$\Phi_{i+1} (\xi) = (\xi - \theta_i) \Phi_i (\xi) - \omega_i \Phi_{i-1} (\xi), \ i \in \{0, 1, \ldots\},$$  \hspace{1cm} (2.15)

where the starting polynomials are defined by $\Phi_{-1}(\xi) = 0$ and $\Phi_0(\xi) = 1$ and $\theta_i, \omega_i \in \mathbb{R}$ are constant parameters with $\omega_i > 0$.

The projection coefficients in the polynomial chaos expansion of a random variable $\kappa \in L^2$ with distribution $\kappa \sim \mu$ are defined as usual by $\varphi_i = \langle \kappa, \Phi_i \rangle / \langle \Phi_i, \Phi_i \rangle$ for all $i \in \{0, 1, \ldots\}$. If $\kappa$ is not a direct function of the basic random variable $\xi$, one uses the fact that both c.d.f.s $\mu, F \sim \mathcal{U}[0, 1]$ are uniform to compute the coefficients

$$\varphi_i = \frac{1}{\langle \Phi_i, \Phi_i \rangle} \int \mu^{-1} \circ F (\xi) \Phi_i (\xi) \ dF (\xi) \ \forall \ i \in \{0, 1, \ldots\},$$  \hspace{1cm} (2.16)

where $\mu^{-1}$ is the generalized inverse distribution function of $\kappa$. Hence, with the polynomial chaos expansion, we can translate any square-integrable random variable $\sim \mu \sim \mathcal{U}[0, 1]$ into a countable series of constant projection coefficients $\{\varphi_i\}_{i=0}^\infty$. For computational reasons, I truncate the series of projection coefficients later on.

For practical reasons, it is important to note that polynomial chaos is easily extended to multivariate distributions by the tensor product rule. To approximate a joint distribution of $n$ random variables denoted by $\kappa \sim \mu$, we simply need to fix $n$ independent basic random variables $\xi^1, \ldots, \xi^n$ and determine their corresponding univariate orthogonal polynomials $\Phi^1, \ldots, \Phi^n$ separately. Then, the polynomial chaos expansion for the multivariate distribution equals

$$\kappa = \sum_{i=0}^\infty \varphi_i \Phi_i (\xi^1, \ldots, \xi^n) = \sum_{i=0}^\infty \varphi_i \sum_{0 \leq i_1, \ldots, i_n \leq i, \ i_1 + \ldots + i_n = i} \Phi^1_{i_1} (\xi^1) \cdot \ldots \cdot \Phi^n_{i_n} (\xi^n).$$  \hspace{1cm} (2.17)

The projection coefficient is computed as in (2.16) for $\xi = (\xi^1, \ldots, \xi^n)$ which reduces to the sum of a composition of integrals due to the independence of the basic random variables.
Let me now summarize which steps are necessary to approximate the space of cross-sectional distributions with a polynomial chaos expansion.

**Before starting the solution algorithm:**

1. Determine how many basic random variables are necessary.
2. Fix the distribution of each basic random variable.
3. For each basic random variable, generate its corresponding orthogonal polynomials using the orthogonality condition (2.14) and the three-term recurrence relation (2.15).
4. Compute the multivariate orthogonal polynomials by multiplying the univariate polynomials according to (2.17).

**During the solution algorithm:**

5. Represent any endogenous distribution by projecting it onto the predetermined polynomial chaos expansion according to (2.16).

In the following, I will explain how to tackle Steps 1 to 3 for the growth model in more detail.

### 2.3.2 Applying Polynomial Chaos to the Growth Model

We want to approximate the cross-sectional distribution of idiosyncratic variables \( \mu' \), i.e. a bivariate distribution. Hence, I define two independent basic random variables \( \xi^z \sim F^z \) and \( \xi^k \sim F^k \) and their corresponding univariate orthogonal polynomials \( \Phi^z \) and \( \Phi^k \).

The cross-sectional distribution of the agents' choice variables can be represented by the unconditional random variable of individual capital denoted by \( \kappa \). Its polynomial chaos expansion reads

\[
\kappa = \sum_{i=0}^{\infty} \varphi_i \Phi_i \left( \xi^z, \xi^k \right) \sim \mu'.
\]

This expression can be rewritten in terms of individual capital conditional on employment status \( \kappa(\text{id}|\text{ag}) \sim \mu'/\mathbb{P}(\text{id}|\text{ag}) \)

\[
\kappa(\text{id}) = \begin{cases} 
\sum_{i=0}^{\infty} \varphi_i \Phi_i \left( \xi^z|\xi^z \leq F^z - 1, \xi^k \right), & z^{id} = 0 \\
\sum_{i=0}^{\infty} \varphi_i \Phi_i \left( \xi^z|\xi^z > F^z - 1, \xi^k \right), & z^{id} = 1,
\end{cases}
\]  

which yields the following definition of the projection coefficient for a cross-sectional distribution \( \mu' \)

\[
\varphi_i(\mu') = \begin{cases} 
\int_{\xi^z \leq F^z - 1} \frac{\mu'(0, \ldots)}{1 - p^\varepsilon} \circ F^k(\xi^k) \Phi_i \left( \xi^z, \xi^k \right) \, dF^k(\xi^k) \, dF^z(\xi^z), & z^{id} = 0 \\
\int_{\xi^z > F^z - 1} \frac{\mu'(1, \ldots)}{p^\varepsilon} \circ F^k(\xi^k) \Phi_i \left( \xi^z, \xi^k \right) \, dF^k(\xi^k) \, dF^z(\xi^z), & z^{id} = 1,
\end{cases}
\]

\[ \frac{1}{\langle \Phi_i, \Phi_i \rangle}. \]

---

5 Any such distribution has to possess finite moments of all orders and be determinate in the Hamburger sense (see Ernst et al., 2012). A distribution is determinate in the Hamburger sense if it uniquely solves the Hamburger moment problem or in other words if it is uniquely determined by the sequence of its moments.
Accordingly, the law of motion of the projection coefficients is \( \varphi'_i = \varphi_i(\mu'') \) where, according to (2.7),

\[
\mu''(z^{id'}, k) = \sum_{z^{id'} \in \mathbb{Z}} \frac{p''(z^{id'}, z^{ag})}{p^{ag}} \mu(z^{id'}, \{ \varphi_i \}_{i=0}^{\infty} \leq k)
\]

with \( \kappa(z^{id'}) \) as in (2.18). The cross-sectional distribution and, therefore, also \( \kappa(z^{id'}) \) are fully defined by the infinite series of projection coefficients \( \{ \varphi_i \}_{i=0}^{\infty} \), which is why I insert these as the third argument of the recursive savings policy.

### A Specific Choice of the Basic Random Variables

We have to determine two independent basic random variable for the growth model \( \xi^k \) and \( \xi^z \). It was illustrated in Xiu and Karniadakis (2002) that the speed of convergence of the polynomial chaos expansion significantly improves if the distribution of the basic random variables is not too far from the distribution we want to approximate. Let me first explain, how I fix the basic random variable corresponding to individual capital and subsequently, the one for the exogenous idiosyncratic shock.

As the cross-sectional distribution in our growth model is an endogenous object, we do not know its shape a priori. We do know, however, that it will have mass points according to Proposition 21. Also, we know that the same growth model without aggregate shocks, i.e., where \( z^{ag} \) is fixed at either 0 or 1, has an endogenous stationary cross-sectional distribution. This case is easy to compute because \( K' = K \) in the agents’ optimization problem (2.5). Therefore, one just has to solve the individual optimization problem at different values of aggregate capital \( K \). In a second step, given these optimal responses, one can compute the stationary distribution as a fixed point of the distribution's law of motion (2.7). I do this using histograms. Naturally, the stationary cross-sectional distributions will have features similar to the distribution of the model with aggregate shocks. Hence, I fix \( \xi^k \) as the cross-sectional distribution of individual capital in the model without aggregate shocks averaged over the two cases of keeping \( z^{ag} \) fixed at 0 and 1. The distribution of the basic random variable \( \xi^k \) is displayed in Figure 2.1. It exhibits several mass points measured as the local extrema in the histogram representation. To obtain an accurate approximation of the stationary distributions in the model without aggregate risk, one should choose a reasonably small bin size for the histogram.

I now move on to fix a distribution for the basic random variable corresponding to the idiosyncratic employment shock. This distribution needs to accommodate both the idiosyncratic shock in the good economic state as well as in the bad aggregate state. Hence, I define it as a discrete distribution with three states

\[
F(\xi^z) = \begin{cases} 
  p^{z^{id}=0 | z^{ag}=1}, & \xi^z = 1 \\
  p^{z^{id}=0 | z^{ag}=0}, & \xi^z = 2 \\
  1, & \xi^z = 3.
\end{cases}
\]

It follows that \( z^{id} | \{ z^{ag} = 0 \} = \mathbb{1}_{\{ \xi^z > 2 \}} \), whereas, \( z^{id} | \{ z^{ag} = 1 \} = \mathbb{1}_{\{ \xi^z > 1 \}} \). Having chosen the two basic random variables, I explain how to generate their orthogonal polynomials next.

### Generating the Corresponding Orthogonal Polynomials

As our basic random variable \( \xi^k \) is represented by a histogram, both basic random variables have discrete distributions where the end points of the bins \( \{ \xi_n \}_{n=1}^{N} \) have probability
Figure 2.1: Distribution for the basic random variable $\xi^k$. Panel A shows a histogram representation with bin size 0.1. A mass point is identified as a bin whose probability is higher than the ones of its direct neighbors, but the global maximum is excluded. Panel B zooms into the left tail of the distribution. Note that the distribution displayed here is the average of the stationary end-of-period distributions from (2.7) of the model without aggregate shocks.

$\{p_n\}_{n=1}^N$. Generally, orthogonal polynomials w.r.t. a discrete distribution with finite support are considered discrete as well in the sense that their maximal degree is $N$. Furthermore, the highest-order polynomial $\Phi_N$ has the points $\{\xi_n\}_{n=1}^N$ as roots.

In Zheng et al. (2015), different methods for generating polynomials corresponding to discrete distributions are compared. Of their suggested methods, I use the Stieltjes method, which performs well in terms of precision. It directly computes the parameters $\theta_i$ and $\omega_i$ in (2.15) using the standard inner product of $L^2$ and is explained in detail in Gautschi (1982). The constant parameters are given by

$$
\theta_i = \frac{\langle \Phi_i, \xi \Phi_i \rangle}{\langle \Phi_i, \Phi_i \rangle}, \ i \in \{0, 1, \ldots\}
$$

$$
\omega_i = \frac{\langle \Phi_i, \Phi_i \rangle}{\langle \Phi_{i-1}, \Phi_{i-1} \rangle}, \ i \in \{1, 2, \ldots\}
$$

with $\langle .., . \rangle$ denoting the standard inner product of $L^2$ w.r.t. the corresponding basic distribution. As these distributions are represented as discrete distributions, the inner product is a sum rather than an integral. The definitions of these parameters follow from inserting the three-term recurrence relation (2.15) into the orthogonality condition (2.14). With the parameters defined as above, the orthogonal polynomials are easily constructed using (2.15).

The orthogonal polynomials of $\xi^k$ and $\xi^z$ are displayed in Figure 2.2. As usual, the number of roots of each polynomial corresponds to its degree. Each first-order polynomial has its root at the mean of the distribution of the corresponding basic random variable. The polynomials of $\xi^z$ are plotted only at its three states $\{1, 2, 3\}$. This graph confirms that the polynomial with maximal degree, which is the third-order polynomial $\Phi_3^z$ in this case, has its roots at the states of the corresponding distribution, i.e. at $\{1, 2, 3\}$.

With the basic random variables defined and the corresponding polynomials generated, the polynomial chaos expansion is fully determined. Any square-integrable distribution
2.3. Discretizing the Space of Distributions

Figure 2.2: Orthogonal polynomials corresponding to the basic random variables of the growth model. Panel A shows the polynomials $\Phi_k^i$ corresponding to individual capital up to order $i \leq 1$, whereas, Panel B displays the polynomials $\Phi_k^i$ of order $i = 2$ and $i = 3$. Panel C plots the polynomials $\Phi_z^i$ corresponding to the idiosyncratic shock up to order $i \leq 3$.

measurable w.r.t. the basic random variables can now be projected. The polynomials with different degrees have different effects in this projection as can be seen in Figure 2.3. In

Figure 2.3: Example distributions resulting from truncated polynomial chaos expansions. The graph displays the histogram representations with bin size 0.1 of distributions resulting from the polynomial chaos expansion truncated at different orders ranging from order 0 to 3. The basic random variable used is $\xi^k$ distributed as in Figure 2.1. The projection coefficients for this example are fixed as $[\varphi_0, \ldots, \varphi_3] = [36, 1, 0.01, 0.0002]$. In this figure, I consider a polynomial chaos expansion $\sum_{i=0}^{\infty} \varphi_i \Phi_i(\xi^k)$ with fixed projection coefficients $\{\varphi_i\}_{i=0}^{\infty}$. The expansion is truncated at increasing order. The zeroth-order polynomial results in a mass point at $\varphi_0$, which, due to its definition, is the mean of the projected distribution. The first-order polynomial term simply stretches or compresses the distribution of the basic random variable depending on its projection coefficient. Summing the zeroth- and first-order term simply centers the stretched/compressed distribution of
the basic random variable around the mean of the projected distribution. Further adding
the second-order polynomial modifies the skewness of the polynomial chaos expansion,
whereas, the third-order polynomial adjusts the kurtosis. Higher orders further refine the
tails. Hence, the polynomial chaos expansion gets closer to the projected distribution the
higher the order of truncation.

2.3.3 Convergence of the Discretized Policy
When using polynomial chaos expansions for the cross-sectional distribution, we can ap-
proximate the policy \( h(z', k; \kappa(z^{id})) \) quite naturally because \( \kappa(z^{id}) \sim \mu'/\mathbb{P}(z^{id}|z^{ag'}) \) is fully defined by the projection coefficients \( \{\varphi_i\}_{i=0}^{\infty} \) due to (2.18). To describe the optimal policy at each realization of the random variable, I write down the discretization of this policy occurs in two steps. First, I truncate the polynomial chaos expansion, and second, I discretize all dimensions and apply the finite element method with first-order Lagrange elements, which amounts to linear interpolation. I denote the truncated policy by \( h_{\text{M}} = h(z', k; \{\varphi_i\}_{i=0}^{\infty}) \). Its interpolant, denoted by \( h_{\text{M,D}} \), is defined on a tensor product of finite grids of the state space elements

\[
D = \{(k_{i_0}, \varphi_{1,i_1}, \ldots, \varphi_{M,i_M}) | i_m = 1, \ldots, I_m < \infty \forall m = 0, \ldots, M \}.
\]

The question is whether the algorithm converges for such a discretized policy. This is shown in the following.

It is clear that the interpolant \( h_{\text{M,D}} \) stays within the admissible set of the policies \( \mathcal{H} \) defined in Proposition 8. Therefore, convergence follows from a vanishing approximation error. The total policy function approximation error is composed of two parts corresponding to the truncation and interpolation error

\[
\|h - h_{\text{M,D}}\|_{L^2} \leq \|h - h_{\text{M}}\|_{L^2} + \|h_{\text{M}} - h_{\text{M,D}}\|_{L^2}.
\]

The following theorem derives bounds on these two parts of the error. The bound on the second part is a well-established result from the theory on finite elements (see e.g., Brenner and Scott, 2007), whereas, the bound on the first part is more involved. To derive it, I follow the methodology of the error analysis in Babuška et al. (2007).

**Theorem 22** (Error bounds of the approximation). Consider the growth model from Section 2.1 with the function space \( \mathcal{H} \) defined in Proposition 8. Consider Algorithm 2.1 with polynomial chaos extension as in Section 2.3, i.e., using the basic random variables \( \xi^z \) and \( \xi^k \) and the corresponding orthogonal polynomials \( \Phi \) to project any square-integrable cross-sectional distribution \( (\zeta', \kappa) \sim \mu' \) with \( (\zeta', \kappa) = \sum_{i=0}^{M} \varphi_i \Phi_i(\xi^z, \xi^k) \). Assume that, for any fixed exogenous shock, start capital and individual capital distribution \( (z', k, \mu') \), the initial guess of the savings policy \( h^0 \) and the Lagrange multiplier \( g^0 \) for the proximal point algorithm are real analytic in the basic random variables and hence, satisfy

\[
\left| \frac{\partial^p}{\partial \xi^j} f \right| \leq c_f^p j!, \quad p \in \{1,2,\ldots\}, \quad j \in \{z,k\}, \quad (2.20)
\]

for some constants \( c_f \) where \( f \) is a handle for \( h^0 \) and \( g^0 \). Furthermore, assume that the initial policy guess \( h^0 \) is real analytic in start capital. Consider the following subsets of the complex plane

\[
\Sigma \left( \tau_{n+1}^i, \Gamma^i \right) = \left\{ x \in \mathbb{C} \left| \inf_{\xi \in \Gamma^i} |x - \xi| \leq \tau_{n+1}^i \right\} , \quad i \in \{z,k\},
\]

where

\[
\Sigma \left( \tau_{n+1}^i, \Gamma^i \right) = \left\{ x \in \mathbb{C} \left| \inf_{\xi \in \Gamma^i} |x - \xi| \leq \tau_{n+1}^i \right\} , \quad i \in \{z,k\},
\]
2.4. Numerical Results

where $\Gamma^i$ is the range of $\xi^i$ and $0 < \tau^i_{n+1} < \frac{\min(1,L_{n+1})}{2A^1_{n+1,i}} < \infty$. $L_{n+1}$ is the value of the second derivative of the Lagrangian $L^A$ as in (2.11) evaluated at the $(n+1)$-th policy iterate and $A^1_{n+1,i}$ is given in (B.4) in the proof. Then, the approximation error bound for the $(n+1)$-th policy iterate resulting from truncating the polynomial chaos expansion at order $M$ and using linear interpolation on a rectangular tensor-product grid

$$D = \{(k_{i_0}, \varphi_{0,i}, \ldots, \varphi_{M,i_M}) | k_{i_n} < k_{i_{n+1}}, \varphi_{m,i_n} < \varphi_{m,i_{n+1}}$$

$\forall i_n \in \{1, \ldots, d_n\}, m \in \{1, \ldots, M\}\}$

with maximum mesh-size $s$ is given by

$$\|h^{n+1} - h^{M,D}\|_{L^2} \leq \sum_{i \in \{z,k\}} b_i \frac{2}{\eta^i - 1} e^{-M \log(\eta^i)} \frac{\min(1, L_{n+1})}{\min(1, L_{n+1}) - 2 \tau^i_{n+1} A^1_{n+1,i}}$$

$$+ b_d s^2 \left( \sum_{j=0}^{M+1} \frac{\|\partial^2 h^M\|_{L^2}}{\|\partial D\|_{L^2}^2} \right)^{\frac{1}{2}},$$

where $b_i, i \in \{z,k,d\}$, are constants and

$$\eta^i = \frac{2 \tau^i_{n+1}}{|\Gamma^i|} + \sqrt{1 + \frac{4 (\tau^i_{n+1})^2}{|\Gamma^i|^2}} > 1, i \in \{z,k\}.$$

**Remark.** The theorem implies that the error from the truncation of the polynomial chaos expansion decreases exponentially when increasing the order of the expansion. Furthermore, the error from the interpolation decreases proportionately with the mesh size of the discretization.

### 2.4 Numerical Results

I compute the recursive equilibrium solution of the accelerated proximal point algorithm explained in Appendix B.3 truncating the polynomial chaos expansion at different orders using Matlab R2016b. Furthermore, to demonstrate that the polynomial chaos expansion can be combined with other solution methods, I also compute the recursive solution via standard policy function iteration. Furthermore, I compute the solutions of existing algorithms for comparison. I choose the algorithm by Krusell and Smith (1998) since it is the most prominent existing method. I use its Matlab implementation by Maliar et al. (2010). Furthermore, there has been an effort to improve on this original algorithm in a special issue of the Journal of Economic Dynamics and Control in January 2010. From these more recent methods, I use the backward induction algorithm by Reiter (2010a) and the explicit aggregation algorithm by den Haan and Rendahl (2010), both implemented in Matlab, as they perform best in the comparison by den Haan (2010).

To ensure comparability, I run all these methods using the same grid for individual capital and the same termination criterion $5e^{-5}$. Additionally, I configure the discretizations of the cross-sectional distribution so that they are as close as possible. The Krusell-Smith and the Reiter method use total aggregate capital, whereas, the den Haan-Rendahl algorithm uses the aggregate capital of the unemployed and employed. In the proximal point

---

6The computations were performed on the Baobab cluster at the University of Geneva.
algorithm as well as the policy function iteration, the total aggregate capital is equivalent to the projection coefficient $\phi_0$ of the polynomial chaos expansion. I use 4 grid points for aggregate capital for the Krusell-Smith, the Reiter and the polynomial chaos based algorithms. Note that the Haan-Rendahl algorithm then has $4 \times 4$ grid points in aggregate capital because it differentiates unemployed and employed aggregate capital. Keep in mind that the proximal point algorithm has additional dimensions to discretize the cross-sectional distribution depending on the order of truncation. The different methods are summarized in Table 2.1. Note that the proximal point algorithm and the policy function iteration is implemented with parallelized Matlab code run on a HPC cluster, whereas, the other three algorithms are implemented with serial code run on a desktop computer. This is the reason for the differences in the number of CPUs used. The compute time is higher for the proximal point algorithm. This is mainly due to the fact that it solves a full optimization problem in each iteration to ensure convergence for a model with unbounded utility function. I argue that the goal of being more accurate and having a theoretically sound algorithm justifies the increased compute time. However, if one can ensure that policy function iteration converges, then this should be the method of choice. If convergence of a faster ad hoc method is not clear, one can use the proximal point algorithm as a benchmark.

In the following, I investigate whether the algorithms using polynomial chaos expansions really yield higher precision than the existing methods. Furthermore, I compare a case when approximate aggregation holds with a case where it fails and show that the polynomial chaos based algorithms do produce reliable results in the latter case as well where existing algorithms run into problems.

<table>
<thead>
<tr>
<th>Algorithm</th>
<th># Grid Points</th>
<th># CPUs</th>
<th>Compute Time</th>
</tr>
</thead>
<tbody>
<tr>
<td>Krusell-Smith (K-S)</td>
<td>$4 \times 80 \times 04$</td>
<td>4</td>
<td>53s</td>
</tr>
<tr>
<td>Reiter (R)</td>
<td>$4 \times 80 \times 04$</td>
<td>4</td>
<td>2m 59s</td>
</tr>
<tr>
<td>den Haan-Rendahl (D-R)</td>
<td>$4 \times 80 \times 16$</td>
<td>4</td>
<td>22s</td>
</tr>
<tr>
<td>Proximal Point Algorithm M=0 (PPA0)</td>
<td>$4 \times 80 \times 04$</td>
<td>20</td>
<td>24s</td>
</tr>
<tr>
<td>Proximal Point Algorithm M=1 (PPA1)</td>
<td>$4 \times 80 \times 12$</td>
<td>20</td>
<td>31s</td>
</tr>
<tr>
<td>Proximal Point Algorithm M=2 (PPA2)</td>
<td>$4 \times 80 \times 24$</td>
<td>20</td>
<td>1m</td>
</tr>
<tr>
<td>Proximal Point Algorithm M=3 (PPA3)</td>
<td>$4 \times 80 \times 48$</td>
<td>20</td>
<td>1m 49s</td>
</tr>
<tr>
<td>Policy Function Iteration M=0 (PFI0)</td>
<td>$4 \times 80 \times 04$</td>
<td>20</td>
<td>6s</td>
</tr>
<tr>
<td>Policy Function Iteration M=1 (PFI1)</td>
<td>$4 \times 80 \times 12$</td>
<td>20</td>
<td>8s</td>
</tr>
<tr>
<td>Policy Function Iteration M=2 (PFI2)</td>
<td>$4 \times 80 \times 24$</td>
<td>20</td>
<td>13s</td>
</tr>
<tr>
<td>Policy Function Iteration M=3 (PFI3)</td>
<td>$4 \times 80 \times 48$</td>
<td>20</td>
<td>18s</td>
</tr>
</tbody>
</table>

Table 2.1: Summary of the algorithms to be compared. In the first column, $M$ denotes the order of truncation of the polynomial chaos expansion. The abbreviation in the parenthesis is the algorithm identifier used in the comparison analysis, which follows. The second column displays the total number of grid points to discretize the policy function. Note that the methods of discretizing the distribution $\mu$ vary across algorithms. If the distribution is discretized with several parameters, the number of grid points is the product of the number of grid points in each parameter.
2.4.1 Precision of the Proximal Point Algorithm vs. Existing Algorithms

One way of comparing these sets of numerical solutions is to analyze their Euler equation errors. There have been two different Euler equation error tests put forward in the literature (see e.g., den Haan, 2010), the standard Euler equation error test and the dynamic Euler equation error test. The standard Euler equation errors are calculated by comparing the numerical solution for optimal consumption $c$ against the explicitly calculated conditional expectation in the Euler equation denoted by $\tilde{c}$. It is the absolute percentage error

$$\epsilon_{SE} = \frac{|c - \tilde{c}|}{\tilde{c}}.$$ 

In contrast to the standard Euler equation error, the dynamic equivalent denoted by $\epsilon_{DE}$ is computed for several consecutive periods. This test is more stringent as the numerical solution and the explicit conditional expectation usually diverges with more periods. I compute the standard and the dynamic Euler equation error for a random sample of aggregate shocks over $N$ periods for the different numerical solutions from Table 2.1. I set the number of periods to $N = 3000$. Note that I compute the standard Euler Equation error test also over multiple periods, but it is reset every period and hence, does not accumulate. The errors’ summary statistics are displayed in Table 2.2. Note that the den Haan-Rendahl algorithm does not seem to work in the configuration used here. The other solution algorithms produce more or less the same numbers in terms of the mean and median, although Reiter and the algorithms based on polynomial chaos improve the maximum error. Furthermore, increasing the order of truncation of the polynomial chaos expansion does improve the mean and median standard Euler equation error slightly.

The advantage of the polynomial chaos algorithms becomes clearer when displaying the full error distribution in terms of boxplots in Figure 2.4. One can see that the existing algorithms produce much wider error distributions for both the standard and the dynamic Euler equation error. It is interesting to observe that the Reiter algorithm, although improving on the extreme points of the error distribution, does not lead to any improvement compared to the Krusell-Smith algorithm. The same is true for the den Haan-Rendahl method, which performs much worse. It seems that the reason why they performed well in the comparison by den Haan (2010) is that they use considerably more grid points, whereas here, I deliberately run all methods on the same discretization. In comparison, all polynomial chaos based solutions produce much narrower error bands. This is mainly due to the better anticipation of the cross-sectional distribution’s law of motion in the polynomial chaos algorithms.

Since consumption is concave and increasing in individual capital, the interpolation error when not taking the absolute value is expected to be negative due to the underestimation of the true consumption away from the interpolation points. Hence, by looking at the error without absolute values in Figure 2.5, we can identify any systematic biases. We see that the standard errors of the algorithms based on polynomial chaos are indeed negative which indicates that the approximation at the interpolation points is good. However, the existing algorithms have a systematic positive bias. The dynamic error for the polynomial chaos based algorithms shows positive bias as well. This is due to the equilibrium effect that less consumption today increases consumption tomorrow. However, not taking the absolute error still does not help in distinguishing between the different truncation orders.
Table 2.2: Euler equation errors for the numerical solutions from Table 2.1. This table displays the summary statistics of the standard Euler equation error $\epsilon SE$ and of the dynamic Euler equation error $\epsilon DE$. It is computed over a random sample of aggregate shocks over $N = 3000$ periods. The initial cross-sectional distribution is the same for all algorithms.

<table>
<thead>
<tr>
<th></th>
<th>SE Mean</th>
<th>SE Median</th>
<th>SE Min</th>
<th>SE Max</th>
<th>DE Mean</th>
<th>DE Median</th>
<th>DE Min</th>
<th>DE Max</th>
</tr>
</thead>
<tbody>
<tr>
<td>PFI0</td>
<td>2.2784e-03</td>
<td>1.0985e-05</td>
<td>2.0314e-03</td>
<td>2.8142e-03</td>
<td>8.3721e-03</td>
<td>6.3071e-03</td>
<td>2.8142e-03</td>
<td>8.5023e-03</td>
</tr>
<tr>
<td>PFI1</td>
<td>2.7826e-03</td>
<td>1.0985e-05</td>
<td>2.0314e-03</td>
<td>2.8142e-03</td>
<td>8.3721e-03</td>
<td>6.3071e-03</td>
<td>2.8142e-03</td>
<td>8.5023e-03</td>
</tr>
<tr>
<td>PFI2</td>
<td>2.7826e-03</td>
<td>1.0985e-05</td>
<td>2.0314e-03</td>
<td>2.8142e-03</td>
<td>8.3721e-03</td>
<td>6.3071e-03</td>
<td>2.8142e-03</td>
<td>8.5023e-03</td>
</tr>
<tr>
<td>PFI3</td>
<td>2.7826e-03</td>
<td>1.0985e-05</td>
<td>2.0314e-03</td>
<td>2.8142e-03</td>
<td>8.3721e-03</td>
<td>6.3071e-03</td>
<td>2.8142e-03</td>
<td>8.5023e-03</td>
</tr>
<tr>
<td>PPA0</td>
<td>1.2769e-04</td>
<td>1.0251e-04</td>
<td>4.0947e-09</td>
<td>2.9345e-03</td>
<td>5.2671e-05</td>
<td>3.8256e-05</td>
<td>2.7046e-11</td>
<td>2.8195e-03</td>
</tr>
<tr>
<td>PPA1</td>
<td>5.4468e-05</td>
<td>3.4197e-05</td>
<td>2.1654e-09</td>
<td>2.8482e-03</td>
<td>6.3027e-05</td>
<td>4.9718e-05</td>
<td>4.4474e-10</td>
<td>2.7854e-03</td>
</tr>
<tr>
<td>PPA2</td>
<td>5.0722e-05</td>
<td>3.136e-05</td>
<td>2.07e-09</td>
<td>2.8446e-03</td>
<td>6.3415e-05</td>
<td>5.0015e-05</td>
<td>8.0581e-11</td>
<td>2.7828e-03</td>
</tr>
<tr>
<td>PPA3</td>
<td>4.8672e-05</td>
<td>3.136e-05</td>
<td>2.07e-09</td>
<td>2.8429e-03</td>
<td>6.3721e-05</td>
<td>5.0331e-05</td>
<td>1.0985e-10</td>
<td>2.7814e-03</td>
</tr>
</tbody>
</table>

This table displays the summary statistics of the standard Euler equation error $\epsilon SE$ and of the dynamic Euler equation error $\epsilon DE$. It is computed over a random sample of aggregate shocks over $N = 3000$ periods. The initial cross-sectional distribution is the same for all algorithms.
2.4. Numerical Results

Figure 2.4: **Boxplots of the Euler equation error distributions of individual capital.** Panel A shows the standard Euler equation error and Panel B displays the dynamic Euler equation error for the numerical solutions from Table 2.1. The error is computed over a finer grid in the dimension of individual capital than the grid on which the solutions are computed. I compute the error for a random sample of aggregate shocks over $N = 3000$ periods. The initial cross-sectional distribution is the same for all algorithms. The red lines mark the medians, whereas, the blue boxes denote the 25th to 75th percentiles. The whiskers indicate the range of the distribution and the red dots outside are outliers.

Figure 2.5: **Boxplots of the Euler equation error distributions of individual capital without absolute value.** Panel A shows the standard Euler equation error and Panel B displays the dynamic Euler equation error for the numerical solutions from Table 2.1. The error is computed over a finer grid in the dimension of individual capital than the grid on which the solutions are computed. I compute the error for a random sample of aggregate shocks over $N = 3000$ periods. The initial cross-sectional distribution is the same for all algorithms. The red lines mark the medians, whereas, the blue boxes denote the 25th to 75th percentiles. The whiskers indicate the range of the distribution and the red dots outside are outliers.

It seems counter-intuitive that the error distributions for the algorithms based on polynomial chaos do not differ much. One would expect that they decrease with increasing order. To understand why, recall that the approximation error bound in (2.21) consists
of two terms. A decrease in error due to the increase in the truncation order of the polynomial chaos might be offset by an increase in the interpolation error. Note that even though I use the same interpolation grid for all algorithms, the interpolation error may still differ as it depends on the curvature of the solution. The higher the curvature, the higher the interpolation error. Therefore, we need to disentangle the truncation error from the interpolation error. This is possible by comparing the prediction of the next-period aggregate capital by the algorithm $K$ with the true next-period aggregate capital $\tilde{K}$. I display the law of motion error, computed by

$$\epsilon_{\text{LoM}} = \frac{K - \tilde{K}}{K},$$

in Figure 2.6. I exclude the den Haan-Rendahl algorithm because it already performed worse than the others in terms of the Euler equation errors. I also exclude the law of motion error for the Reiter algorithm because it was not possible to extract the prediction of next-period aggregate capital from Reiter’s Matlab implementation. The figure shows that indeed, the law of motion error becomes smaller when the truncation order is increased. In particular, we observe an exponential decrease as predicted by the error bound. Therefore, I conclude that the Euler equation error is indeed dominated by the interpolation error which offsets the decrease of the law of motion error when the curvature of the solution increases with the truncation order. Furthermore, the law of motion error is within the region of the truncation criterion when truncating at second or higher order. This indicates that a polynomial chaos expansion up to second order yields sufficient precision.

Overall, the error analysis shows that the polynomial chaos based algorithms outperform the existing algorithms and that the polynomial chaos expansion up to second order
suffices to approximate the growth model. Recall that order zero implies that the optimal policies depend solely on aggregate capital. Order one and higher, however, imply a dependence on the full approximated distribution. Hence, to approximate the rational expectations equilibrium of the growth model sufficiently, the agents need to know more than the aggregate capital. However, a crude approximation of the cross-sectional distribution seems to be enough.

2.4.2 Reasons for Approximate Aggregation

Let me now compare the implications of the different numerical solutions. As the Euler equation errors for the den Haan-Rendahl algorithm are large, I will compare the polynomial chaos algorithms only to the Krusell-Smith and the Reiter algorithm. The largest conceptual difference between these algorithms is that the Krusell-Smith method assumes bounded rationality in terms of a rule of thumb, i.e., a parametric law of motion for the aggregate variables depending on a finite number of moments. The polynomial chaos based algorithms, however, use the nonparametric law of motion of the aggregate variables stemming from the cross-sectional distribution. The Reiter algorithm lies conceptually in between the former two. It maps the set of moments to a parameterized cross-sectional distribution rather than a rule of thumb to compute the prediction for aggregate variables.

To compare the implications of these conceptual differences, I look at the stationary cross-sectional distributions. I cannot compute the full stationary state distribution for this model though, since this is a distribution of distributions \( P(z', k, \mu) \). However, I can consider the expected conditional cross-sectional distribution \( E_{\mu}(P(z', k|\mu)) \), which is essentially the average stationary cross-sectional distribution. It is computed as a fixed point of the cross-sectional distribution’s law of motion and displayed in Figure 2.7. We can see that the distributions of the proximal point algorithm with different orders of truncation are almost indistinguishable. They are further to the right than the distribution of the Krusell-Smith algorithm. As expected, the distribution of the Reiter algorithm is in between the former two. Furthermore, the distribution by the Krusell-Smith algorithm has a much thicker tail to the right than the other distributions. Panel B serves as a robustness check. It displays the solutions of the Krusell-Smith algorithm with several moments. As the number of moments increases, the distributions also seem to converge. However, they do not converge to the distribution produced by the proximal point algorithm.

The fact that the distributions of the different proximal point algorithms are so close explains the approximate aggregation result from Krusell and Smith (1998). In terms of stationary distributions, higher orders do not matter in this calibration of the growth model. This is confirmed when computing the changes in the stationary distributions for increasing truncation order, as displayed in Figure 2.8. The change from order one to two is already smaller than the termination threshold \( 5e^{-5} \) of the algorithm. This means that the solution of this calibration of the growth model is not sensitive to errors in the law of motion of aggregate capital. Note that the expected ergodic distribution is used in Krusell and Smith (1998) to update the parameter estimate for their law of motion. Since this ergodic distribution does not change substantially with a different law of motion, they did not find any changes in the solution when using more than one moment.

Hence, I conclude that approximate aggregation does hold in this calibration of the model. Moreover, already a first order truncation of the polynomial chaos results in a good approximation in this model. An interesting question, though, remains. By how
Figure 2.7: Average cross-sectional distributions produced by the Krusell-Smith algorithm and the polynomial chaos algorithms from Table 2.1. This graph displays the average stationary cross-sectional distribution, i.e., the expectation of the stationary distribution of the state space \((z', k, \mu)\) conditional on \(\mu\). It displays histogram approximations of the distributions with bin size 0.1. Panel A compares the p.d.f.s of the benchmark Krusell-Smith algorithm using the first moment with the proximal point algorithm using polynomial chaos. Panel B compares the benchmark proximal point algorithm truncated at second order with the Krusell-Smith algorithm using several moments. Note that only the distributions of the proximal point algorithms are displayed because they are identical with the ones produced by policy function iteration.

Figure 2.8: Changes in the average cross-sectional distributions produced by the proximal point algorithms from Table 2.1. This graph displays the changes in the average stationary cross-sectional distribution, i.e., the expectation of the expected ergodic distribution of the state space \((z', k, \mu)\) conditional on \(\mu\), when the order of truncation of the polynomial chaos expansion is increased. It shows the differences in p.d.f.s with a histogram approximation with bin size 0.1.
much would the solutions differ in a model where approximate aggregation does not hold. To investigate this, I do not even have to change the model. In the following section, I show that approximate aggregation fails for a different calibration of the benchmark growth model.

2.4.3 A Case where Approximate Aggregation Fails

In the previous calibration taken from den Haan et al. (2010), the unemployment benefit is set to 15% of gross wage. In the following, I set it to 65%. Therefore, the higher unemployment insurance leads to more redistribution in the new calibration and hence, better risk sharing. As in the previous calibration, the law of motion error decreases in an exponential manner with increasing truncation order as can be seen in Figure 2.9. Interestingly, the errors of the Krusell-Smith algorithm and the zeroth order truncation are smaller than in the first calibration. This makes sense as fewer agents will hit the borrowing constraint meaning that the kink in the policy has less impact. This effect does also explain the narrower Euler equation error distributions displayed in Figure 2.10. To investigate, which order yields sufficient precision, I also plot the changes in the stationary distributions in Figure 2.11. Note that I skip the approximation which truncates at order one because it did not converge. We can see that the distribution does not change too much when comparing order 2 and three. Therefore, the approximation with second order seems to be sufficient.

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7This figure does in fact correspond to the 2015 OECD median of the net replacement rate in the initial unemployment phase of an average-wage household with two children and one earner (see http://www.oecd.org/els/benefits-and-wages-statistics.htm).
Figure 2.10: **Boxplots of the Euler equation error distributions of individual capital.** Panel A shows the standard Euler equation error and Panel B displays the dynamic Euler equation error for the numerical solutions from Table 2.1 for 65 % unemployment benefit rate. The error is computed over a finer grid in the dimension of individual capital than the grid on which the solutions are computed. I compute the error for a random sample of aggregate shocks over $N = 3000$ periods. The whiskers indicate the range of the distribution and the red dots outside are outliers.

Figure 2.11: **Changes in the average cross-sectional distributions produced by the proximal point algorithms from Table 2.1 for 65 % unemployment benefit rate.** This graph displays the changes in the average stationary cross-sectional distribution, i.e., the expectation of the expected ergodic distribution of the state space $(z', k, \mu)$ conditional on $\mu$, when the order of truncation of the polynomial chaos expansion is increased. It shows the differences in p.d.f.s with a histogram approximation with bin size 0.1.

That approximate aggregation does not hold in this calibration becomes clear when looking at the stationary distributions. The failure of approximate aggregation results from the stark differences in the expected ergodic distributions as can bee seen in Figure
2.12. The distribution resulting from the Krusell-Smith algorithm exhibits much fatter
tails compared to the distributions induced by the polynomial chaos base algorithms.
Furthermore, given it’s methodology, it is not surprising that the distribution from Reiter’s
method is closer to ours than to Krusell-Smith’s distribution. This graph confirms that
bounded rationality in the sense of using only the first moment to predict the law of motion
can in fact be a substantial contributor to inequality. Panel B shows that the Krusell-
Smith method does not converge for this calibration when the number of moments is
increased which exemplifies that approximate aggregation fails in this instance.

One question remains when considering Panel A of Figure 2.12. Why does the Krusell-
Smith algorithm produce such a different expected ergodic distribution, even though its
law of motion errors are even smaller than in the previous calibration? The reason is
that the sensitivity to changes in aggregate capital, and thus, errors in its prediction, is
much higher in the case of more redistribution. All agents are affected by idiosyncratic
risk solely in their wages, whereas, aggregate risk impacts both wages and the rental rate.
Note that for average to wealthy agents, changes in the rental rate have a larger impact on
wealth relative to changes in wage. When idiosyncratic risk is large, this disparity is still
small. However, when idiosyncratic risk is hedged by a higher unemployment insurance,
this disparity between poor and rich agents in their sensitivity towards changes in the
rental rate becomes much stronger. Therefore, even small errors in the law of motion lead
to sizable differences in the agents’ reactions. When this law of motion error is systematic,
it then results in a divergence of the wealth distribution.
2.5 Economic Insights

Two interesting economic results emerge by comparing the expected ergodic distributions of the proximal point algorithm for the two different calibrations as can be seen Figure 2.13. When comparing both graphs, one can see that the expected ergodic distribution in

![Figure 2.13: Average cross-sectional distributions produced by the polynomial chaos algorithm for low and high unemployment benefit.](image)

the case of a high unemployment benefit has fatter tails than the distribution in the case of a low benefit. Moreover, the mode of the distribution for the higher unemployment benefit is further to the right and has a lower probability than for low benefits. This result is consistent for both the model with and without aggregate shocks. Therefore, albeit the better risk sharing due to more redistribution, the volatility of the cross-sectional distribution, which can be interpreted as systemic risk, increases. This result mirrors the volatility paradox in Brunnermeier and Sannikov (2014). In their paper, they analyze a model with two types of agents who face aggregate risk and find that low-risk environments where idiosyncratic risk is hedged are conductive of greater systemic risk and that. The reason that the setup herein is comparable is the following. The cross-sectional distribution in our model at any particular point in time is just a screenshot of the economic state across agents at that point. However, the expected stationary cross-sectional distribution over the infinite time horizon represents all states an agent may reach over the infinite time horizon and their respective probabilities. The volatility of this distribution can, hence, be interpreted as the volatility of the individual capital holdings of each agent over time. This interpretation does confirm the volatility paradox of Brunnermeier and Sannikov (2014) in the Aiyagari-Bewley economy. Furthermore, this effect occurs rather smoothly as can be seen in Figure 2.14. The graph displays the distributions resulting from an

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8 A similar observation has been made in Krueger et al. (2016), Section 6.3, who show that savings of poor agents are lower and consumption is higher when there is higher unemployment insurance.
Figure 2.14: Average cross-sectional distributions produced by the polynomial chaos algorithm for different levels of the unemployment benefit. This graph displays the average stationary cross-sectional distribution, i.e., the expectation of the stationary distribution of the state space \((z', k, \mu)\) conditional on \(\mu\) for different unemployment benefit rates \(\nu\). The proximal point algorithm for up to 15% unemployment benefit is truncated at first order, whereas, the proximal point algorithm for higher benefits is truncated at second order.

array of different unemployment benefit rates. It shows that they change gradually and smoothly when the level of benefits changes. In Appendix B.4, I show that this result is robust to changes in the model parameters.

The second economic insight which follows from Figure 2.13 emerges from evaluating the impact of aggregate risk across the different levels of risk-sharing. In the case of the low unemployment benefit displayed in Panel A of Figure 2.13, the expected stationary cross-sectional distribution of the model with aggregate risk lies within the two distributions resulting from the model without aggregate risk. I compute two distributions when aggregate risk is absent as I fix the productivity level to one of the two aggregate states. Naturally, one would expect the distribution of the model with aggregate risk to lie in between these two distributions as the productivity level varies stochastically between the bad and good state. However, when considering the case of more redistribution through a higher unemployment benefit which is displayed in Panel B of Figure 2.13, we see that this is not necessarily the case. The model with aggregate risk produces an expected stationary cross-sectional distribution with fatter tails on the left and right than both models where aggregate risk is absent. This means that risk-sharing of idiosyncratic risk across agents amplifies the effect of aggregate risk in this model featuring ex-post heterogeneity.

To explain why, one has to look at what risk-sharing means in this model. With a higher level of risk-sharing, employed agents pay increased taxes and unemployed agents receive higher unemployment benefits. The increase in unemployment benefits is much higher though than the increase in taxes as there are many more employed than unemployed agents. With a higher income at the disposal of unemployed agents, rich unemployed agents whose consumption is already at a high level will primarily increase their savings, whereas, poor unemployed agents who have a high marginal propensity to consume will
increase their consumption. Due to the higher insurance, poor agents will also decrease their precautionary savings for the sake of higher consumption. This explains the higher weights on the poor and rich agents in the cross-sectional distribution in Figure 2.13 when the unemployment benefits are high. In Appendix B.4, I show that this result is robust to changes in the subjective discount factor and the depreciation rate of capital. However, the amplification effect changes with increasing risk aversion and increasing elasticity in the production function. In both of these cases, the agents place a higher value on capital. The reason is that they want to insure aggregate risk with higher capital holdings if risk aversion is high and they want to insure against the higher sensitivity of production w.r.t. aggregate capital when the elasticity is high. Both of these incentives dampen the slope of the marginal propensity to consume such that it will be lower for poor agents and higher for rich agents compared to a lower risk aversion or a lower production elasticity. The lower marginal propensity to consume of poor agents leaves them room to increase their savings after only moderately increasing their consumption. Rich agents will primarily increase their consumption as they already have high savings. This leads to the opposite direction in amplification as the cross-sectional distribution for the model with aggregate risk displays lower weight on poor and rich agents compared to the model without aggregate risk. Overall, risk-sharing among ex-post heterogeneous agents amplifies the effect of aggregate risk in the model. The direction of this amplification depends on the slope of the marginal propensity to consume.
Chapter 3

An Approach to Compute Solutions to an Asset Pricing Model

Abstract
In this chapter, I extend the numerical method developed in the previous two chapters to a model with a zero-net supply condition. This implies that equilibrium asset prices are implicitly given, and therefore, have to be solved for in addition to the optimal policies of the agents. By adding this additional step, I compute a fully rational equilibrium with policies and prices dependent on the whole cross-sectional distribution. Economically, I find that risk sharing through trading in the bond amplifies aggregate risk significantly. The bond prices in the model with aggregate risk differ substantially from the model where aggregate risk is absent as the motives to trade change with aggregate risk. Furthermore, the conditional price volatility and skewness decrease with tighter borrowing constraints.

The chapter proceeds as follows. In the next section, I present the Huggett model with aggregate risk. In Section 2, I extend the iterative procedure and the discretization technique for the cross-sectional distribution to this model with a zero-net supply equilibrium condition. In Section 3, I analyze the precision of the numerical method. Lastly, I analyze which economic insights this novel computational method yields for asset prices. Appendix C contains information on the calibration and robustness checks.

3.1 The Huggett Economy with Aggregate Risk

In this chapter, I extend my algorithm for models featuring asset trading. The simplest example of such a model is the Huggett (1993) model which constitutes an endowment economy where agents can invest or borrow through a risk-free bond in zero-net supply. Consider a discrete-time infinite-horizon model with a continuum of agents of measure one. As in the Aiyagari-Bewley economy in Chapter 2, there are two kinds of exogenous shocks, an aggregate shock and an idiosyncratic shock. The aggregate shock characterizes the state of the economy with outcomes in \( Z^{ag} = \{0, 1\} \) standing for a bad and good state, respectively. The idiosyncratic shock with outcomes in \( Z^{id} = \{0, 1\} \) indicates that an agent receives a low or a high endowment, respectively. It is i.i.d. across agents
conditional on the aggregate shock. I denote the compound exogenous process \( (z^a_t, z^id_t)_{t \geq 0} \) by \((z_t)_{t \geq 0} \in \mathcal{Z}\) with \( \mathcal{Z} = \mathcal{Z}^a \times \mathcal{Z}^id \). The transition probabilities are exogenously given by a four-by-four matrix.

The security market in this model consists of a risk-free single-period bond which can be bought or sold for a price \( p \) and pays out one unit of the consumption good to the buyers one period later. An agent’s asset holding is denoted by \((a_t)_{t \geq 0}\). The bond’s price which is an aggregate endogenous variable is defined by the zero-net supply condition

\[
0 = \sum_{z^id = 0}^{1} \int_{-\infty}^{\infty} ad \mu_t (z^id, a) \quad \forall \ t \geq 0,
\]

where \( \mu_t \) is the cross-sectional distribution of idiosyncratic exogenous and endogenous variables at the beginning of time \( t \), i.e., before the agents choose their optimal current bond holdings. It is simply the probability distribution of individual asset allocations across the low and high endowment agents given the trajectory of aggregate shocks

\[
\mu_t (z^id, a) = \mathbb{P} \left( \{ z^id_t = z^id \} \cap \{ a_{t-1} \leq a \} \mid z^a_0, \ldots, z^a_t \right)
\]

for all \( t \geq 0 \), \( z^id \in \mathcal{Z}^id \) and \( a \in \mathbb{R} \). The aggregate shocks cause the cross-sectional distribution to vary over time, which is indicated by the time subscript of \( \mu_t \).

Each agent chooses her bond allocation and consumption such that they satisfy certain constraints. First, individual consumption must be positive at all times \( c_t > 0 \), \( t \geq 0 \), and bond holdings are subject to a borrowing constraint \( a_t \geq \bar{a} \), \( t \geq 0 \), where \( \bar{a} < 0 \). Second, given an initial bond holding \( a_{t-1} \geq \bar{a} \) and an initial cross-sectional distribution \( \mu_{t-1} \) with zero mean and support in \([\bar{a}, \infty)\), each agent adheres to a budget constraint, which equates individual consumption and the current value of bond holdings to the agent’s current endowment and settlement of the previous period bond holdings\(^1\)

\[
a_t p_t + c_t = e (z_t) + a_{t-1} \quad \forall \ t \geq 0.
\]

The endowment \( e \) is a positive constant whose level depends on the exogenous shock. It is higher in the good economic state and it is higher for agents with \( z^id = 1 \) compared to the other agent group.

Assume that all agents have time-separable CRRA utility with a risk aversion coefficient \( \gamma > 1 \) and time preference parameter \( \beta \in (0, 1) \). Then, given an agent’s initial bond holding \( a_{t-1} \geq \bar{a} \) and the initial cross-sectional distribution \( \mu_{t-1} \) with zero mean and support in \([\bar{a}, \infty)\), the individual optimization problem reads

\[
\max_{\{c_t, a_t\} \in \mathbb{R}^2} \mathbb{E} \left[ \sum_{t=0}^{\infty} \beta^t c_t^{1-\gamma} - 1 \right] \quad \text{s.t.} \quad a_t p_t + c_t = e (z_t) + a_{t-1} \forall \ t \geq 0 \quad c_t > 0, \quad a_t \geq \bar{a} \forall \ t \geq 0.
\]

\(^1\) Note that I specify the time line slightly differently than Huggett (1993). There, \( a_t \) in the budget constraint 3.2 is denoted by \( a_{t+1} \) because it is the bond allocation, which is settled in period \( t + 1 \). In contrast to that notation, however, I want to emphasize the time period, at which the agent optimally chooses the magnitude of her bond allocation. Taking this view, the optimal consumption and bond holding choice have the same time subscript. My time line, therefore, indicates which filtration the endogenous variables are adapted to.
In a competitive equilibrium, the individual problems are solved subject to the price defining equilibrium condition (3.1) that the bond is in zero-net supply. As in the previous chapters, I consider a particular competitive equilibrium of recursive form. To define a recursive equilibrium, I switch to prime-notation for convenience, where a prime denotes variables in the current period and variables with no prime refer to the previous period.

**Definition 23** (Recursive equilibrium). A solution to the agents’ individual optimization problems (3.3) subject to the equilibrium condition (3.1) given an initial cross-sectional distribution of asset holdings \( \mu_{-1} \) with mean zero and support in \([\bar{a}, \infty)\) is called recursive if there exist functions \( g_p: Z^{ag} \times P(Z^{id} \times \mathbb{R}) \to \mathbb{R} \) and \( g_i: \mathbb{Z} \times \mathbb{R} \times P(Z^{id} \times \mathbb{R}) \to \mathbb{R} \), \( i \in \{c, a\} \), such that, for any point in time, the equilibrium price, current optimal bond holding and consumption choices equal \( p' = g_p(z^{ag'}, \mu') \), \( a' = g_a(z', a, \mu') \) and \( c' = g_c(z', a, \mu') \), respectively, for any agent with previous-period bond holding \( a \) who observes the current-period exogenous shock \( z' = (z^{ag'}, z^{id'}) \) and the beginning-of-current-period cross-sectional distribution \( \mu' \).

**Remark.** Note that I solely work with the equilibrium prices \( p' = g_p(z^{ag'}, \mu') \) and the asset allocation policy \( a' = g_a(z', a, \mu') \) in the following as the consumption policy follows directly due to the budget constraint.

In order to obtain a full description of equilibrium, we need to define a consistent law of motion of \( \mu' \) to \( \mu'' \). As the exogenous shocks are defined similarly as in the Aiyagari-Bewley model from Chapter 2, the transition of the cross-sectional distribution is defined equivalently. Given a fixed distribution \( \mu' \) over the cross-section of individual bond holdings at the beginning of the current period and a recursive equilibrium, the distribution changes in two steps \( \mu' \to \tilde{\mu}' \to \mu'' \). In the first step, the agents implement their optimal bond allocation, which leads to the end-of-current-period distribution

\[
\tilde{\mu}'(z^{id'}, a) = \mathbb{P}\left( \left\{ \zeta' = z^{id'} \right\} \cap \left\{ g_a(z^{ag'}, \zeta', a, \mu') \leq a \right\} \right) \mid z^{ag'}, \mu', \alpha, \mu''
\]

where \((\zeta', a) \sim \mu'\) is a random variable distributed according to the cross-sectional distribution. In the second step, the next-period shocks \( z'' \) for all agents realize and shift the quantities of low and high endowment agents depending on the outcome of the aggregate shock. Agents who previously received a high endowment either keep receiving a high endowment or switch to a low endowment, the same holds for the agents who previously received a low endowment. Therefore, the distribution at the beginning of the next period \( \mu'' \) is given by

\[
\mu''(z^{id''}, a) = \sum_{z^{id'} \in Z^{id}} \frac{p(z^{ag''}, z^{id''})|z^{ag''}, z^{id'}}{p(z^{ag'}, z^{id'})} \tilde{\mu}'(z^{id'}, a)
\]

\[
= \sum_{z^{id'} \in Z^{id}} \frac{p(z^{ag''}, z^{id''})|z^{ag''}, z^{id'}}{p(z^{ag'}, z^{id'})} \cdot \mathbb{P}\left( \left\{ \zeta' = z^{id'} \right\} \cap \left\{ g_a\left(z^{ag'}, \zeta', a, \mu'\right) \leq a \right\} \right) \mid z^{ag'}
\]

for all \( z^{id''} \in Z^{id} \) and \( a \in \mathbb{R} \). The multipliers in front of the end-of-current-period distribution are the probabilities that the employment status changes from \( z^{id} \) to \( z^{id''} \) given the observed trajectory of \( z^{ag} \) to \( z^{ag''} \).

Now that all model ingredients are defined, the next section lays out the methodology to solve for the recursive equilibrium in this model. The parameter values of the model are set according to the calibration of Krusell et al. (2011) and can be found in Appendix C.1.
3.2 Computing the Equilibrium

3.2.1 Extending the Proximal Point Algorithm for the Huggett Economy

Establishing a theory of solving this equilibrium problem comparable to Chapter 1 is outside the scope of this thesis and one objective for my future research. Therefore, I just explain the practical approach I take to compute the numerical results. In Chapter 1, I introduce an iterative procedure named the proximal point algorithm which converges to the solution of the equilibrium model if the rates of return are explicitly given through the equilibrium condition. Even though this is not the case in the Huggett model, I can still use the proximal point algorithm to compute a solution for the bond holdings given a guess of the equilibrium price function \( g_p \). Let me recall some of notions first before I extend the proximal point algorithm to the Huggett model.

I show in Chapter 1 that the recursive equilibrium can be written in terms of random variables.

**Definition 24** (Recursive equilibrium with random variables). The recursive equilibrium in Definition 23 can be rewritten using functions \( h_a : Z \times R \times L_{t-1} \rightarrow R \) such that \( a' = g_a(z',a,\mu') = h(z',a,a(z'|id)) \) and \( h_p : Z \times L_{t-1} \rightarrow R \) such that \( p' = g_p(z',\mu') = h_p(z',\alpha(z'|id)) \) for any agent with previous-period bond holding \( a \) who observes the current-period exogenous shock \( z' = (z'|ag',z'|id) \) and the beginning-of-current-period conditional cross-sectional random variable \( \alpha(z'|id) \in L_{t-1} \). The function for consumption follows from the budget constraint.

Using this functional form of the recursive equilibrium, I define the model’s Euler equation operator \( T : H_t \rightarrow C(L_P^a) \) such that

\[
T(h) = - h_p \left( z', \alpha(z'|id) \right) \frac{\partial}{\partial \mu} \left( e(z') + \alpha(z'|id) - h_a \left( z', \alpha(z'|id) \right) h_p \left( z', \alpha(z'|id) \right) \right) \\
+ \sum_{z'' \epsilon Z} p' e \left( z'' \right) \frac{\partial}{\partial \mu} \left( e \left( z'' \right) + \alpha' \left( z'' | id \right) - h_a \left( z'', \alpha' \left( z'' | id \right) \right) h_p \left( z'', \alpha' \left( z'' | id \right) \right) \right),
\]

where \( \alpha(z'|id) \sim \mu' / \mathbb{P}(z'|id|z'|id) \) denotes the current-period conditional random variable of individual bond holdings and

\[
\alpha' \left( z'' | id \right) = \mathbb{E}(z'' | id|z'|id) \left[ h_a \left( z', \alpha(z'|id) \right) \right]
\]

denotes the next-period conditional bond holding random variable which accounts for the agents’ changing idiosyncratic states. Recall that the individual bond holdings in the recursive functional form is substituted by the corresponding random variable in the Euler equation operator which is why it depends on \( h_a(z', \alpha(z'|id)) \) instead of \( h_a(z', a, \alpha(z'|id)) \). Given a price function \( h_p \), this yields the model’s first-order condition

\[
T(h_a; h_p) + y = 0 \text{ a.e. with } h_a \perp y \geq 0. \tag{3.5}
\]

There exists a Lagrangian \( L : H_t \times C(L_P^a) \rightarrow [-\infty, \infty] \) such that \( T \) maximizes \( L \) in its second argument. According to (Lemma 5.1, Ghoussoub, 2008), the Lagrangian associated with \( T \) is given by

\[
L(h_a, p) = \sup_{g \epsilon H_t} \left\{ \langle p, g \rangle + \langle T(g), h_a - g \rangle \right\} \tag{3.6}
\]
and maximizing the Lagrangian over \( p \) for a given \( h_a \) yields \( p^* = T(h_a; h_p) \) with the function value \( L(h_a, p^*) = \langle p^*, h_a \rangle = \langle T(h_a; h_p), h_a \rangle \). For notational convenience, I denote \( L(h_a, p^*) \) by \( L(h_a) \).

As in the previous chapters, I follow Rockafellar (1976b) for defining the proximal point algorithm’s update. This algorithm iterates on the resolvent of the first-order condition associated with the Lagrangian (3.6). Hence, each iteration on the resolvent updates the agents’ optimal choices for individual capital \( h \) as well as the Lagrange multiplier \( y \) for the inequality constraint of (3.3). The \( (n + 1) \)-th iterate of the agent’s optimal choices, i.e., \( h_{a+1}^n \), is the minimizer of the Lagrangian augmented by terms featuring the \( n \)-th iterate. The augmented Lagrangian is a function \( L^A : H_a \times L^2(\mathcal{Z}^{id} \times \mathbb{R}, \mathcal{B}(\mathcal{Z}^{id} \times \mathbb{R}), \mu) \rightarrow [-\infty, \infty] \) given by

\[
L^A (h_a, y; z', \alpha(z^{id}), h_a^n, h_p^n) = L(h_a)
+ \frac{1}{2\lambda} \|h_a - h_a^n\|_{L_p}^2
+ \left\{ -y (h_a - \bar{a}) + \frac{1}{2} \|h_a - \bar{a}\|_{L_p}^2, h_a - \bar{a} \leq \frac{y}{\lambda},
-\frac{1}{2\lambda} \|y\|_{L_p}^2, h_a - \bar{a} > \frac{y}{\lambda} \right\}
\]

where \( L \) as in (3.6), \( \lambda > 0 \) is the step size parameter of the proximal point algorithm. The first line of the augmented Lagrangian features the Lagrangian corresponding to the Euler equation (3.5). The second line consists of the objective’s proximal point augmentation, which transforms the first-order condition into its resolvent. The last line corresponds to the inequality constraint. It also consists of the Lagrange term and the augmentation, but it is defined piecewise to account for the case of a binding constraint.

So far, I have explained how to derive the optimal policy \( h_a \) given that the price function \( h_p \) is known. Hence, the question which remains is how to find the price function \( h_p \) such that the equilibrium zero net supply condition

\[
0 = \mathbb{E}^P \left[ h_a \left( z', \alpha(z^{id}) \right) \mid h_p \right]
\]

is satisfied. I implement this joint problem of finding \( h_p \) and \( h_a \) with an outer and inner loop construction. In the outer loop, I use a nonlinear solver to update the price function from its \( n \)-th iterate to the \( (n + 1) \)-th iterate. This means that, given the iterate \( h_p^n \), the nonlinear solver finds the update \( h_p^{n+1} \) which solves (3.8) where, for any step in the nonlinear solver, the policy \( h_a \) inside the equilibrium condition is produced by an inner loop. For this inner loop, I use the proximal point algorithm corresponding to the Euler equation operator

\[
T(h) = -h_p^{n+1} \left( z', \alpha(z^{id}) \right) \frac{\partial}{\partial c} u \left( e(z') + \alpha(z^{id}) - h_a \left( z', \alpha(z^{id}) \right) h_p^{n+1} \left( z', \alpha(z^{id}) \right) \right) + \sum_{z'' \in \mathcal{Z}} p(z'' \mid z') \frac{\partial}{\partial c} u \left( e(z'') + \alpha'(z^{id}) - h_a \left( z'', \alpha'(z^{id}) \right) h_p^n \left( z'', \alpha'(z^{id}) \right) \right).
\]

With all ingredients at hand, I now state the algorithm to approximate a recursive equilibrium of the Huggett model in Algorithm 3.1. Even though convergence results are not established yet, in practice, I find that this procedure does work reasonably well.

### 3.2.2 Applying Polynomial Chaos to the Huggett Economy

Similarly to the growth model, I divide the basic random variable into components \( \xi^z \) and \( \xi^c \) corresponding to the exogenous and endogenous agent-specific states, respectively.
Algorithm 3.1 Extended proximal point algorithm for the Huggett model

\[ A \] Initialization
1: Set \( n = 0 \). Initialize the equilibrium prices \( h^n_p, \) the agents’ choices of individual bond holdings and the Lagrange multiplier \( H^n = (h^n_a, y^n). \)
2: Set the parameter \( \lambda > 0. \)
3: Set the termination criterion small \( \tau \) and the initial distances larger \( d > \tau \).

\[ B \] Iterative Procedure

\[ B.1 \] Outer loop (prices)
4: Solve the equilibrium condition (3.8) with a nonlinear solver which calls the inner loop for the corresponding policy at each iteration.

\[ B.2 \] Inner loop (policy)
5: Set \( \tilde{H} = H^n. \)
6: while \( d > \tau \) do
7: Update \( H^{n+1} \) by
   \[ h^{n+1}_a(z', \alpha(z'')) \approx \arg\min_{h \in H_a} L^A \left( h, y^n; z', \alpha(z''), \tilde{h}_a, h^n_p, h^{n+1}_p \right) \]
   \[ y^{n+1}(z', \alpha(z'')) = \max \left\{ 0, y^n(z', \alpha(z'')) - \lambda \left[ h^{n+1}_a(z', \alpha(z'')) - \bar{a} \right] \right\} \]
   where \( L^A \) is defined as in (3.7) with \( L \) as in (3.6) and \( T \) as in (3.9).
8: Compute the distance \( d = \| H^{n+1} - \tilde{H} \|_{L^2}. \)
9: Set \( \tilde{H} = H^{n+1}. \)
10: end while

Therefore, the law of motion of the projection coefficients is equivalent to the one in the growth model. Let me now elaborate on how I define these components in detail. Since the structure of the exogenous shock in the Huggett economy is defined exactly as in the Aiyagari-Bewley model, I define the basic random variable \( \xi_z \) as before to equal a three state random variable with a c.d.f.

\[ F(\xi_z) = \begin{cases} p^{z'' = 0 | z'' = 1} , & \xi_z = 1 \\ p^{z'' = 0 | z'' = 0} , & \xi_z = 2 \\ 1 , & \xi_z = 3. \end{cases} \]

Note that the probability at state two in the calibration of Krusell et al. (2011) is zero, which means that this basic random variable has only two effective states.

The basic random variable \( \xi_a \) is defined differently from the growth model. As for the growth model, I first compute the stationary bond holdings distribution for the Huggett economy without aggregate risk. By doing so, it is apparent that the conditional distributions of the agents with high and low endowment are very different from each other. This is expected as the majority of agents with a high endowment are on the buy side and the agents with the low endowment are mostly on the sell side of the bond market to smooth consumption. Therefore, to achieve higher precision for the polynomial chaos expansion, I define the basic random variable as a bivariate variable \( \xi_a = (\xi_1, \xi_2) \), where the first component is defined by the stationary distribution of the model without aggregate risk of the agents with low endowment averaged over the two cases of staying in the good and bad economic state, respectively. Accordingly, the second component is given by the stationary distribution of the agents with high endowment. Note that it is necessary for
the polynomial chaos approach that $\xi_1^a$ and $\xi_2^a$ are independent. The distributions of these two components are displayed in Figure 3.1.

![Figure 3.1: Distribution for the basic random variable $\xi^a = (\xi_1^a, \xi_2^a)$. Panel A shows a histogram representation with bin size 0.1 of the distribution of the agents with low endowment, whereas, Panel B displays the distribution of the agents with high endowment. Note that the distributions displayed here are the average of the stationary end-of-period distributions of the model without aggregate shocks.](image)

Once the basic random variables are defined, the corresponding orthogonal polynomials can be generated. As in Chapter 2, I use the three-term recurrence equation (2.15) and the Stieltjes method (see Gautschi, 1982) to compute the orthogonal polynomials for $\xi^z$, $\xi_1^a$ and $\xi_2^a$ separately. The results are illustrated in Figure 3.2. As usual, the number of roots of each polynomial corresponds to its degree. Each first-order polynomial has its root at the mean of the distribution of the corresponding basic random variable. The

![Figure 3.2: Orthogonal polynomials corresponding to the basic random variables of the Huggett model. Panel A shows the polynomials $\Phi_{i1}^{a1}$ corresponding to individual bond holdings up to order $i \leq 3$, whereas, Panel B displays the polynomials $\Phi_{i2}^{a2}$ of order $i \leq 3$. Panel C plots the polynomials $\Phi_i^z$ corresponding to the idiosyncratic shock up to order $i \leq 2$.](image)
polynomials of $\xi^z$ are plotted only at its two states \{1, 3\}. This graph confirms that the polynomial with maximal degree, which is the second-order polynomial $\Phi^z_2$ in this case, has its roots at the states of the corresponding distribution, i.e. at \{1, 3\}.

### 3.3 Numerical Results

I compute the recursive equilibrium solution of the accelerated proximal point algorithm explained in Appendix B.3 combined with a nonlinear solver to find the equilibrium prices truncating the polynomial chaos expansion at different orders using Matlab R2016b.

Furthermore, to demonstrate that the polynomial chaos expansion can be combined with other solution methods, I also compute the recursive solution via standard policy function iteration. To ensure comparability, I run all these methods using the same grid for individual capital and the same termination criterion $5e^{-5}$. The different methods are summarized in Table 3.1. Note that I start with the truncation at first order rather

<table>
<thead>
<tr>
<th>Algorithm</th>
<th># Grid Points for $z' \times k \times \mu$</th>
<th># CPUs</th>
<th>Compute Time</th>
</tr>
</thead>
<tbody>
<tr>
<td>Proximal Point Algorithm M=1 (PPA1)</td>
<td>$4 \times 80 \times 05$</td>
<td>20</td>
<td>5m 17s</td>
</tr>
<tr>
<td>Proximal Point Algorithm M=2 (PPA2)</td>
<td>$4 \times 80 \times 20$</td>
<td>20</td>
<td>8m 28s</td>
</tr>
<tr>
<td>Proximal Point Algorithm M=3 (PPA3)</td>
<td>$4 \times 80 \times 40$</td>
<td>20</td>
<td>33m 24s</td>
</tr>
<tr>
<td>Proximal Point Algorithm M=4 (PPA4)</td>
<td>$4 \times 80 \times 80$</td>
<td>20</td>
<td>2h 28m 12s</td>
</tr>
<tr>
<td>Policy Function Iteration M=1 (PFI1)</td>
<td>$4 \times 80 \times 05$</td>
<td>20</td>
<td>15s</td>
</tr>
<tr>
<td>Policy Function Iteration M=2 (PFI2)</td>
<td>$4 \times 80 \times 20$</td>
<td>20</td>
<td>31s</td>
</tr>
<tr>
<td>Policy Function Iteration M=3 (PFI3)</td>
<td>$4 \times 80 \times 40$</td>
<td>20</td>
<td>1m 15s</td>
</tr>
<tr>
<td>Policy Function Iteration M=4 (PFI4)</td>
<td>$4 \times 80 \times 80$</td>
<td>20</td>
<td>3m 31s</td>
</tr>
</tbody>
</table>

Table 3.1: **Summary of the algorithms to be compared.** In the first column, $M$ denotes the order of truncation of the polynomial chaos expansion. The abbreviation in the parenthesis is the algorithm identifier used in the comparison analysis, which follows. The second column displays the total number of grid points to discretize the policy function. When the distribution is discretized with several parameters, the number of grid points is the product of the number of grid points in each parameter.

than at zeroth order because the zeroth order polynomial chaos represents the mean of the cross-sectional distribution which is fixed at zero in equilibrium due to the zero-net supply condition. The compute time is higher for the proximal point algorithm. This is mainly due to the fact that it solves a full optimization problem in each iteration of the inner loop which then has to be done multiple times to solve the outer loop. I argue that the goal of being more accurate and having a theoretically sound algorithm justifies the increased compute time. However, if one can ensure that policy function iteration converges, then this should be the method of choice. If convergence of a faster ad hoc method is not clear, one can use the proximal point algorithm as a benchmark. Generally, the compute time is higher than for the Aiyagari-Bewley growth model in Chapter 2 because the Huggett economy requires the additional step of solving for the equilibrium price in form of an outer loop.

In the following, I investigate the precision of the algorithms. Furthermore, I examine the order of truncation of the polynomial chaos expansion.

---

2 The computations were performed on the Baobab cluster at the University of Geneva.
3.3.1 Precision

Similarly to the Aiyagari-Bewley growth model, I compute the standard and the dynamic Euler equation error for a random sample of aggregate shocks over $N$ periods for the different numerical solutions from Table 3.1. I set the number of periods to $N = 3000$. Note that I compute the standard Euler Equation error test also over multiple periods, but it is reset every period and hence, does not accumulate in contrast to the dynamic Euler equation error test. The errors’ summary statistics are displayed in Table 3.2. We can see from the error statistics that the increase in the order of the polynomial chaos expansion decreases the standard Euler equation error, but has no such effect in the dynamic Euler equation error. This is illustrated when displaying the full error distribution in terms of boxplots in Figure 3.3. The dynamic Euler equation error does only improve slightly in terms of its maximum range.

Apart from the Euler equation errors, I consider the law of motion errors. As the Huggett economy features asset prices which are indirectly defined through the zero-net supply condition, there are two ways of looking at the law of motion of aggregate variables. Since the equilibrium prices are computed by a recursive price function, we can first consider the corresponding Euler equation errors. Instead of computing the error for the problem’s consumption function, we can simply compute the Euler equation error using the problem’s price function. These error distributions are displayed in Figure reffig4H. A second way of evaluating the law of motion error of prices is the deviation of the cross-sectional distribution’s mean which should equal zero in theory over the simulation of $N = 3000$ periods. The cross-sectional distribution is computed using the distribution’s law of motion stated in (3.4). The boxplots of this law of motion error are displayed in Figure 3.5. Both variants of the law of motion error move closer towards zero with the
The initial cross-sectional distribution is the same for all algorithms.

Table 3.2: Euler equation errors for the numerical solutions from Table 3.1. This table displays the summary statistics of the standard Euler equation error $\epsilon_{SE}$ and of the dynamic Euler equation error $\epsilon_{DE}$. It is computed for a random sample of aggregate shocks over $N = 3000$ periods. The initial cross-sectional distribution is the same for all algorithms.

<table>
<thead>
<tr>
<th>Algorithm</th>
<th>SE Mean</th>
<th>SE Median</th>
<th>SE Min</th>
<th>SE Max</th>
<th>DE Mean</th>
<th>DE Median</th>
<th>DE Min</th>
<th>DE Max</th>
</tr>
</thead>
<tbody>
<tr>
<td>PPA1</td>
<td>4.7608e-02</td>
<td>4.6198e-02</td>
<td>1.8649e-08</td>
<td>4.4125e-02</td>
<td>3.2303e-02</td>
<td>4.8432e-09</td>
<td>6.3854e-01</td>
<td></td>
</tr>
<tr>
<td>PPA2</td>
<td>3.0465e-02</td>
<td>2.9183e-02</td>
<td>3.6783e-08</td>
<td>4.2968e-02</td>
<td>3.3806e-02</td>
<td>2.6762e-08</td>
<td>5.4033e-01</td>
<td></td>
</tr>
<tr>
<td>PPA3</td>
<td>2.4299e-02</td>
<td>2.3017e-02</td>
<td>3.6365e-08</td>
<td>4.0766e-02</td>
<td>3.3085e-02</td>
<td>7.1553e-09</td>
<td>5.1258e-01</td>
<td></td>
</tr>
<tr>
<td>PPA4</td>
<td>2.3853e-02</td>
<td>2.2733e-02</td>
<td>4.1868e-08</td>
<td>4.1058e-02</td>
<td>3.3226e-02</td>
<td>2.0422e-09</td>
<td>5.0460e-01</td>
<td></td>
</tr>
<tr>
<td>PFI2</td>
<td>3.0381e-02</td>
<td>2.9069e-02</td>
<td>3.1653e-08</td>
<td>4.2931e-02</td>
<td>3.3777e-02</td>
<td>2.3655e-08</td>
<td>5.3761e-01</td>
<td></td>
</tr>
<tr>
<td>PFI3</td>
<td>2.4217e-02</td>
<td>2.2924e-02</td>
<td>1.0226e-08</td>
<td>4.0727e-02</td>
<td>3.3057e-02</td>
<td>1.2460e-08</td>
<td>5.0834e-01</td>
<td></td>
</tr>
<tr>
<td>PFI4</td>
<td>2.3853e-02</td>
<td>2.2733e-02</td>
<td>4.1868e-08</td>
<td>4.1058e-02</td>
<td>3.3226e-02</td>
<td>2.0422e-09</td>
<td>5.0460e-01</td>
<td></td>
</tr>
</tbody>
</table>
3.3. Numerical Results

Figure 3.4: **Boxplots of the Euler equation error distributions for the equilibrium prices.** Panel A shows the standard Euler equation error and Panel B displays the dynamic Euler equation error for the numerical solutions from Table 3.1. I compute the error for a random sample of aggregate shocks over \( N = 3000 \) periods. The initial cross-sectional distribution is the same for all algorithms. The red lines mark the medians, whereas, the blue boxes denote the 25\(^{th}\) to 75\(^{th}\) percentiles. The whiskers indicate the range of the distribution and the red dots outside are outliers.

Figure 3.5: **Boxplots of the law of motion error distributions of aggregate bond holdings.** This figure shows the low of motion error \( \epsilon_{\text{LoM}} \) produced by the polynomial chaos algorithms from Table 3.1. The red lines mark the medians, whereas, the blue boxes denote the 25\(^{th}\) to 75\(^{th}\) percentiles. The whiskers indicate the range of the distribution and the red dots outside are outliers.

increase of the truncation order of the polynomial chaos expansion which implies that the method generally works.
3.3.2 Failure of Approximate Aggregation

To evaluate whether approximate aggregation fails for the Huggett model, I again compute the expected stationary cross-sectional distribution $E^\mu(\mathcal{P}(z',a|\mu))$. This distribution is displayed in Figure 3.6. We can see that the largest change in the average stationary distribution occurs with the first and second-order approximations, but third and fourth-order approximations are very close. This is confirmed when zooming into the left tail, i.e., the area where mass points occur in the distribution. However, when looking at the change of the stationary distribution in Figure 3.7, the criterion suggested for the Aiyagari-Bewley model, where I suggested to take the termination criterion of the algorithm as the criterion when to truncate the polynomial chaos, seems too strict for the Huggett model. According to Figure 3.7, even a fourth-order approximation is not sufficient using this criterion. Therefore, the failure of approximate aggregation is much stronger for the Huggett economy than for the Aiyagari-Bewley model with high unemployment benefits. One of the main reasons is that the borrowing constraint has a much larger impact in the Huggett economy resulting in mass points as becomes obvious from Figure 3.6.

As the change of the stationary distribution is not too far from the termination criterion in the areas where there are fewer mass points, I still truncate at fourth order for interpreting the asset prices in the Huggett model in the following section. The reached conclusions are the same for the solution truncated at third order.

Figure 3.6: Average cross-sectional distributions produced by the polynomial chaos algorithms from Table 3.1. This graph displays the average stationary cross-sectional distribution, i.e., the expectation of the stationary distribution of the state space $(z',a,\mu)$ conditional on $\mu$. It displays histogram approximations of the distributions. Panel A compares the p.d.f.s of the proximal point algorithm using polynomial chaos truncated at different orders. Panel B zooms into the left tail of the average cross-sectional distributions. Note that only the distributions of the proximal point algorithms are displayed because they are very similar to the ones produced by policy function iteration.
3.4 Economic Insights

The pressing question in the Huggett model is how idiosyncratic risk in combination with aggregate risk affects asset prices. Due to the computational difficulty, this question has not been fully answered. Huggett (1993) investigates the effect of idiosyncratic risk on asset prices but does not look at aggregate risk. Krusell et al. (2011) consider aggregate risk in addition to idiosyncratic risk but limit themselves to a borrowing constraint at $\bar{a} = 0$ and, therefore, an autarkic economy without trade.

In the following, I compare bond prices from the model without aggregate risk, i.e., where the aggregate shock is fixed to one of its states, and the model with aggregate risk. The prices corresponding to the expected stationary cross-sectional distribution are displayed in Table 3.3. Let me start with the bond prices in the model without aggregate risk. I compute both solutions, i.e., the aggregate state being fixed to the bad and to the good state. In both cases, the mean-variance ratio of the income process is the same. However, as the average level of the income process is lower in the bad state, the average marginal propensity to consume is much higher which increases the precautionary savings motive for the agents. Therefore, the demand is higher which results in a higher price. Furthermore, tightening the borrowing constraint from $\bar{a} = -2.5$ to $\bar{a} = -0.5$ increases the price level in both cases which is in line with the existing literature. It is interesting that the prices are above one in all these cases. This implies that agents with high income are willing to pay a fee to store money for future consumption. This is intuitive as the idiosyncratic risk gives rise to the precautionary savings motive. On the opposite side, agents with low income are willing to sell the bond to increase their current consumption. However, they ask for a premium to pay for the risk of hitting the borrowing constraint in the next period which would limit their future consumption. Note that there is no
### Table 3.3: Equilibrium prices corresponding to the average stationary cross-sectional distributions for various levels of the borrowing constraint.

This table displays the prices corresponding to the average stationary cross-sectional distribution, i.e., the expectation of the stationary distribution of the state space \((z', a, \mu)\) conditional on \(\mu\). It compares the model without aggregate risk which is either fixed to the bad or good state, respectively, and the model with aggregate risk where the prices are conditional on the aggregate state.

<table>
<thead>
<tr>
<th>Borrowing Constraint (\bar{a})</th>
<th>No Aggregate Risk</th>
<th>Aggregate Risk</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>bad</td>
<td>good</td>
</tr>
<tr>
<td>-0.5</td>
<td>1.0440</td>
<td>1.0161</td>
</tr>
<tr>
<td>-1.0</td>
<td>1.0247</td>
<td>1.0111</td>
</tr>
<tr>
<td>-1.5</td>
<td>1.0150</td>
<td>1.0080</td>
</tr>
<tr>
<td>-2.0</td>
<td>1.0099</td>
<td>1.0057</td>
</tr>
<tr>
<td>-2.5</td>
<td>1.0063</td>
<td>1.0039</td>
</tr>
</tbody>
</table>

The question is what happens to the price level when aggregate risk is added. It is natural to assume that the expected stationary price lies in between the two equilibrium price levels of the model without aggregate risk. However, this is not confirmed by the numerical solutions. To show this, I compute the stationary state distribution. As before, this distribution is a distribution of distributions. I compute the expected conditional state distribution given that the economy is in a particular aggregate state \(E_\mu(P(z', a, \mu|z_{ag}'))\) and its corresponding equilibrium price. When we consider the stationary distribution given that the economy is in a bad state, the corresponding equilibrium price is lower than in the model without aggregate fixed to either state. The corresponding equilibrium price conditional on the economy being in the good state, on the contrary, is much higher than in the model without aggregate risk. Hence, the price levels conditional on the aggregate shock are reversed compared to the model without aggregate risk. Furthermore, the average of both price levels in the model with aggregate risk is higher than in the model without aggregate risk. These are interesting observations which seem puzzling at first. I explain the reasons for this behavior in the following.

The increased average price level is explained when we consider the income process of the model with aggregate risk. It has a lower mean-variance ratio than in the model without aggregate risk. Even though the income level in the model with aggregate risk is the average of the income levels of the model without aggregate risk, the variance shifts nonlinearly. It is higher than the average of the variances of the model without aggregate risk. This leads to an on average stronger precautionary savings motive in the model with aggregate risk. Hence, there is more demand for the bond on average which results in the higher average equilibrium price level. However, this does not explain the different price levels when conditioning on the aggregate shock. When conditioning on the aggregate shock, the borrowing constraint strongly amplifies the impact of aggregate risk. Given that the economy is in a bad state, the impact of the borrowing constraint on agents with low endowment is less severe because their ability to pay back debt in the next period is increased due to the higher average next-period income. Therefore, these agents have
a higher ability to borrow which increases the supply of the bond and lowers the price compared to the model without aggregate risk. This effect is so strong that the price is even lower than in the model without aggregate risk fixed at the good economic state. The mechanism is mirrored for the equilibrium price conditional on the economy being in the good state. In that case, the borrowing constraint has a much stronger grip on the agents with low income as their ability to pay back debt is decreased due to the lower average next-period income. This leads to a decrease in supply and an increase in the price. The effect is so strong that the agents who want to buy the bond are willing to pay a significant fee for storing their wealth. The disparity between the prices in the two aggregate states illustrates the forces at play. In the good state, the demand for storing wealth from agents with high income is much stronger than for consuming future income when agents have low endowment. Thus, the bond buyers pay a fee to invest to give an incentive to other agents to sell. On the contrary, the supply in the bad state is much stronger than the demand which results in a price lower than one. The agents with low endowment who want to consume future wealth today are willing to pay an interest rate to incentivize other agents to buy the bond. Overall, this effect is another example of the amplification of aggregate risk through ex-post heterogeneity. In fact, it is a much stronger showcase of this effect than the Aiyagari-Bewley model with high risk-sharing in Chapter 2.

Tightening the borrowing constraint in the model with aggregate risk increases the average price level unconditional and conditional on the aggregate state of the economy. For further insights on prices, I consider the stationary distribution of equilibrium prices. It can be approximated by a histogram of the equilibrium prices over the simulation of the Huggett economy which is also used to compute the Euler equation errors. Figure 3.8 displays

![Figure 3.8: Stationary price distribution for different borrowing constraints.](image)

This graph shows an approximation of the stationary state distribution of equilibrium prices. It displays a histogram of the equilibrium prices from a simulation of the Huggett economy for different borrowing constraints over \( N = 3000 \) periods.

the approximated stationary bond price distributions for the Huggett economy with various borrowing constraints. In fact, the equilibrium price distributions conditional on the aggregate shock are disjoint in this calibration of the model. It is interesting to see what
happens to this distribution when the borrowing constraint is tightened. The price level goes up in general. However, the price volatility conditional on the aggregate shock drops considerably as the conditional price distribution becomes tighter which is confirmed by Table 3.4. Furthermore, the conditional price skewness decreases with tightening borrowing constraints. It is negative conditional on the good state for constraints tighter than $\bar{a} = -2$ and positive, otherwise. Kurtosis decreases with tightening borrowing constraints which is mainly due to the decreased volume. Tightening the borrowing constraint in the Huggett economy means that the agents have a lower capacity to share idiosyncratic risk because agents with low endowment who want to consume future endowments by selling the bond are more constrained in doing so. Therefore, the volatility paradox holds for the Huggett economy as well. More risk sharing in form of laxer borrowing constraints leads to higher price volatility. The results of the amplification of aggregate risk and the volatility paradox are robust to parameter changes which is shown in Appendix C.2.

<table>
<thead>
<tr>
<th>Borrowing Constraint $\bar{a}$</th>
<th>-0.5</th>
<th>-1.0</th>
<th>-1.5</th>
<th>-2.0</th>
<th>-2.5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean</td>
<td>0.9779</td>
<td>0.9759</td>
<td>0.9668</td>
<td>0.9575</td>
<td>0.9499</td>
</tr>
<tr>
<td>St. dev.</td>
<td>0.0054</td>
<td>0.0114</td>
<td>0.0120</td>
<td>0.0128</td>
<td>0.0130</td>
</tr>
<tr>
<td>Skewness</td>
<td>0.0095</td>
<td>0.0448</td>
<td>0.0617</td>
<td>0.1464</td>
<td>0.3242</td>
</tr>
<tr>
<td>Kurtosis</td>
<td>1.2320</td>
<td>1.6381</td>
<td>2.0371</td>
<td>2.2008</td>
<td>2.2150</td>
</tr>
</tbody>
</table>

**Panel B**

| Mean | 1.1911 | 1.1761 | 1.1527 | 1.1364 | 1.1238 |
| St. dev. | 0.0064 | 0.0160 | 0.0160 | 0.0161 | 0.0150 |
| Skewness | -0.4347 | -0.3711 | -0.2290 | -0.0529 | 0.1745 |
| Kurtosis | 1.4485 | 1.8941 | 2.2509 | 2.1279 | 2.1216 |

Table 3.4: Simulated stationary distribution of equilibrium prices. This graph displays summary statistics of the simulated stationary distribution of equilibrium prices of the Huggett economy with aggregate risk for different borrowing constraints over a simulation of $N = 3000$ periods. Panel A presents the statistics conditional on the bad state, whereas, Panel B displays them conditional on being in the good state.
Dynamic stochastic general equilibrium models with aggregate and idiosyncratic risk pose numerous theoretical and practical challenges. The main reason for this is that the cross-sectional distribution of agent-specific variables becomes an infinite-dimensional element of the state space. This thesis makes several contributions to the literature on heterogeneous agent models. First, I establish existence of recursive equilibria for production economies with a continuum of agents with unbounded utility facing idiosyncratic shocks in combination with aggregate risk. Instead of relying on compactness arguments to establish the fixed point, I use the monotonicity of the equilibrium problem and arguments from convex analysis. This makes it possible to accommodate unbounded utility. A great advantage of this approach is that one can easily examine whether the equilibrium is unique. This is very difficult otherwise when using fixed point theory based on compactness. I apply the methodology based on monotonicity to establish existence and uniqueness of the recursive equilibrium for the Aiyagari-Bewley economy with aggregate risk.

Second, I propose a novel computational method to solve these equilibrium models. There are two elements to my numerical method which set this algorithm apart from existing methods. First, an iterative procedure to solve for the recursive equilibrium emerges quite naturally from the theory on existence. This procedure is ensured to converge for production economies. Second, rather than approximating the law of motion of aggregate variables with a more or less parametric formula, I directly approximate the cross-sectional distribution of individual variables. I use a projection technique which extends orthogonal polynomial projection from spaces of smooth functions to the space of square-integrable random variables. This technique is known as generalized polynomial chaos and can be interpreted as a probabilistic polynomial projection method. I derive approximation error bounds which imply that the error in the equilibrium solution decreases geometrically when the order of the polynomials is increased.

Lastly, I put this theoretically founded algorithm to the test. I compute solutions to the Aiyagari-Bewley economy with aggregate risk and unemployment insurance, and, with an extension of the algorithm, the Huggett economy with aggregate risk. Even though these models are standard heterogeneous agent models in growth theory and asset pricing, my computed solutions yield interesting new economic implications. First, I show that approximate aggregation in the growth model which is widely assumed to hold does, in fact, fail for sufficiently high but not unrealistic levels of unemployment insurance. Second, I find that risk sharing of idiosyncratic risk among agents amplifies aggregate risk. In the growth model, high levels of unemployment insurance lead to fatter tails in the expected stationary capital distribution. Even though the magnitude of this effect is small, it supports the failure of approximate aggregation. The amplification effect is much stronger for any level of risk sharing in the form of bond trading in the Huggett model with aggregate risk. The equilibrium prices differ significantly from the prices in models where aggregate risk is absent. They are lower in recessions and higher in booms.

My
third and last economic result emerges by comparing low and high levels of idiosyncratic risk sharing. I find that the expected stationary distribution of individual capital in the Aiyagari-Bewley model has fatter tails, whereas, the stationary distribution of equilibrium bond prices in the Huggett model has larger volatility the higher the level of risk sharing. This means that systemic risk in the equilibrium problem increases with risk sharing, a result which recovers for these standard models what was coined the volatility paradox in Brunnermeier and Sannikov (2014).
Appendices
Appendix A

Appendix of Chapter 1

A.1 Proofs

A.1.1 Proof of Proposition 8

Proof. It is well known that the subspace of continuous functions with bounded variation within $L^2$ is complete and hence, a Hilbert space itself. With condition (i), we take yet another subset of functions. It is easy to see that any limiting element $h^\ast$ of a Cauchy sequence $h^n \in H_c, n \in \{1, 2, \ldots\}$, satisfies conditions (i) as well. The subspace $H_c$ is, therefore, complete and a Hilbert space itself.

A.1.2 Proof of Lemma 10

Proof. I compute the Gâteaux derivative of the Euler equation error to show monotonicity of $T_i$. Let me first rewrite $T_i$ in a simplified form

$$T_i[h_x; \tilde{h}] = h_c\left( z', \chi\left( z'^{id'} \right) \right)^{-\gamma}$$

$$- \mathbb{E}(z'^{id'}; z') \left[ \beta \left( 1 + f_i \left( z'', \mathbb{E}\left[ \chi'\left( z'^{id'} \right) \right] \right) \right) h_c\left( z'', \chi'(z'^{id'}) \right)^{-\gamma} \right],$$

where $i \in \{1, \ldots, n\}$, $h_c$ is given by the budget constraint and

$$\chi'(z'^{id'}) = \int_{\mathbb{Z}^{id}} h_x\left( z', \chi\left( z'^{id'} \right) \right) \mathbb{P}\left( z'^{id'} \mid dz'^{id}, z^{ag'}, z^{ag''} \right).$$

Its Gâteaux derivative is given by

$$\delta T_i[h_x; \tilde{h}] = \gamma h_c\left( z', \chi\left( z'^{id'} \right) \right)^{-\gamma-1} \sum_j \tilde{h}_j$$

$$+ \mathbb{E}(z'^{id'}; z') \left[ \beta \left( 1 + f_i \left( z'', \mathbb{E}\left[ \chi'\left( z'^{id'} \right) \right] \right) \right) \gamma h_c\left( z'', \chi'(z'^{id'}) \right)^{-\gamma-1} \delta h_c[h_x; \tilde{h}] \right]$$

$$- \mathbb{E}(z'^{id'}; z') \left[ \beta \delta \left( f_i \left( z'', \mathbb{E}\left[ \chi'\left( z'^{id'} \right) \right] \right); \tilde{h} \right) h_c\left( z'', \chi'(z'^{id'}) \right)^{-\gamma} \right]$$

for $i \in \{1, \ldots, n\}$. The first two terms are non-negative $P$-a.s. for any $h_x, \tilde{h} \in H_c$ due to the conditions (i)-(ii) in Proposition 8. Since Assumption 9 holds, we obtain monotonicity as $\langle \delta T[h_x; \tilde{h}], \tilde{h} \rangle \geq 0$. Furthermore, because $T$ is continuous in $h_x$, we can apply (Corollary 20.28, Bauschke and Combettes, 2017) and obtain maximal monotonicity. \qed
A.1.3 Proof of Lemma 11

Before I start the proof, let me state some preliminaries. I need to show that the Lagrangian in (1.9) is a saddle function. Let me first define what a saddle function is in this context.

Definition 25 (Saddle function (see Rockafellar, 1970)). (i) Let C and D be Hilbert spaces over \( \mathbb{R} \). A saddle-function is an everywhere-defined function \( L : C \times D \to [-\infty, \infty] \) such that \( L(c, d) \) is a convex function of \( c \in C \) for any \( d \in D \) and a concave function of \( d \in D \) for any \( c \in C \).

(ii) A saddle function is called proper if there exists a point \((c, d) \in C \times D \) with \( L(c, \tilde{d}) < +\infty \) for any \( \tilde{d} \in D \) and \( L(\tilde{c}, d) > -\infty \) for any \( \tilde{c} \in C \).

(iii) The operator associated with the saddle function \( L \) is defined as the set-valued mapping  
\[
T_L(c, d) = \{(v, w) | L(\tilde{c}, d) - \langle \tilde{c}, v \rangle + \langle d, w \rangle 
\geq L(c, d) - \langle c, v \rangle + \langle d, w \rangle 
\geq L(c, \tilde{d}) - \langle c, v \rangle + \langle \tilde{d}, w \rangle \forall (\tilde{c}, \tilde{d}) \in C \times D \},
\]

where \( \langle ., . \rangle \) denotes the Hilbert space inner product. A saddle point is a point \((c^*, d^*) \in C \times D \) such that \( 0 \in T_L(c^*, d^*) \).

Note that if our Lagrangian satisfies all properties of a saddle function, then the first-order conditions coincide with the operator \( T_L \). This operator can be further characterized by the following Corollary.

Corollary 26 (Rockafellar (1970)). Let \( C \) and \( D \) be Hilbert spaces over \( \mathbb{R} \). If \( L(c, d) \) is a proper saddle function on \( C \times D \), which is lower semicontinuous in its convex element \( c \in C \) and upper semicontinuous in its concave element \( d \in D \), then the operator \( T_L \) associated with \( L \) is maximal monotone.

Proof of Proposition 11. According to (Lemma 5.1 Ghoussoub, 2008), the Lagrangian \( L_T \) is convex and lower semicontinuous in \( h \in \mathcal{H} \). It follows that the Lagrangian of the constrained problem \( L \) is convex and lower semicontinuous in \( h \in \mathcal{H} \) and concave and upper semicontinuous in \( y \). The Lagrangian is also a proper function because \( L(h, y) > -\infty \) for any \( h \in \mathcal{H} \) when \( y = 1_{\{h<\varepsilon\}} \). Conversely, a security holding \( h \) can be constructed such that \( L(h, y) < \infty \) for all \( y \in C(L_0^2) \). We simply need \( \mathbf{T}[h] < \infty \). This is achieved by setting consumption to a positive number \( h_c = A < \varepsilon \) which is less than the endowment \( \varepsilon \). Choosing the distribution \( \chi \) such that the rate of returns are finite \( f < \infty \), e.g., setting \( \chi = \varepsilon > 0 \), leads to \( h = \frac{1}{n}(e + \sum_{j=1}^n(1 + f^j)e - A) \). Thus, Corollary 26 holds such that the first-order conditions summarized by \( T_L \) define a maximal monotone operator. The operator \( \mathbf{M} \) is the subgradient of \( L(., y) \) at a Lagrange multiplier \( y \in C(L_0^2) \) and, therefore, equals \( v \) in the definition of \( T_L \). Hence, maximal monotonicity of \( \mathbf{M} \) follows trivially. \( \square \)

---

\( ^1 \) The operator \( T_L \) is closely related to the subdifferential of the saddle function \( L \) as \( v \) equals the subgradient of \( L(., d) \) at \( c \in C \) and \( w \) is the subgradient of \( -L(c, .) \) at \( d \in D \).
A.1.4 Proof of Lemma 12

\textit{Proof.} Given that \( h_x \in \mathcal{H}_t \) solves the Euler equation (1.7), we obtain a sequence of security holdings and consumption which by construction satisfy the first-order condition of the individual optimization problem and the equilibrium condition by

\[
x_{t+1}^* = h_x \left( z_{t+1}, x_t^*, \chi_{t+1} \left( z^{id} \right) \right)
\]

\[
e_{t+1}^* = e(z_{t+1}) + \sum_{j=1}^{n} \left\{ (1 + f_j \left( z_{t+1}, E \left[ \chi_{t+1} \left( z^{id} \right) \right] \right)) x_t^j - x_{t+1}^j \right\}
\]

where \( \chi_{t+1}(z^{id}) \) as in (1.5). Suppose that there is another arbitrary feasible series of security holdings and consumption in \( L^2_p \) such that \( x_t \geq \bar{x}, t \geq 0 \), and \( c_t > 0, t \geq 0 \), satisfying the budget constraint. Then, we get

\[
E \left[ \sum_{t=0}^{T} \beta^t u(c_t^*) - u(c_t) \right] \\
\geq E \left[ \sum_{t=1}^{T} \beta^t \left\langle \delta u(c_t^*; (x_{t-1}^* - x_{t-1})), (x_{t-1}^* - x_{t-1}) \right\rangle \right] \\
+ E \left[ \sum_{t=0}^{T-1} \beta^t \left\langle \delta u(c_t^*; (x_t^* - x_t)), (x_t^* - x_t) \right\rangle \right] \\
= E \left[ \sum_{t=0}^{T-1} \beta^t \left\langle \delta \left( u(c_t^*) + \beta u(c_{t+1}^*; (x_{t+1}^* - x_t)), (x_t^* - x_t) \right) \right\rangle \right] \\
+ \beta^T E \left[ \left\langle \delta u(c_T^*; (x_T^* - x_T)), (x_T^* - x_T) \right\rangle \right],
\]

where the first term equals zero because the first-order condition holds \( P \)-a.e. For the same reason, the second term yields

\[
\beta^T E \left[ \left\langle \delta u(c_T^*; (x_T^* - x_T)), (x_T^* - x_T) \right\rangle \right] \\
= - \beta^{T+1} E \left[ \left\langle \delta u(c_{T+1}^*; (x_{T+1}^* - x_T)), (x_{T+1}^* - x_T) \right\rangle \right] \\
= + \beta^{T+1} E \left[ \left\langle (c_{T+1}^*)^{-\gamma} (x_{T+1}^* - x_T), (x_{T+1}^* - x_T) \right\rangle \right] \geq 0.
\]

As \( c_{T+1} \) is feasible, the marginal utility is positive. The Gâteaux derivative is negative due to the budget constraint. Hence, \( c_t^* \) is optimal. This concludes the proof. \( \square \)

A.1.5 Proof of Lemma 13

\textit{Proof.} It is well known from convex analysis that the solution to a strictly convex optimization problem is unique (see e.g. Bauschke and Combettes, 2017). The Lagrangian in (1.9) is indeed strictly convex in \( h \) as \( \left\langle \delta T[h; \hat{h}], \hat{h} \right\rangle > 0 \). This is obvious from the Gâteaux derivative in (A.1) as the first term is strictly positive due to the fact that consumption is bounded from above by \( e(z') + \sum_{j=1}^{n} \{(1 + f_j(z', E[\chi])) - \bar{x}\} < \infty \) for the constrained problem. \( \square \)

A.1.6 Proof of Proposition 14

\textit{Proof.} To check that Assumption 9 holds, I show that

\[
\sum_{j=1}^{2} \left\langle E^{(z'|z)} \left[ \delta f_j \left( z''; E[h]; \hat{h} \right) \right], \hat{h} \right\rangle \leq 0.
\]
Note that any direction $\tilde{h}$ has to equal zero in its second component which corresponds to the labor supply since this is not a choice variable. It is exogenously fixed at one. Therefore,

$$\left\langle \mathbb{E}(z' z') \left[ \delta f^1 \left( z'' , \mathbb{E} [ h ] , \tilde{h} \right) \right] , \tilde{h} \right\rangle \leq 0.$$

It is easy to see that the Gâteaux derivative of the return on capital at any $z''$ is negative

$$\left\langle \delta f^1 \left( z'' , \mathbb{E} [ h ] , \tilde{h} \right) , \tilde{h} \right\rangle = -(1 - \alpha) a \left( 1 + z'' a - (1 - z'' a) \right) \left( \pi p'' \right)^{1 - \alpha} (K'')^{\alpha - 2} \mathbb{E} [ \tilde{h} ]^2 \leq 0$$

because the inner product computes the expectation with respect to individual capital which equals the aggregation over capital holdings denoted by the aggregation operator $\mathbb{E}$. This concludes the proof. \hfill \Box

A.1.7 Proof of Theorem 16

\textbf{Proof of Theorem 16.} I construct two savings policies $h$ at which the left-hand side of the Euler equation is positive and negative, respectively. The idea is to use the two polar strategies save everything/consume nothing and save nothing/consume everything. From these two strategies, one can then in a last step construct a set of policy functions such that the convex hull of its image in the Euler equation operator contains zero.

Let me first define the candidate policy

$$h (z', k, \mu) = (1 - \epsilon) \left( I (z', k, K) + (1 - \delta) k \right),$$

where productive income $I$ is as in (1.11). This implies that aggregate capital and current and next-period consumption are given by

$$K' = (1 - \epsilon) \left( \mathbb{E}^n [ I (z', k, K) ] + (1 - \delta) K \right)$$

$$c (z', k, \mu) = \epsilon \left( I (z', k, K) + (1 - \delta) k \right)$$

$$c' (z', k, \mu) = \epsilon \left[ (1 - \epsilon)^a I (z'' , h|_{e=0} , K'|_{e=0}) + (1 - \epsilon)(1 - \delta) h|_{e=0} \right].$$

Note that, due to the definition of $c'$, its first variation equals $(1 - \delta + R (z'', K'))$. Hence, the first-order condition is given by

$$\frac{\partial}{\partial c} u (c) - \beta \sum_{z'' \in Z} p z'' | i' (1 - \delta + R (z'', K')) \frac{\partial}{\partial c'} u (c').$$

\textbf{Positive FOC:} This outcome is equivalent to

$$1 > \beta \sum_{z'' \in Z} p z'' | i' (1 - \delta + R (z'', K')) \left( \frac{c}{c'} \right)^\gamma,$$

where

$$\frac{c}{c'} = \frac{I (z', k, K) + (1 - \delta) k}{(1 - \epsilon)^a I (z'' , h|_{e=0} , K'|_{e=0}) + (1 - \epsilon)(1 - \delta) h|_{e=0}}.$$

I let $\epsilon > 0$ go to zero which is equivalent to the save everything/consume nothing strategy. Clearly, this strategy is admissible $h \in \mathbb{H}^*_c$ with $c^* = \min c (z', k, \mu) > 0$ getting closer and closer to zero. Also, it is easy to see that $c/c'$ is an increasing function in individual capital in this case. I can compute its limit by applying l'Hôpital’s rule

$$\lim_{k \to \infty} \frac{c}{c'} = \frac{1}{1 - \delta + R (z'', K'|_{e=0})}.$$
If $\gamma = 1$, the right side of (A.2) equals $\beta < 1$ which results in the positive value of the first-order condition. When $\gamma > 1$, the right hand side is an increasing function of $K'$. It goes to zero when $K' \to 0$ and to $\beta(1-\delta)^{1-\gamma}$ when $K' \to \infty$. This also results in a positive value of the first-order condition by assumption. Hence, the trick is to choose $\epsilon^* > 0$ of the admissible set small enough.

**Negative FOC:** This outcome is equivalent to

$$1 < \beta \sum_{z'' \in \mathcal{Z}} p^{z''} (1 - \delta + R(z'', K')) \left( \frac{c}{c'} \right)^{\gamma}.$$  

I now let $\epsilon \to 1$ which corresponds to the save nothing/consume everything strategy. It is obvious that $c/c' \to +\infty$ in this case which gives us a negative value of the first-order condition. It remains to check that this strategy is admissible, i.e. $h \in \mathcal{H}_{\ast \ast}$. Condition (i) of Proposition 8 is trivial, whereas, condition (ii) requires more care. Clearly, the left side of condition (ii) equals zero. Let me now analyze the first Gâteaux of income at $\kappa = (1 - \epsilon)[I(z', k, K) + (1-\delta)k]$ w.r.o.g. in the direction of $\tilde{\kappa}^\epsilon = (1 - \epsilon)\tilde{\kappa}$

$$\delta I(z', \kappa; \tilde{\kappa}^\epsilon) = \frac{\partial}{\partial K'} I(z', \kappa) \cdot \langle \tilde{\kappa}^\epsilon, 1 \rangle + R(z', \kappa) \tilde{\kappa}^\epsilon = (1 - \epsilon)^{\alpha} \delta I(z', \kappa|_{\epsilon = 0}; \tilde{\kappa}) ,$$

i.e., I can pull out $\epsilon$ from the derivative. It follows that the first derivative of income converges to zero for $\epsilon \to 1$. This shows that $h \in \mathcal{H}_{\ast \ast}$.

**Convex Hull:** The last step consists of constructing a set of functions such that the convex hull of the image of that set contains zero. To do so, I define the savings policy by

$$h(z', k, \mu) = (1 - \epsilon(z', k, \mu))(I(z', k, K) + (1-\delta)k),$$

where $\epsilon(z', k, \mu)$ is a continuous piecewise linear tent function which equals 1 everywhere except on $z'_\text{fix} \times (k^* - \Delta, k^* + \Delta) \times (K^* - \Delta, K^* + \Delta)$ and $\epsilon(z'_\text{fix}, k^*, K^*) = 0$. Now, define a grid of $k^*_n = n\Delta$ and $K^*_m = m\Delta$ such that we obtain a set of tent functions $\epsilon_{nm}$. The question is which value the Euler equation operator has for such a tent function strategy. Similar to the analysis above, I compute the ratio of current-to-future consumption

$$\frac{c}{c'} = \frac{\epsilon_{nm}I(z', k, K) + (1-\delta)k}{\epsilon_{nm}I(z', h, K')|_{\epsilon = 0} + (1 - \epsilon_{nm})(1 - \delta)h|_{\epsilon = 0}},$$

where $\epsilon_{nm} = \epsilon_{nm}(z', k, K)$ and $\epsilon'_{nm} = \epsilon_{nm}(z', h, K')$. Hence, whether $T$ is positive or negative for a particular triplet $(z', k, K)$ depends on the values of $\epsilon_{nm}$ and $\epsilon'_{nm}$. Using a limit analysis as above, I distinguish four cases.

- $\epsilon_{nm} = 1, \epsilon'_{nm} \geq 0$: $\frac{c}{c'} \to \infty \Rightarrow T < 0$
- $\epsilon_{nm} = 0, \epsilon'_{nm} \geq 0$: $\frac{c}{c'} = 0$ or $\lim_{k \to \infty} \frac{c}{c'} = \frac{1}{1 - \delta + R(z', K'|_{\epsilon = 0})} \Rightarrow T > 0$
- $0 < \epsilon_{nm} < 1, \epsilon'_{nm} = 0$: $\frac{c}{c'} \to \infty \Rightarrow T < 0$
- $0 < \epsilon_{nm} < 1, \epsilon'_{nm} > 0$: $T$ can be positive or negative

Hence, the Euler equation operator evaluated at the tent function strategy has a positive value at $(z', k^*_n, K^*_m)$ and in the close vicinity of that point. It is negative elsewhere. Thus, the Euler equation operator evaluated at the tent function strategy is a tent function itself.
This implies that one can find a convex combination of $T_{nm}$ and the Euler equation operator $T < 0$ at the save nothing/consume everything strategy which equals zero at $(z', k^*_n, K^*_m)$. Therefore, the convex hull of the image of the set consisting of the two polar strategies and the set of tent function strategies with the multipliers $\epsilon_{nm}$ contains zero when the mesh size $\Delta \to 0$. Applying Corollary 7 ensures existence. Applying Lemma 13 yields uniqueness and concludes the proof.
Appendix B

Appendix of Chapter 2

B.1 The Model’s Parameter Values

I base my analysis for the growth model on the model and its parameters used in den Haan et al. (2010). They take the basic model from Krusell and Smith (1998) and add unemployment benefits to it. The probabilities of the exogenous shocks are set to roughly match the postwar U.S. time series of macroeconomic aggregates. This means that the average duration of the good and bad economic state, i.e., the business cycle is eight quarters, respectively and that unemployment lasts on average 1.5 quarters in good times and 2.5 quarters in bad times. This is achieved by defining the aggregate shock as an uniformly distributed Markov chain such that the good and bad economic state both occur with probability 0.5. The exogenous idiosyncratic shock is defined by an employment probability of $p^e = 0.9$ in the recession state and a probability of $p^e = 0.96$ in the boom state. The full transition matrix of the exogenous shocks is given in Table B.1.

<table>
<thead>
<tr>
<th></th>
<th>$z' = (0, 0)$</th>
<th>$z' = (0, 1)$</th>
<th>$z' = (1, 0)$</th>
<th>$z' = (1, 1)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$z = (0, 0)$</td>
<td>0.525</td>
<td>0.35</td>
<td>0.03125</td>
<td>0.09375</td>
</tr>
<tr>
<td>$z = (0, 1)$</td>
<td>0.038889</td>
<td>0.836111</td>
<td>0.002083</td>
<td>0.122917</td>
</tr>
<tr>
<td>$z = (1, 0)$</td>
<td>0.09375</td>
<td>0.03125</td>
<td>0.291667</td>
<td>0.583333</td>
</tr>
<tr>
<td>$z = (1, 1)$</td>
<td>0.009115</td>
<td>0.115885</td>
<td>0.024306</td>
<td>0.850694</td>
</tr>
</tbody>
</table>

Table B.1: Transition matrix of the exogenous shocks for the Aiyagari-Bewley Model. This table displays the transition matrix which is taken from the calibration in den Haan et al. (2010).

The remaining parameters which are used for the computations are given in Table B.2. The discount factor $\beta$, the risk aversion $\gamma$ and the elasticity $\alpha$ are set to standard values as in Krusell and Smith (1998). The depreciation rate $\delta$ reflects a period of one quarter. The change in productivity $a$ is set such that productivity in a boom is two percent above the value of productivity in a recession. The time endowment factor $\pi$ is chosen to normalize labor supply to one in the bad economic state. den Haan et al. (2010) fix the unemployment benefit $\nu$ at 15%.
Table B.2: **Parameter Values for the Aiyagari-Bewley Model.** This table displays the parameter values which are taken from the calibration in den Haan et al. (2010).

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\beta$</td>
<td>0.99</td>
</tr>
<tr>
<td>$\gamma$</td>
<td>1</td>
</tr>
<tr>
<td>$\delta$</td>
<td>0.025</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>0.36</td>
</tr>
<tr>
<td>$a$</td>
<td>0.01</td>
</tr>
<tr>
<td>$\pi$</td>
<td>$\frac{1}{2}$</td>
</tr>
<tr>
<td>$\nu$</td>
<td>0.15</td>
</tr>
</tbody>
</table>

**B.2 Proofs**

**B.2.1 Proof of Theorem 20**

Assume w.l.o.g. that $T$ is maximal monotone. If it is not maximal, work with its maximal monotone extension $\bar{T}$ instead. This property is very useful because the resolvent of a maximal monotone operator is firmly nonexpansive. This fact is due to Minty (1962). It is well known that any firmly nonexpansive operator is equivalent to a mixture $(1/2)\text{Id} + (1/2)\mathbf{R}$ of the identity operator $\text{Id}$ and a nonexpansive operator $\mathbf{R}$ (see e.g., Bauschke and Combettes, 2017, Remark 4.34 (iii)). Weak convergence of the iteration of such a mixture to its fixed point is well established (see e.g., Zeidler, 1986a, Proposition 10.16). This procedure is also known as damped fixed-point iteration. Hence, iterating on the resolvent of a maximal monotone operator yields the proximal point algorithm.

In Rockafellar (1976a), this convergence argument is made precise for Lagrange problems like the one I consider. The author furthermore shows that the proximal point algorithm converges to an optimum of the Lagrangian even if the update of the optimal policy is only approximate. Salzo and Villa (2012) extend this result to different concepts of approximation. Let me define which kind of approximation applies in this work.

**Definition 27** (Resolvent approximation). Let $\mathcal{C}$ be a Hilbert space over $\mathbb{R}$. Consider the resolvent $(\text{Id} + \lambda T_L)^{-1}(c)$ of an operator $\lambda T_L$ associated with a saddle function $L$ at $c \in \mathcal{C}$ with $\lambda > 0$. The approximation with $\epsilon$-precision of this resolvent at $c \in \mathcal{C}$ is defined as $\tilde{c} \in (\text{Id} + \lambda T_L^{\epsilon^2/(2\lambda)})^{-1}(c)$ where

$$T_L^{\epsilon^2/(2\lambda)}(c) = \left\{ v \left| L(c) - L(\tilde{c}) + \langle \tilde{c} - c, v \rangle \leq \frac{\epsilon^2}{2\lambda} \forall \tilde{c} \in \mathcal{C} \right. \right\}.$$ 

It is denoted by $\tilde{c} \approx (\text{Id} + \lambda T_L)^{-1}(c)$.

**Proof of Theorem 20.** Generally, convergence of the proximal point algorithm is well established as explained above. It remains to show that our specific approximate policy

---

1 Resolvent (see e.g., Bauschke and Combettes, 2017): Let $\mathcal{E}$ be a Hilbert space. The resolvent of an operator $T : \mathcal{E} \rightarrow \mathcal{E}$ is the operator $(\text{Id} + T)^{-1}$ where $\text{Id}$ is the identity operator.

2 Nonexpansiveness (see e.g., Bauschke and Combettes, 2017): Let $\mathcal{E}$ be a Hilbert space. An operator $T : \mathcal{E} \rightarrow \mathcal{E}$ is called nonexpansive if it is Lipschitz continuous with constant 1. It is called firmly nonexpansive if for all $e, \bar{e} \in \mathcal{E}$ it holds that $\|T(e) - T(\bar{e})\| \leq \langle e - \bar{e}, T(e) - T(\bar{e}) \rangle$.

3 This definition corresponds to the type 2 approximation with $\epsilon$-precision in Salzo and Villa (2012). Note that the approximation operator is an approximate subdifferential operator. This is the case because I minimize the controls for fixed Lagrange multipliers rather than computing a minimax problem immediately in Algorithm 2.1.
update satisfies Definition 27. As \( \max_{h \in \mathcal{H}}(h^{n+1} - h) = h^{n+1} \leq X(z', k, K') \), Equation (2.12) implies that

\[
\langle h^{n+1} - h, \nabla \tilde{L}(h^{n+1}, y^n; z', k, h^n) - v \rangle \leq \frac{\epsilon^2}{2\lambda}, \quad \forall h \in \mathcal{H},
\]

where

\[
\tilde{L}(h^{n+1}, y^n; z', k, h^n) = L^A(h^{n+1}, y^n; z', k, h^n) - \frac{1}{2\lambda} \| h - h^n \|_2^2
\]

\[
v = \frac{1}{\lambda}(h^n - h^{n+1})
\]

with \( L^A \) as in (2.11). Adding a zero and applying the definition of the gradient (or more generally, an element of the subdifferential) then implies

\[
\begin{align*}
&\left[ \tilde{L}(h^{n+1}, y^n; z', k, h^n) - \tilde{L}(h, y^n; z', k, h^n) \right] \\
&- \left[ \tilde{L}(h^{n+1}, y^n; z', k, h^n) - \tilde{L}(h, y^n; z', k, h^n) \right] \\
&+ \langle h - h^{n+1}, v - \nabla \tilde{L}(h^{n+1}, y^n; z', k, h^n) \rangle \leq \frac{\epsilon^2}{2\lambda}
\end{align*}
\]

\[
\Rightarrow \begin{cases}
\tilde{L}(h^{n+1}, y^n; z', k, h^n) - \tilde{L}(h, y^n; z', k, h^n) + \langle h - h^{n+1}, v \rangle \leq \frac{\epsilon^2}{2\lambda} \\
\tilde{L}(h^{n+1}, y^n; z', k, h^n) - \tilde{L}(h, y^n; z', k, h^n) \\
+ \langle h - h^{n+1}, \nabla \tilde{L}(h^{n+1}, y^n; z', k, h^n) \rangle \leq 0
\end{cases}
\]

for all \( h \in \mathcal{H} \). Note that \( \tilde{L}(., y^n; z', k, h^n) = L(., y^{n+1}; z', k) \) with \( L \) as in (2.10). Hence, we have that \( v \in T^{x/2(2\lambda)}_L(h^{n+1}) \), which concludes the proof as \( h^{n+1} \in \left( \text{Id} + \lambda T^{x/2(2\lambda)}_L \right)^{-1}(h^n) \).

---

### B.2.2 Proof of Proposition 21

**Proof.** Let us denote the support of the marginal distribution w.r.t. \( k \) of the cross-sectional distribution by \( \text{supp} \mu^k \). The minimum value of \( k \), which has positive probability, is denoted by \( k = \min_k \text{supp} \mu^k \). First, let us show that the constraint has positive probability \( \delta \in \text{supp} \mu^k \). Because of \( p_{z, z'} > 0 \), eventually we have \( z^{ag} \) in the previous and the current period. Suppose that the start capital, at which the constraint starts binding, is not in the support \( k^* < k \leq k^* \). Applying the optimal capital savings function, I obtain that \( k' = g_k(\tilde{z}, k, \mu) \leq k^* \). By induction, this contradicts \( k^* \notin \text{supp} \mu^k \). Now let us show that there is a mass point at \( \delta \). Assume that \( \delta < k^* = k^* \). Because \( g_k \) is continuous and strictly increasing to the right of its kink, there exists an interval \([k^*, k^*]\) with \( k := \max\{k \geq \delta \mid g_k(\tilde{z}, k, \mu) = k^* \} > k^* \) and positive measure \( \mu^k([k^*, k^*]) > 0 \). Due to \( p_{z, z'} > 0 \), a strictly positive part of this mass will stay at \( \tilde{z} \) and have future value \( \delta \). Hence, \( \mu^k(\delta) > 0 \) and \( k' = k^* \). This yields the mass point at the constraint for the cross-sectional distribution. Using the same reasoning, one can easily see that this mass point at zero propagates to higher levels of individual capital at \( \tilde{z} \in \mathcal{Z} \). \( \square \)
B.2.3 Proof of Theorem 22

In order to prove Theorem 22, I first need to establish that any iterate of the optimal policy \( h_{n+1} \), \( n \geq 0 \), as computed in the proximal point algorithm is analytic in the basic random variables \( \xi^z \) and \( \xi^k \), i.e. there exist constants \( c_{h_{n+1},j} \) such that

\[
\left\| D^p_j h_{n+1} \right\| \leq c_{h_{n+1},j} p!, \quad p \in \{1, 2, \ldots\}, \quad j \in \{z, k\},
\]

where the \( p \)-th derivative is denoted by \( D^p_j \). Before I show analyticity, let me specify how exactly the policy depends on the basic variables. In our discretization of the optimal savings policy, I impose a grid on the exogenous shocks, the individual start capital and the projection coefficients. Hence, at each grid point of \( h(z', k, \{\varphi_i\}_{i=0}^M) \), the states \( z' \), the individual capital and the projection coefficients are fixed. However, the policy implicitly depends on the basic random variables through the Euler equation because it contains the next-period aggregate capital

\[
K' = \sum_{\xi^z=1}^3 \int_{-\infty}^\infty \! h \left( z^{ag'}, 1 \{\xi^z>2-z^{ag'}\}, \kappa, \{\varphi_i\}_{i=0}^M \right) dF^k(\xi^k) \, dF^z(\xi^z),
\]

where, due to (2.18),

\[
\kappa = \begin{cases} 
\frac{1}{1-p^e} \sum_{i=0}^M \varphi_i \Phi_i(\xi^z, \xi^k) \quad ; \xi^z \leq 2 - z^{ag'} \\
\frac{1}{p^e} \sum_{i=0}^M \varphi_i \Phi_i(\xi^z, \xi^k) \quad ; \xi^z > 2 - z^{ag'}
\end{cases}
\]

is a function of the basic random variables. Therefore, the savings policy is a function of \( \xi^z \) and \( \xi^k \).

**Proposition 28 (Analytic policies).** Under the assumptions of Theorem 22, all iterates of the savings policy and the Lagrange multipliers as functions of \( \xi^z \) and \( \xi^k \) admit analytic extensions in the complex plane, namely in the region \( \Sigma(\tau_{n+1}^j, \Gamma^j) \), \( j \in \{z, k\} \), given in Theorem 22. Furthermore, it holds that the \( (n+1) \)-th policy iterate

\[
\max_{x \in \Sigma(\tau_{n+1}^j, \Gamma^j)} \left| h_{n+1} \right(x) \right| \leq \frac{\min(1, L_{n+1})}{\min(1, L_{n+1}) - 2\tau_{n+1}^j A_{n+1,j}^1}.
\]

is bounded in the region \( \Sigma(\tau_{n+1}^j, \Gamma^j) \), \( j \in \{z, k\} \).

**Proof.** The proof now proceeds in two steps. First, I establish that all iterates of the policy and hence, Lagrange multipliers are real analytic functions of the basic random variables. Second, I construct the complex analytic extension.

**Real analytic:** Equation (2.12) implies that the \( (n+1) \)-th iterate of the savings policy \( h_{n+1} \) in the proximal point algorithm solves the following first-order condition

\[
X \frac{\partial}{\partial h_{n+1}^A} L_A(h_{n+1}, y^n; z', k, h^n) = e \tag{B.1}
\]

with constant \( \|e\| \leq \frac{\epsilon^2}{2^{\lambda}} \) for any fixed exogenous shock and start capital \( (z', k) \). Now, let us take the derivatives of the first-order condition (B.1) w.r.t. \( \xi^z \) and \( \xi^k \). It is obvious that \( X \) and \( e \) do not depend on the basic random variables. The partial derivative of the augmented Lagrangian, however, does due to its dependence on \( K' \) and because the
optimal policies and hence, also the Lagrange multipliers depend on $\xi^*$ and $\xi^k$ as can be seen in
\[
\frac{\partial}{\partial h_{n+1}} L^A \left( h_{n+1}^+, y^n; z', k, h^n \right) = \frac{\partial}{\partial c} u \left( I(z', k, K) + [1 - \delta] k - h_{n+1}^+ \right) - \beta \sum_{z'' \in \mathcal{Z}} \left\{ \sum_{l=1}^{p(z'')} \left[ 1 - \delta + R(z'', K') \right] \right\} \left\{ I \left( z'', h_{n+1}, K' \right) + [1 - \delta] h_{n+1}^+ - h_{(n+1)'}^+ \right\} \right\}
\]  
(B.2)

I now investigate the derivatives of (B.1) w.r.t. the basic random variable $\xi^j$, $j \in \{z, k\}$. Trivially,
\[ D^p_j \left( \frac{\partial}{\partial h_{n+1}} L^A \right) = 0, \ p \in \{1, 2, \ldots\}. \]

It follows that
\[ D^{p-1}_j \left( \frac{\partial^2 L^A}{\partial h_{n+1} \partial K'} D^1_j K' \right) + D^{p-1}_j \left( \frac{\partial^2 L^A}{\partial h_{n+1}^2} D^1_j h_{n+1}^+ \right) = \frac{1}{\lambda} D^p_j h^n + 1 \{ h_{n+1}^+ \leq \frac{\nu_n}{\lambda} \} D^p_j y^n. \quad (B.3) \]

Let us first analyze the derivative of $K'$. It is easy to see from $D^1_j K' = \partial / \partial K K' D^1_j K$ that all derivatives of $K'$ are composed of the derivatives of the optimal policy w.r.t. start capital and the derivatives of $\kappa$ w.r.t. the basic random variables. The latter component is obviously analytic. Hence, $K'$ is analytic in the basic random variables if the optimal policy is analytic in start capital. This fact is easily established by induction when taking derivatives of the first-order condition w.r.t. $k$ and taking into account that $h^0$ is analytic in $k$. I exploit the fact that products, sums and compositions of analytic functions are analytic. Hence, there is a $c_{K', j}$ such that
\[
\left\| D^p_j K' \right\| \leq c_{K', j} p! \ p \in \{1, 2, \ldots\}. \]

Furthermore, note that it follows from (B.2) that $\partial / \partial h_{n+1} L^A$ is analytic in $K'$ and $h_{n+1}^+$. I can now show analyticity of the optimal policy by induction in two dimensions: First increasing the iterate of the policy, then increasing the order of the derivative. Assume that all iterates $h^j$, $j \leq n$, are analytic and that the derivatives of $h_{n+1}^+$ w.r.t. the basic random variables up to order $p - 1$ are bounded as required for analyticity. This implies that the derivatives w.r.t. $K'$ and $h_{n+1}^+$ of the first-order condition also satisfy the analyticity condition up to order $p - 1$ with coefficients $c_{L_{h^j}, j}^{K', j}$ and $c_{L_{h^j}, j}^{h_{n+1}, j}$. W.l.o.g., choose $c_{L_{h^j}, j}^{K', j} \geq c_{L_{h^j}, j}^{h_{n+1}, j}$. Applying the product rule, I rewrite (B.3) as
\[
\frac{\partial^2 L^A}{\partial h_{n+1}^2} D^p_j h_{n+1}^+ = \frac{1}{\lambda} D^p_j h^n + 1 \{ h_{n+1}^+ \leq \frac{\nu_n}{\lambda} \} D^p_j y^n
\]

\[
- \sum_{l=0}^{p-1} \binom{p - 1}{l} D^{p-l}_j K' D^l_j \left( \frac{\partial^2 L^A}{\partial h_{n+1} \partial K'} \right)
\]

\[
- \sum_{l=1}^{p-1} \binom{p - 1}{l} D^{p-l}_j h_{n+1}^+ D^l_j \left( \frac{\partial^2 L^A}{\partial h_{n+1}^2} \right).
\]
Dividing by \(p!\), taking norms and denoting \(R_{n+1,j}^p = \|D_j^n h^{n+1}\|/p!\) leads to

\[
\frac{\partial^2 L^A}{\partial h^{n+1}_j^2} R_{n+1,j}^p \leq \frac{1}{\lambda} c_{h^n,j}^p + 1 \{h^{n+1} \leq \frac{\tau^n}{\lambda}\} c_{y^n,j}^p + \sum_{l=0}^{p-1} c_{K^{l,j}}^p c_{L_{hK^{l,j}}}^A \]

\[
+ \sum_{l=1}^{p-1} R_{n+1,j}^{p-l} c_{L_{hh}^l}^A, \]

\[
\leq \frac{1}{\lambda} c_{h^n,j}^p + 1 \{h^{n+1} \leq \frac{\tau^n}{\lambda}\} c_{y^n,j}^p + \max \left(2c_{K^{l,j}}, 2c_{L_{hK^{l,j}}}^A\right)^p =: A_{n+1,j}^p
\]

\[
+ \sum_{l=1}^{p-1} R_{n+1,j}^{p-l} c_{L_{hh}^l}^A. \tag{B.4}
\]

Note that, due to convexity, \(L_{n+1} > 0\). Solving this recursion yields

\[
R_{n+1,j}^p \leq \frac{A_{n+1,j}^p}{L_{n+1}} + \sum_{l=0}^{p-1} \frac{A_{n+1,j}^{p-l} c_{L_{hh}^l}^p}{L_{n+1}} \leq 2p \frac{A_{n+1,j}^p}{L_{n+1}} \leq \left(\frac{2A_{n+1,j}^1}{\min(1, L_{n+1})}\right)^p,
\]

where \(A_0 = \|e\|/X\). Hence, I obtain a uniform bound for all derivatives of the optimal policy. Analyticity follows by induction.

**Complex continuation:** I define the following power series for the \((n + 1)\)-th iterate in terms of the basic random variable \(\xi^j, j \in \{z, k\}\), on the complex plane

\[
h^{n+1}(x) = \sum_{p=0}^{\infty} \frac{(x - \xi^j)^p}{p!} D_j^n h^{n+1}(\xi).
\]

Taking norms leads to

\[
|h^{n+1}(x)| = \sum_{p=0}^{\infty} \left|\frac{(x - \xi^j)^p}{\min(1, L_{n+1})}\right|^p.
\]

This series converges for all \(|x - \xi^j| \leq \tau_{n+1}^j < \frac{\min(1, L_{n+1})}{2A_{n+1,j}^1}\) such that

\[
|h^{n+1}(x)| \leq \frac{\min(1, L_{n+1})}{\min(1, L_{n+1}) - 2\tau_{n+1}^j A_{n+1,j}^1}.
\]

Therefore, by continuation the iterates can be extended analytically in the whole region \(\Sigma(\tau_{n+1}^j, \Gamma^j)\), which concludes the proof.

**Remark.** Note that I follow the proof of Theorem 4.1 in Babuška et al. (2007) with bounded range of the basic random variables for our proof of Theorem 22. I use bounded range since I choose a histogram approximation of the basic random variables. For other types of approximation, one might need to modify the error bound estimates to accommodate an unbounded range. I refer to Babuška et al. (2007) for that case.

**Proof of Theorem 22.** The last term of the bound is the interpolation error from tensor-product finite elements of order 1 on a rectangular discretization \(D\). It is well established (see e.g., Brenner and Scott, 2007, Theorem 4.6.14). The error bound due to truncation
The Accelerated Proximal Point Algorithm for the Growth Model

The idea behind the acceleration is to approximate the highly nonlinear augmented Lagrangian with a sequence of simple convex quadratic functions \( \phi_n \) such that the difference to the Lagrangian is reduced by a fraction \( (1 - \alpha^n) \) in every iteration step

\[
\phi^{n+1} - L^A \leq (1 - \alpha^n)(\phi^n - L^A).
\]

The update for the agents’ optimal choice \( h \) is then determined such that the following condition is satisfied

\[
L^A \left( h^{n+1}, y^n, z_{ag}^n, \zeta^n, \kappa, h^n \right) \leq \hat{\phi}^{n+1} = \min_h \phi^{n+1}(h),
\]

where \( \phi^{n+1} \) is of the form \( \phi^{n+1}(h) = \hat{\phi}^{n+1} + (A^{n+1}/2)\|h - \nu^{n+1}\|_2^2 \).

Salzo and Villa (2012) show that this is achieved by Algorithm B.1. Furthermore, they show that this algorithm has a convergence rate of \( O(n^{-2}) \) if the resolvent approximation precision increases by \( e^n = O(1/n^q) \) with \( q > 3/2 \).
Algorithm B.1 Accelerated proximal point algorithm for the growth model

\begin{algorithm}
\begin{algorithmic}[1]
\State \textbf{A Initialization}
\State 1. Set $n = 0$. Initialize the agents’ choices of consumption and individual capital and the Lagrange multipliers $H^n = (h^n, y^n)$. Set $\nu^n = h^n$.
\State 2. Set the parameters $\lambda > 0$, $A^n > 0$ and $b \in [0, 2)$.
\State 3. Set the resolvent approximation precision $\{\epsilon^n\}_{n=0}^\infty$.
\State 4. Set the termination criterion small $\tau > 0$ and the initial distance larger $d > \tau$.
\State \textbf{B Iterative Procedure}
\State 5. \While{$d > \tau$}
\State 6. Update $\alpha^n = \frac{1}{2} \left( \sqrt{(b\lambda A^n)^2 + 4b\lambda A^n} - b\lambda A^n \right)$.
\State 7. Update $x^n = (1 - \alpha^n)h^n + \alpha^n\nu^n$.
\State 8. Update $H^{n+1}$ by
\begin{align*}
h^{n+1} \left( z^{ag'}, \zeta', \kappa, \right) & \approx \arg \min_h L^A \left( h, y^n; z^{ag'}, \zeta', \kappa, x^n \right) \\
y^{n+1} \left( z^{ag'}, \zeta', \kappa, \right) & = \max \left( 0, y^n \left( z^{ag'}, \zeta', \kappa, \right) - \lambda h^{n+1} \left( z^{ag'}, \zeta', \kappa, \right) \right)
\end{align*}
where $L^A$ is defined as in (2.11) with $L$ as in (2.10) and $T$ as in (2.8).
\State 9. Update $A^{n+1} = (1 - \alpha^n)A^n$.
\State 10. Update $\nu^{n+1} = \nu^n - \frac{\alpha^n}{(1 - \alpha^n)\lambda A^n} (x^n - h^{n+1})$.
\State 11. Compute the distance $d = \|H^{n+1} - H^n\|_{L^2}$.
\State 12. Set $n = n + 1$.
\State \EndWhile
\end{algorithmic}
\end{algorithm}
B.4 Robustness Checks

This section investigates whether the two economic results from Chapter 2 are robust to changes in the model parameters. I start with the volatility paradox and then, move on to the amplification due to ex-post heterogeneity.

B.4.1 The Volatility Paradox

It can be seen in Figure B.1 that the volatility paradox holds for various parameter choices meaning that the expected stationary cross-sectional distribution in the case of a high unemployment benefit, i.e., more redistribution, has fatter tails. The first row of plots looks at a varying subjective discount factor $\beta$. It seems that the cross-sectional distributions converge as $\beta$ goes to one. The second row confirms the result for different risk aversion values $\gamma$, whereas, the third row plots the distributions for various production function elasticities $\alpha$. Lastly, the depreciation rate of capital $\delta$ is varied and the result still holds. Overall, the volatility paradox in the Aiyagari-Bewley economy seems to be a very robust result.

B.4.2 Amplification due to Ex-Post Heterogeneity

Figure B.2 shows that the amplification of aggregate risk due to higher risk-sharing among ex-post heterogeneous agents, in the sense that the distribution of model with aggregate risk features more weight on poor and rich agents than for the model where aggregate risk is absent, is robust to changes in the subjective discount factor $\beta$ and the depreciation rate of capital $\delta$. This effect also persists for low production elasticity $\alpha = 0.2$ and for low risk aversion $\gamma = 2$, although at a much smaller scale. However, the cases of high risk aversion $\gamma = 4$ and high elasticity $\alpha = 0.45$ differ. In these cases the distribution of the model with aggregate risk puts lower weight on poor and rich agents compared to the model where aggregate risk is absent. Therefore, there is still amplification of aggregate risk but in the opposite direction. The reason for this is that in a model with high risk aversion or high production elasticity, the agents value capital savings more as they want to insure against aggregate risk more and they want to insure a higher sensitivity of production to aggregate capital, respectively. This is reflected in the higher average levels of individual capital. Due to the increased importance of capital savings, the slope of the marginal propensity to consume is less steepe compared to the other calibrations. With more redistribution, all unemployed agents increase their consumption and savings. However, in these calibrations, poor agents increase their savings and rich agents increase their consumption more relative to the other calibrations. This reverses the direction of the amplification effect.
Figure B.1: Robustness of the volatility paradox across varying model parameters. This graph displays the average stationary cross-sectional distribution, i.e., the expectation of the stationary distribution of the state space \((z', k, \mu)\) conditional on \(\mu\) for two different unemployment benefit rates \(\nu\) across different model parameters. Each row of plots varies one model parameter and leaves the others unchanged. The benchmark plot corresponds to the calibration used in the main text. The proximal point algorithm for 15% unemployment benefit is truncated at first order, whereas, the proximal point algorithm for 65% is truncated at second order.
Figure B.2: Robustness of the amplification of aggregate risk due to ex-post heterogeneity across varying model parameters. This graph displays the average stationary cross-sectional distribution, i.e., the expectation of the stationary distribution of the state space \((z', k, \mu)\) conditional on \(\mu\) for two different unemployment benefit rates \(\nu\). All graphs plot the expected stationary distributions for the model with aggregate risk as well as the models without aggregate risk which fix the productivity level at one aggregate state. The proximal point algorithm for 15% unemployment benefit is truncated at first order, whereas, the proximal point algorithm for 65% is truncated at second order. In each row, one model parameter is varied between two values one lower and one higher than in the benchmark calibration used in the main text. The other parameters are kept the same as in the benchmark calibration.
Appendix C

Appendix of Chapter 3

C.1 The Model’s Parameter Values

I choose the parameters in the Huggett economy similar to a quantitative example in Krusell et al. (2011). At any time point, there are agents with high endowment and agents with low endowment. The fraction of each is set to 0.5. Similarly, the good and bad economic state have equal probability. The probability of staying in the same aggregate state is set to 0.43 which matches the time series of the U.S. per capita consumption growth rate over the time period 1889-1978. The other transition probabilities are chosen to match features of the household-level labor income in the Panel Study of Income Dynamics. The probability of keeping the same income level given that tomorrow’s economy is in a boom is 0.984, whereas, it is 0.942 when tomorrow’s economy is in a recession. The full transition matrix of the exogenous shocks can be derived from that and is given in Table C.1.

\[
\begin{bmatrix}
0.4051 & 0.0249 & 0.5609 & 0.0091 \\
0.0249 & 0.4051 & 0.0091 & 0.5609 \\
0.5369 & 0.0331 & 0.4231 & 0.0069 \\
0.0331 & 0.5369 & 0.0069 & 0.4231 \\
\end{bmatrix}
\]

Table C.1: Transition matrix of the exogenous shocks for the Huggett Model. This table displays the transition matrix which is taken from the calibration in Krusell et al. (2011).

The endowment process I use is a simplified version of the one used in Krusell et al. (2011). Instead of letting aggregate endowment vary stochastically, I keep it fixed at one. In a boom, the aggregate endowment is multiplied with a growth rate of 1.054, whereas, it is multiplied with 0.982 in a recession. These growth rates match the U.S. per capita consumption growth rate over the time period 1889-1978. Each agent receives a share of aggregate endowment \(\epsilon_h > \epsilon_l\) whose level depends on the idiosyncratic shock. The ratio of high-to-low endowment \(\epsilon_h/\epsilon_l\) is set to 2.06 to match the household-level labor income in the Panel Study of Income Dynamics. The levels of the idiosyncratic share then follow by normalizing their expectation to one, i.e. \(0.5\epsilon_h + 0.5\epsilon_l = 1\). Thus, the endowment process is given by

\[
e(z) = e_1(z^{ag}) e_2(z^{id}),
\]
where
\[ e_1(z^{ag}) = \begin{cases} 0.982, & z^{ag} = 0 \\ 1.054, & z^{ag} = 1 \end{cases}, \quad e_2(z^{id}) = \begin{cases} 2, & z^{id} = 0 \\ 3.12, & z^{id} = 1 \end{cases}. \]

The first term stands for the aggregate endowment and the second term is the idiosyncratic share of aggregate endowment.

I set the remaining parameters to standard values, i.e., \( \beta = 0.99 \) and \( \gamma = 2 \). The borrowing constraint for the calibration to check the precision of the algorithm in the main text is set to \( \bar{a} = -1.5 \), but it is varied to derive the economic results in the main text.

## C.2 Robustness Checks

This section investigates whether the two economic results from Chapter 3 are robust to changes in the model parameters. As this model has fewer parameters, I only have to vary the subjective discount factor \( \beta \) and the risk aversion \( \gamma \). I start with the amplification due to ex-post heterogeneity and then, move on to the volatility paradox.

### C.2.1 Amplification due to Ex-Post Heterogeneity

To see whether the amplification is robust to parameter changes, I compare bond prices from the model without aggregate risk, i.e., where the aggregate shock is fixed to one of its states, and the model with aggregate risk. The prices corresponding to the expected stationary cross-sectional distribution are displayed in Table C.2. This table confirms

<table>
<thead>
<tr>
<th>( \beta ) ( \gamma )</th>
<th>No Aggregate Risk bad</th>
<th>good</th>
<th>Aggregate Risk average bad</th>
<th>good</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Panel A</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \beta = 0.9 )</td>
<td>0.9566</td>
<td>0.9266</td>
<td>0.9864</td>
<td>0.8926</td>
</tr>
<tr>
<td>( \beta = 0.999 )</td>
<td>1.0523</td>
<td>1.0262</td>
<td>1.0950</td>
<td>0.9871</td>
</tr>
<tr>
<td>( \gamma = 1 )</td>
<td>1.0136</td>
<td>1.0022</td>
<td>1.0370</td>
<td>0.9836</td>
</tr>
<tr>
<td>( \gamma = 3 )</td>
<td>1.0792</td>
<td>1.0386</td>
<td>1.1394</td>
<td>0.9756</td>
</tr>
<tr>
<td><strong>Panel B</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \beta = 0.9 )</td>
<td>0.9330</td>
<td>0.9196</td>
<td>0.9347</td>
<td>0.8217</td>
</tr>
<tr>
<td>( \beta = 0.999 )</td>
<td>1.0235</td>
<td>1.0156</td>
<td>1.0733</td>
<td>0.9787</td>
</tr>
<tr>
<td>( \gamma = 1 )</td>
<td>1.0024</td>
<td>0.9972</td>
<td>1.0312</td>
<td>0.9861</td>
</tr>
<tr>
<td>( \gamma = 3 )</td>
<td>1.0289</td>
<td>1.0191</td>
<td>1.0296</td>
<td>0.9832</td>
</tr>
</tbody>
</table>

Table C.2: Equilibrium prices corresponding to the average stationary cross-sectional distributions for various parameter values. This graph displays the prices corresponding to the average stationary cross-sectional distribution, i.e., the expectation of the stationary distribution of the state space \((z', a, \mu)\) conditional on \( \mu \). It compares the model without aggregate risk which is either fixed to the bad or good state, respectively, and the model with aggregate risk where the prices are conditional on the aggregate state. Panel A displays the prices for the model with strict borrowing constraint \( \bar{a} = -0.5 \), whereas, Panel B displays the prices for the model with a laxer borrowing constraint \( \bar{a} = -1.5 \).

that results stated in the main text are robust to changes in \( \beta \) and \( \gamma \). In particular, price levels for the tighter borrowing constraint are higher. The prices of the model with
aggregate risk always lie outside the interval of prices of the models without aggregate risk and the price conditional on the bad state is lower than in the good state. Hence, the amplification of aggregate risk due to risk sharing among ex-post heterogeneous agents is a very robust result in the Huggett economy. Furthermore, price levels increase with a higher subjective discount factor. This is intuitive as a high discount factor means that agents value future consumption more. Hence, they are less willing to sell bonds which essentially converts future consumption into current consumption which increases prices. Similarly, the amplification effect is stronger for a higher discount factor. Price levels also increase with higher risk aversion. This is due to the increased precautionary savings motive which implies a higher demand for buying bonds.

C.2.2 The Volatility Paradox

I now investigate whether the volatility paradox in the Huggett economy is robust to parameter changes. Figure C.1 shows that the conditional price distribution is much wider for the laxer borrowing constraint across all parameter choices. This is confirmed by Table C.3 which displays the summary statistics of the conditional price distributions. Across parameters, the average price is higher for the tighter borrowing constraint and the volatility decreases. This confirms that the volatility paradox, i.e., that risk increases with a higher level of risk sharing. The results for skewness and kurtosis are less straight-
Table C.3: Simulated stationary distribution of equilibrium prices for different parameter values. This graph displays summary statistics of the simulated stationary distribution of equilibrium prices of the Huggett economy with aggregate risk for different borrowing constraints and various parameter values over a simulation of $N = 3000$ periods.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>$\beta = 0.9$</th>
<th>$\beta = 0.999$</th>
<th>$\gamma = 1$</th>
<th>$\gamma = 3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\tilde{\alpha} = 0.5$</td>
<td>Mean</td>
<td>0.8923</td>
<td>0.9862</td>
<td>0.9837</td>
</tr>
<tr>
<td></td>
<td>St. dev.</td>
<td>0.0042</td>
<td>0.0053</td>
<td>0.0013</td>
</tr>
<tr>
<td></td>
<td>Skewness</td>
<td>0.0603</td>
<td>0.1658</td>
<td>0.2779</td>
</tr>
<tr>
<td></td>
<td>Kurtosis</td>
<td>1.4405</td>
<td>1.2384</td>
<td>2.8053</td>
</tr>
<tr>
<td>$\tilde{\alpha} = 1.5$</td>
<td>Mean</td>
<td>1.0793</td>
<td>1.2019</td>
<td>1.0901</td>
</tr>
<tr>
<td></td>
<td>St. dev.</td>
<td>0.0052</td>
<td>0.0064</td>
<td>0.0012</td>
</tr>
<tr>
<td></td>
<td>Skewness</td>
<td>0.6152</td>
<td>-0.4374</td>
<td>-0.2245</td>
</tr>
<tr>
<td></td>
<td>Kurtosis</td>
<td>1.9029</td>
<td>1.4559</td>
<td>2.7404</td>
</tr>
<tr>
<td>$\tilde{\alpha} = -1.5$</td>
<td>Mean</td>
<td>0.8783</td>
<td>0.9753</td>
<td>0.9852</td>
</tr>
<tr>
<td></td>
<td>St. dev.</td>
<td>0.0116</td>
<td>0.0120</td>
<td>0.0036</td>
</tr>
<tr>
<td></td>
<td>Skewness</td>
<td>0.3504</td>
<td>0.0632</td>
<td>0.0151</td>
</tr>
<tr>
<td></td>
<td>Kurtosis</td>
<td>2.2578</td>
<td>2.0263</td>
<td>2.0548</td>
</tr>
</tbody>
</table>

forward. Kurtosis decreases with the tighter borrowing constraint for all parameters but $\gamma = 1$. Skewness decreases in the good state for all parameters except $\beta = 0.9$. 


Appendix D

Supplementary Material

D.1 The Matlab Code

The algorithms in Chapters 2 and 3 have been implemented with Matlab. The code can be downloaded from https://github.com/zschorli/Proehl_SolvingHeterogAgentModels. In the following, I outline the structure of the code and which steps should be taken to reproduce the results of Chapters 2 and 3.

D.1.1 The Folder Structure

The overall code is divided into several folders. Two of them contain novel code implementing the algorithm for the two models presented in this thesis.

1. "Proehl_GrowthModel": This folder contains the code implementing the proximal point algorithm and the policy function iteration using polynomial chaos expansions to solve the Aiyagari-Bewley economy. Furthermore, stationary distributions and Euler equation errors are computed and the figures in the main text can be reproduced.

2. "Proehl_HuggettModel": This folder contains the code implementing the proximal point algorithm and the policy function iteration using polynomial chaos expansions to solve the Huggett economy. Furthermore, stationary distributions and Euler equation errors are computed and the figures in the main text can be reproduced.

The other folders contain the code of existing algorithms. This code was downloaded from http://www.wouterdenhaan.com/datasuite.htm. It was then modified to set the appropriate grids and model parameters for the code comparison.

3. "DenHaanRendahl": This folder contains the code from the paper


4. "KrusellSmithByMaliarMaliarValli1mom": This folder contains the code from the paper

5. "KrusellSmithByMaliarMaliarValli_2mom": This folder contains the code from the same paper as Folder 4 but it is modified to accommodate second moments.

6. "KrusellSmithByMaliarMaliarValli_3mom": This folder contains the code from the same paper as Folder 4 but it is modified to accommodate second and third moments.

7. "KrusellSmithByMaliarMaliarValli_4mom": This folder contains the code from the same paper as Folder 4 but it is modified to accommodate second, third and fourth moments.


**D.1.2 Instructions to Reproduce all Results**

You can simply run "main.m" in the folders "Proehl_GrowthModel" and "Proehl_HuggettModel", respectively, to experiment with the code. However, if you want to fully reproduce the results in the thesis, some additional steps are necessary. First, you should take the following steps to reproduce the results for the Aiyagari-Bewley model in Chapter 2.

1. In Folder 1: Set the variable "case_nr" equal to one in line 36 of "main.m" and comment out the results section in "main.m". Then run "main.m".

2. In Folder 3: Define the appropriate path to Folder 1 in line 105 of "ExplicitAggr.m". Run "ExplicitAggr.m" to produce the result files "DR_Sol1.mat" to "DR_Sol4.mat". Copy the result file ending with number "Sol..." to "Proehl_GrowthModel/res_case...".

3. In Folder 4: Define the appropriate path to Folder 1 in line 133 of "MAIN.m". Run "MAIN.m" to produce the result files "KS1_Sol1.mat" to "KS1_Sol4.mat". Copy the result file ending with number "Sol..." to "Proehl_GrowthModel/res_case...".

4. In Folder 5: Define the appropriate path to Folder 1 in line 127 of "MAIN.m". Run "MAIN.m" to produce the result files "KS2_Sol1.mat" to "KS2_Sol4.mat". Copy the result file ending with number "Sol..." to "Proehl_GrowthModel/res_case...".

5. In Folder 6: Define the appropriate path to Folder 1 in line 127 of "MAIN.m". Run "MAIN.m" to produce the result files "KS3_Sol1.mat" to "KS3_Sol4.mat". Copy the result file ending with number "Sol..." to "Proehl_GrowthModel/res_case...".

6. In Folder 7: Define the appropriate path to Folder 1 in line 127 of "MAIN.m". Run "MAIN.m" to produce the result files "KS4_Sol1.mat" to "KS4_Sol4.mat". Copy the result file ending with number "Sol..." to "Proehl_GrowthModel/res_case...".

7. In Folder 8: Define the appropriate path to Folder 1 in line 191 of "setparam.m". Run "main.m" to produce the result files "R_Sol1.mat" to "R_Sol4.mat". Copy the result file ending with number "Sol..." to "Proehl_GrowthModel/res_case...".

8. In Folder 1: Run the results section in "main.m".
9. In Folder 1: Vary the variable "case_nr" in line 36 of "main.m" to run the different model configurations for the robustness checks.

10. In Folder 1: Run "plots.m" to reproduce the figures.

Second, you should take the following steps to reproduce the results for the Huggett model in Chapter 3.

1. In Folder 2: Set the variable "case_nr" equal to one in line 29 of "main.m" and run "main.m".

2. In Folder 2: Vary the variable "case_nr" in line 29 of "main.m" to run the different model configurations for the robustness checks.

3. In Folder 2: Run "plots.m" to reproduce the figures.
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