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COULOMB EFFECT ON SUPERCONDUCTING TRANSITION TEMPERATURE

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ABSTRACT: The effect of the Coulomb repulsion on a superconductor described by an effective attraction is studied using a Hubbard-Stratonovich transformation to eliminate both interactions. In the absence of Coulomb repulsion the gap equation is derived for a constant order parameter. Then the dielectric function is calculated for a one- and two-band model, and an acoustic plasmon mode is shown to exit in the latter. Finally the Coulomb effect on the transition temperature $T_c$ is studied in the mentioned models. While the result for the one-band model is a simple decrease of $T_c$, the situation in the two-band model is more involved and will be investigated separately.

1. INTRODUCTION

In the theory of superconductivity the adverse effect of Coulomb repulsion on the transition temperature $T_c$ is well known [1]; it is the reason for studying screening mechanisms. A simple way of screening may be obtained in a two-band model in which heavy carriers are shielded by light ones and the associated collective mode is an acoustic plasmon [2] (the latter is also produced in superlattices [3]). Acoustic plasmons have been shown to give rise to an enhanced pairing in a model where pairs are formed between the light carriers [4]. In another model light “spectator electrons” are responsible for boson pair formation and condensation heavy holes [5]. A quite different mechanism of screening consists in the exchange of pairs of carriers between the conduction band and an empty upper band [6].

In view of the complicated nature of the problem of superconductivity in the presence of Coulomb repulsion we here wish to apply the powerful method of the Hubbard-Stratonovich transformation. The hope was that in eliminating both the effective attraction responsible for superconductivity and the Coulomb repulsion in favor of auxiliary fields, viz. the gap function and an effective Coulomb potential, respectively, the calculation of the Coulomb effect on $T_c$ would become sufficiently simple to lead to explicit results even in the two-band case.

For one band this problem has been treated long ago by Rice [7] using the method just described. His aim was to derive an equilibrium Ginzburg-Landau functional modified by Coulomb repulsion. He, however, not find any significant modifications for the charged system as compared to the uncharged one.

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In this paper we first review the Hubbard-Stratonovich method showing in particular that the effective attraction resulting in singlet superconductivity and the Coulomb repulsion are essentially the only forms of interaction for which this method works (Section 2). This implies that, in contrast to the explicit electron-phonon coupling, inclusion of retardation effects is possible in this formalism only via an ad hoc frequency-dependent coupling function.

Keeping first only the effective attraction, a simple derivation of the gap equation for a constant order parameter is given (Section 3). At this stage it is interesting to note that the problem of the cross-over from weakly to strongly bound doubly charged bosons [8] has recently also been analyzed with great success in this formalism [9, 10] although the above remark about retardation effects may reduce the generality of this analysis.

We next consider the case of pure Coulomb repulsion and give a compact derivation of the dielectric function in random phase approximation (RPA) for the case of a one- and two-band model (Section 4). Finally the Coulomb effect on \( T_c \) is calculated for these models (Section 5). While the one-band case is simple and intuitively convincing, the two-band model (with superconducting light carriers) gives rise to rather complicated expressions whose influence on \( T_c \) is not obvious and which will be analyzed in a separate publication.

2. HUBBARD-STRATONOVICH ELIMINATION OF INTERACTIONS

The interactions considered in this paper are the effective attraction

\[
H_{\text{attr}} = -\frac{1}{V} \sum_q g_q b_q^+ b_q
\]

leading to singlet-pair superconductivity, where

\[
b_q^+ = \sum_k \phi_{k-q/2}^+ \chi_{k}^+ \quad (2.2)
\]

is a general pair operator, \( \phi_{k-q} = \phi_{k}^* \) and the Coulomb repulsion

\[
H_{\text{Coul}} = +\frac{1}{V} \sum_{q \neq 0} U_q n_q^+ n_q
\]

where

\[
n_q = n_{-q} = \sum_{k\sigma} a_{k\sigma}^+ a_{k\sigma}
\]

is the density and \( U_q = 4\pi e^2 / q^2 \) the Coulomb potential in Fourier space.

In terms of the total Hamiltonian

\[
H = H_0 + H_{\text{attr}} + H_{\text{Coul}}
\]

where

\[
H_0 = \sum_{k\sigma} \epsilon_k a_{k\sigma}^+ a_{k\sigma}
\]

(2.6)
describes free carriers and \( \xi_k = (k^2/2m) - \mu, \mu \) being the chemical potential, the respective functions are

\[
Z = e^{-\beta \Omega} = \text{Tr} e^{-\beta H}; \quad Z_0 = e^{-\beta \Omega_0} = \text{Tr} e^{-\beta H_0}.
\]  
(2.7)

Defining the unperturbed average of an operator \( A \) as \( \langle A \rangle = \text{Tr} (e^{-\beta H_0} A)/\text{Tr} e^{-\beta H_0} \) a simple operator relation then yields (see, e.g. Sec. 5 of Ref. 11)

\[
\frac{Z}{Z_0} = \langle T \exp \left( - \int_0^\beta \! d\tau H_{\text{int}} (-i\tau) \right) \rangle
\]  
(2.8)

where \( T \) is the imaginary-time ordering operator, \( \tau = it \) and \( A(t) = e^{iH_0 t} A e^{-iH_0 t} \).

The Hubbard-Stratonovich transformation makes use of the identity [12]

\[
e^a z^a z^a = \frac{b^2}{\pi} \int \! d^2 \zeta e^{-b^2 |\zeta|^2} e^{-a \beta (\zeta^a - z^a \zeta^a)}
\]  
(2.9)

where \( a^2, b^2 \) are real and \( z, \zeta \) complex numbers, respectively. Discretizing the integral in (2.8) by points \( \tau_j \) of equal distance \( \delta \tau \) both, the effective attraction (2.1) and the Coulomb repulsion (2.3) then appear in the left-hand side of the identity (2.9). The fact that non-commuting hermitean conjugate operators are replaced by complex conjugate numbers may be taken care of by Feynman's well-known ordering-label technique [13, 12, 7].

Thus in the case of Eq. (2.1) we identify \( z = b_q (-i\tau), a^2 = \delta \tau g_q /V, b^2 = \delta \tau V /g_q \) and \( \zeta = \Delta_q (\tau) \) whose meaning is the gap function. Eqs. (2.8, 2.9) now yield

\[
\frac{Z_{\text{attr}}}{Z_0} = \text{const} \int D^2 \Delta \exp \left\{ - \int_0^\beta \! d\tau \sum_q \frac{V}{g_q} |\Delta_q (\tau)|^2 \right\} W_{\text{attr}}[\Delta, \Delta^*]
\]  
(2.10)

where

\[
W_{\text{attr}}[\Delta, \Delta^*] = \langle T \exp \left[ \int_0^\beta \! d\tau \sum_q \left( b_q (-i\tau) \Delta_q (\tau) + b^*_q (-i\tau) \Delta^*_q (\tau) \right) \right] \rangle
\]  
(2.11)

Similarly we identify in the case (2.3) \( z = n_q (-i\tau), a^2 = - \delta \tau U_q /V, b^2 = \delta \tau V /U_q \) and \( \zeta = \Phi_q (\tau) \) which may be interpreted as an effective Coulomb potential. This leads to

\[
\frac{Z_{\text{Coul}}}{Z_0} = \text{const} \int D^2 \Phi \exp \left\{ - \int_0^\beta \! d\tau \sum_{q \neq 0} \frac{V}{U_q} |\Phi_q (\tau)|^2 \right\} W_{\text{Coul}}[\Phi]
\]  
(2.12)

where

\[
W_{\text{Coul}}[\Phi] = \langle T \exp \left[ \int_0^\beta \! d\tau \sum_{q \neq 0} n_q (-i\tau) \Phi_{q^*} (-\tau) \right] \rangle
\]  
(2.13)

It is useful to replace the restriction \( q \neq 0 \) in the sum occurring in Eqs. (2.12, 14) by the condition
\Phi_0(\tau) = 0.

(2.14)

It is from the above examples that the Hubbard-Stratonovich only applies to positive hermitean forms \([12]\) and thus has a rather limited domain of application. In particular any interaction given by a three-pronged vertex like the electron-phonon coupling is excluded. This because the bosonic field attached to such a vertex carries non-zero momentum and hence the remaining operators cannot be mutually hermitean conjugate. Thus it is impossible, for example, to include retardation effects into the theory of superconductivity as is done in Eliashberg theory (see, e.g. Sec. of Ref. 11), except for an ad hoc frequency dependence of the coupling constant \(g\).

3. EFFECTIVE ACTION AND GAP EQUATION FOR A LOCAL ATTRACTION

Writing the functional integral in Eq. (2.10) as \(\int D^2\Delta \exp[S_{\text{attr}}]\) the effective action is given by

\[ S_{\text{attr}}[\Delta, \Delta^*] = \int_0^\beta d\tau \sum_q \left[ \frac{\nu}{g_q} |\Delta_q(\tau)|^2 - \ln W_{\text{attr}}[\Delta, \Delta^*] \right]. \]

(3.1)

\(W_{\text{attr}}\) given by Eq. (2.11) may be calculated by restricting the perturbation series to connected diagrams which leads to the well-known result (see, e.g. Sec. of Ref. 11),

\[ \ln W_{\text{attr}} = \left\langle \text{Tr} \exp \left( \sum_q \left[ b_{q,\ast}(-i\tau)\Delta_q(\tau) + b_q(-i\tau)\Delta_q^\ast(\tau) \right] \right) \right\rangle_{\text{con}}. \]

(3.2)

It is easy to see that, due to the form (2.2) of the operators \(b_q, b_q^\ast\), the only connected diagrams are closed loops with an even number of alternating vertices labelled by \(\Delta_{q}\). The first term of this series is \(\beta V L(q) |\Delta(q)|^2\) where

\[ L(q) = \frac{1}{\beta V} \sum_k G(k)G(q-k) \]

(3.3)

and

\[ \Delta(q, iq_0) = \beta^{-1} \int_0^\beta d\tau \Delta_q(\tau) e^{i\tau q_0}. \]

(3.4)

Here

\[ G(k, ik_0) = -\int_0^\beta d\tau \left\langle a_k(-i\tau) a_k^\ast(-i\tau) \right\rangle e^{i\tau k_0} = -\frac{1}{ik_0 - \delta_k}. \]

(3.5)

is the unperturbed one-particle Greens's function and the four-component notation \(k \equiv (k, ik_0), q \equiv (q, iq_0)\) has been used.

Unfortunately it is not possible to sum the series (3.2) in closed form, except in the case of homogeneous system for which

\[ \Delta_q(\tau) = \Delta \delta_{q,0} \Rightarrow \Delta(q) = \Delta \delta_{q,0}. \]

(3.6)
The one finds
\[
\ln W_{\text{attr}} = - \sum_{N=1}^{\infty} \frac{(-1)^N}{N} \sum_k \{ G(k)G(-k) \}^N |\Delta|^2N
\]  
and using (3.5),
\[
\Delta^* \frac{\partial}{\partial \Delta^*} \ln W_{\text{attr}} = \sum_k \left[ 1 - \frac{1}{1 + G(k)G(-k)|\Delta|^2} \right] \frac{\tanh(\beta E_k/2)}{2E_k} = \beta |\Delta|^2 \sum_k \frac{\tanh(\beta E_k/2)}{2E_k}
\]
where
\[
E_k^2 = \xi_k^2 + |\Delta|^2.
\]  
Assuming in addition locality of the attraction (2.1, 5), meaning that, for all \( q \), \( g_q = g \) and \( \phi = 1 \), Eq. (3.1) yields
\[
\Delta^* \frac{\partial}{\partial \Delta^*} S_{\text{attr}}(\Delta) = \beta V |\Delta|^2 \left\{ \frac{1}{g} - \frac{1}{V} \sum_k \frac{\tanh(\beta E_k/2)}{2E_k} \right\}
\]  
and the stability (saddle point) condition \( \partial S_{\text{attr}}/\partial \Delta^* = 0 \) becomes the gap equation
\[
\frac{1}{g} = \frac{1}{V} \sum_k \frac{\tanh(\beta E_k/2)}{2E_k}.
\]  
The possibility of such a calculation was first indicated by Hubbard in ref. 12; actual calculations were done by Rice and others (see Ref. 7).

Note that Eqs. (3.3, 3.6) cover all the time-independent terms given in Ref. 10. In particular, to order \( |\Delta|^2 \) the effective action (3.10) is
\[
S_{\text{attr}}(\Delta) = \beta V \left( \frac{1}{g} - L(0) \right) |\Delta|^2
\]  
where Eqs. (3.3, 3.5) give
\[
L(0) = \frac{1}{V} \sum_k \frac{\tanh(\beta \xi_k/2)}{2\xi_k}.
\]  
This determines the transition temperature \( T_c = \beta_c^{-1} \) by the condition \( 1/g = L(0) \).

4. DIELECTRIC FUNCTION FOR ONE- AND TWO-BAND MODELS

In analogy to Eq. (3.1) we may write the effective action due to the Coulomb repulsion (2.3) as
\[
S_{\text{Coul}}[\Phi] = \frac{\beta}{\int_0^\infty d\tau \sum_{q \neq 0} V |\Phi_q(\tau)|^2 - \ln W_{\text{Coul}}[\Phi].
\]  
Keeping in \( \ln W_{\text{coul}} \) only terms quadratic in \( \Phi \) one finds
\[ S_{\text{Coul}}[\Phi] = \beta V \sum_q \left[ \frac{1}{U_q} - I(q) \right] |\Phi(q)|^2 \]  

(4.2)

where

\[ I(q) = \frac{2}{\beta V} \sum_k G(k) G(k - q) \]  

(4.3)

and

\[ \Phi(q, iq_0) = \Phi^*(q_0, -q) = \beta^{-1} \int_0^\beta d\tau \Phi_\tau(q) e^{iq_0 \tau}. \]  

(4.4)

Now the dielectric function in random phase approximation is given by

\[ \varepsilon^{RPA} = 1 - U_q I(q, \omega) = \frac{U_q}{U^{RPA}(q, \omega)} \]  

(4.5)

thus defining the renormalized Coulomb potential in RPA, \[ U^{RPA}(q, \omega) = \frac{u(|q|, \omega)}{2N(0)} \] where \[ N(0) = 3n/4\mu \] is the density of states at the Fermi level \[ |q| = k_F = \frac{mv_F}{n} \] and \[ n = k_F^2/3\pi \] the electron density. Evaluation of the expression (4.3) yields

\[ I(q, \omega) = I(q, -\omega) = \frac{1}{\beta V} \sum_k \frac{f_0(\xi_k) - f_0(\xi_{k} - q)}{\xi_k - \xi_{k} - q - \omega} \]  

(4.6)

where \[ f_0(\xi) = (e^{\beta \xi} + 1)^{-1} \] is the Fermi distribution function. In particular, for \[ |q| \ll k_F, \] which is the dominant domain because of the singular nature of \[ U_q, \] \[ \frac{U_q I(q, \omega)}{\omega} \]  

(4.7)

where

\[ \psi(x) = -1 + \frac{1}{2x} \ln \frac{1 + x}{1 - x} \]  

(4.8)

and

\[ q_D^2 = 8\pi^2 N(0) = \frac{3\omega_{pl}^2}{v_F^2} \]  

(4.9)

defines the Debye screening length \[ 2\pi/q_D, \omega_{pl} = \sqrt{4\pi^2 n/m} \] being the plasma frequency.

The physical interesting limits of Eqs. (4.5, 4.7, 4.8) are the static screening (isothermal) limit where

\[ u(|q|, \omega) = \frac{q_D^2}{q^2 + q_D^2}; \ |\omega| \ll v_F |q| \]  

(4.10)

and the plasmon limit where

\[ u(|q|, \omega) = \frac{q_D^2}{q^2 + \omega^2}; \ |\omega| >> v_F |q| \]  

(4.11)
Since the lattice constant $a = 4\pi/2k_F$ one has $q_0^2/k_F^2 = 8a/\pi\sqrt{a_B}$ \( \gg 1 \) where $a_B = 4m^2$ is the Bohr radius, and it follows from Eq. (4.10) that, for $|\omega| \ll v_F |q|$, \( U^{RPA} = \Delta U = \text{const} \) or $u = 1$. This is the approximation of the Hubbard model (see, e.g., Sec. 33 of Ref. 11).

We now consider a model consisting of light (L) and heavy (H) carriers characterized by masses $m_L$, $m_H$ satisfying

$$\frac{m_L}{m_H} \ll 1$$  \hspace{1cm} (4.12)

but such that the respective Fermi momenta $k_b$ are of the same order of magnitude, $k_L = k_H$. It then follows that the densities $n_b = k_b^2/3\pi^2$ are also comparable in size, $n_L = n_H$ whereas the chemical potentials $\mu_b = k_b^2/2m_b$ and the velocities $v_b = \sqrt{m_b}$ and the plasma frequencies squared $\omega_b^2 = 4e^2n_b/m_b$ are such that $\mu_L/\mu_H = v_L/v_H = \omega_L^2/\omega_H^2 = m_H/m_L$. On the other hand, the densities of state at the Fermi energy $N_b(0) = 3n_b/4\pi\hbar^2$ and the Debye wavevectors squared $q_b^2 = 3\omega_b^2/v_b^2$ satisfy $N_L(0)/N_H(0) = q_L^2/q_H^2 = m_L/m_H$. Comparing the lattice constant $a = \pi/2k_L$ with the Bohr radius $a_B = 1/m_Le^2$ one deduces the relative magnitudes

$$q_H^2 \gg q_L^2 \gg k_L^2$$  \hspace{1cm} (4.13)

Eq. (4.3) now has the modified form

$$I(q) = \frac{2}{B\gamma} \sum_{b=L,H} \sum_k G_b(k) G_b(k - q)$$  \hspace{1cm} (4.14)

where $G_b(k)$ is the Green’s function of free b-carriers. The renormalized Coulomb potential in RPA, multiplied by $2N_L(0)$, is then found from Eqs. (4.5-7) to be

$$u_L(|q|,\omega) = 2N_L(0)U^{RPA}(q,\omega) = \frac{q_L^2 - q_H^2}{q_L^2} \left[ \frac{v_H}{\omega} - \left( \frac{v_L}{\omega} \right) \right]^{-1}$$  \hspace{1cm} (4.15)

This result had been obtained by Abrikosov, Gor'kov and Dzyaloshinskii using diagrammatic techniques (see Sec. 22 in Ref. 14). As discussed in Ref. 14, Eq. (4.15) has three domains of physical interest, namely the static screening (isothermal) limit where

$$u_L(|q|,\omega) = \frac{q_L^2}{q_H} = \text{const}; \ |\omega| \ll v_H |q|$$  \hspace{1cm} (4.16)

the acoustic-plasmon domain where

$$u_L(|q|,\omega) = \frac{\omega^2}{\omega^2 + s^2 q^2} \ ; \ v_H |q| \ll |\omega| \ll v_L |q|$$  \hspace{1cm} (4.17)

and the optical-plasmon limit where

$$u_L(|q|,\omega) = \frac{q_L^2}{q^2} \frac{\omega^2}{\omega^2 - \omega_L^2} \ ; \ v_L |q| \ll |\omega|$$  \hspace{1cm} (4.18)
\( s \) in Eq. (4.17) is the velocity of the acoustic plasmon,

\[
\frac{\omega_H}{q_L} = \sqrt{\frac{U_H v_H}{3}}. \tag{4.19}
\]

**5. COULOMB EFFECT ON \( T_c \) IN ONE- AND TWO-BAND MODELS**

We now combine the two interactions (2.1) and (2.3) to obtain the total partition function, divided by \( Z_0 \),

\[
\frac{Z}{Z_0} = \left\langle \exp \left\{ \int_0^\beta d\tau H_{\text{attr}}(-i\tau) \right\} \exp \left\{ \int_0^\beta d\tau' H_{\text{Coul}}(-i\tau') \right\} \right\rangle. \tag{5.1}
\]

Applying the Hubbard-Stratonovich transformation to both exponentials we obtain

\[
\frac{Z}{Z_0} = \text{const} \int D^2 \Delta \int D^2 \psi \times
\]

\[
\times \exp \left\{ \int_0^\beta d\tau \sum_q \left[ \frac{V}{U_q} |\Delta_q(\tau)|^2 + \frac{V}{U_q} |\Phi_q(\tau)|^2 \right] W[\Delta, \Delta^*; \Phi] \right\}. \tag{5.2}
\]

where

\[
W[\Delta, \Delta^*; \Phi] = \left\langle \exp \left\{ \int_0^\beta d\tau \sum_q b_q^*(-i\tau) \Delta_q(\tau) + b_q(-i\tau) \Delta_q^*(\tau) \right\} \right\rangle. \tag{5.3}
\]

In analogy to (2.11), (3.2) Eq. (5.3) may be calculated by restricting the perturbation series to connected diagrams so that \( \ln W \) is again given by closed loops. Assuming as before homogeneity (3.6), the condition (2.14) implies that to second order in \( \Delta \) and \( \Phi_q(\tau) \), the only loops are those labelled by \( \Delta \Delta^*, \Phi\Phi^*, \Delta\Phi\Delta^*\Phi^*, \Delta\Phi^*\Delta^*\Phi^* \) and \( \Delta\Delta^*\Phi\Phi^* \). Here the last two loops may be considered as renormalization of \( G(k) \) due to Coulomb effects and will be neglected although their influence on \( T_c \) is not obviously unimportant. The contribution of the first two loops to the effective action \( S_{\text{eff}} \) defined by \( Z/Z_0 = \text{const} \int d^2 \Delta d^2 \Phi \exp(-S_{\text{eff}}) \) being by Eqs. (3.12) and (4.2), the third loop then is the only that remains to be calculated. One finds

\[
S_{\text{eff}}[\Delta, \Phi] = S_{\text{attr}}[\Delta] + S_{\text{Coul}}[\Phi] - 2\beta V \sum_q J(q) |\Delta|^2 |\Phi(q)|^2, \tag{5.4}
\]

where

\[
J(q) = \frac{1}{\beta V} \sum_k G(k) G(-k) G(k-q) G(q-k). \tag{5.5}
\]

Since \( \Phi(q) \) has no immediate physical meaning we may integrate in out. This
results in an effective action \( \delta S_{\text{eff}} \) defined by

\[
\left( \prod_q \int d\Phi(q) \right) \exp \left( -S_{\text{eff}}[\Delta, \Phi] \right) = \text{const} \exp \left( -S_{\text{eff}}(\Delta) - \delta S_{\text{eff}}(\Delta) \right). \tag{5.6}
\]

Here we have transformed the original functional integral over \( \Phi_q(t) \) into discrete integrations over the variables \( \Phi(q) \) which according to Eq. (4.4) is the same up to a constant. Taking into account the relation (4.5) one finds

\[
\delta S_{\text{eff}}(\Delta) = \sum_q \ln \left( \frac{\beta V}{\pi} \left[ \frac{1}{U RPA(q)} + 2J(q) |\Delta|^2 \right] \right). \tag{5.7}
\]

Adding to this the expression (3.12) we finally arrive at the result

\[
S_{\text{eff}}(\Delta) = \beta V \left[ \frac{1}{g} - L(0) + Q \right] |\Delta|^2 + \sum_q \ln \frac{\beta V}{\pi U RPA(q)} \cdot \tag{5.8}
\]

Here

\[
Q(T) = \frac{2}{\beta V} \sum_q J(q) U RPA(q) \tag{5.9}
\]

is the renormalization of \( T_c \) due to the Coulomb effect as expressed by the condition

\[
L(0) = \frac{1}{g_{\text{ren}}} = \frac{1 + g Q(T_c)}{g}. \tag{5.10}
\]

This renormalization depresses \( T_c \) for \( Q > 0 \) and hence is of the McMillan-type (see Ref. 1), but, as effective attraction.

In the one-band case we may neglect the plasmon contribution (4.11) and adopt the Hubbard approximation \( U_{RPA} - U = \text{const} \) for Eq. (4.10). Since from Eqs. (5.5) and (3.3) it follows that

\[
\sum_q J(q) = \beta V [L(0)]^2 \tag{5.11}
\]

one deduces from (5.9, 10)

\[
\frac{1}{L(0)} = g_{\text{ren}} = \frac{g}{1 + 2U[L(0)]^2g} = g (1 - 2U[L(0)]^2g) = g - 2U. \tag{5.12}
\]

This is exactly the reduction of \( T_c \) one would expect in a negative-\( g \), positive-\( U \) Hubbard model (the factor of 2 is due to spin).

When there are two bands one first has to decide which carriers become superconducting; we assume that these are the light ones. Then transforming the Matsubara sum over \( q_0 \) in (5.9) into a contour integral and deforming the latter so as to enclose the real \( \omega \)-axis on both sides of the origin, one finds using (4.15) (see, e. g. Sec. 10 of Ref. 11)

\[
Q = \frac{1}{\pi N_L(0)V} \sum_q \int_{-\infty}^{\infty} n_0(\omega) \Im \{u_L(\omega) J_L(q, \omega)\} \tag{5.13}
\]

discussed in the Introduction, it does not take into account retardation effects in the
where \( n_0(\omega) = (e^{\beta \omega} - 1)^{-1} \) is the Bose distribution function and \( \mathcal{S}(\omega) \equiv [f(\omega + i\epsilon) - f(\omega - i\epsilon)]/2i \). To proceed further we first must evaluate the \( k_0 \)-sum in Eq. (5.5); it yields

\[
J_L(q, \omega) = J_L(q, -\omega) = \frac{1}{V} \sum_k \frac{\tanh(\beta \xi_{Lk}/2)}{2\xi_{Lk}^2 \xi_{Lk} - q} \times \\
\times \left\{ \frac{\xi_{Lk} \xi_{Lk} - q}{(\xi_{Lk} + \xi_{Lk} - q)^2 - \omega^2} - \frac{\xi_{Lk} - \xi_{Lk} - q}{(\xi_{Lk} - \xi_{Lk} - q)^2 - \omega^2} \right\}.
\] (5.14)

Since both \( u_L \) and \( J_L \) are even functions in \( \omega \), Eq. (5.13) may be cast into the form

\[
Q = \frac{1}{\pi 2N_L(0)V} \sum_q \int_0^\infty d\omega \coth(\frac{\beta \omega}{2}) \mathcal{S} [u_L(|q|, \omega)J_L(q, \omega)].
\] (5.15)

\( Q \) naturally splits into two parts, \( Q = Q_r + Q_i \), containing \( u_L \mathcal{S} J_L \) and \( \mathcal{S} u_L J_L \), respectively. Since \( J_L \), Eq. (5.14), has only simple poles and \( \mathcal{S}(\omega - \omega_0)^{-1} = -\pi \delta(\omega - \omega_0) \) one finds, setting \( q = k + k' \),

\[
Q_r = \frac{1}{2N_L(0)V^2} \sum_{kk'} \frac{\tanh(\beta \xi_{Lk}/2)}{\xi_{Lk} \xi_{Lk'}} \begin{pmatrix} \beta(\xi_{Lk} + \xi_{Lk'}) \\ \coth(\xi_{Lk} + \xi_{Lk'}) \end{pmatrix} \times \\
\times \left\{ u_L(|k + k'|, \xi_{Lk} + \xi_{Lk'}) - \coth(\xi_{Lk} + \xi_{Lk'}) \frac{\beta(\xi_{Lk} + \xi_{Lk'})}{2} u_L(|k + k'|, \xi_{Lk} - \xi_{Lk'}) \right\}.
\] (5.16)

Making use of the identity \( \coth(\alpha \pm \beta) = [1 \pm \tanh(\alpha \tanh(\beta)]/[\tanh(\alpha \pm \tanh(\beta)] \) we obtain after some algebra

\[
Q_r = \frac{1}{2N_L(0)V^2} \sum_{kk'} \frac{1}{\xi_{Lk} \xi_{Lk'}} \times \\
\times \left\{ \left[ 1 + \frac{\tanh(\beta \xi_{Lk}/2)}{\tanh(\beta \xi_{Lk}/2)} \right] u_L(|k + k'|, \xi_{Lk} + \xi_{Lk'}) - \\
- \left[ 1 - \frac{\tanh(\beta \xi_{Lk}/2)}{\tanh(\beta \xi_{Lk}/2)} \right] u_L(|k + k'|, \xi_{Lk} - \xi_{Lk'}) \right\}.
\] (5.17)

Transforming the double sum in Eq. (5.17) into integrals over the variables \( x = \xi_{Lk}/\mu_L \) and \( x' = \xi_{Lk'}/\mu_L \) and approximating the density of states by \( N_L(0) \) yields the expression
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\[ Q_r = \frac{N_L(0)}{2} \int \frac{dx dx'}{x x'} \left\{ \left[ 1 + \tanh \frac{x}{x_o} \tanh \frac{x'}{x_o} \right] u_L(x, x', x + x') - \left[ 1 - \tanh \frac{x}{x_o} \tanh \frac{x'}{x_o} \right] u_L(x, x', x - x') \right\} \]

(5.18)

where \( w u_L \) is the bandwidth and the average means average over the angle between \( k \) and \( k' \) and is defined as

\[ (u_L(x, x', y)) = \frac{1}{2} \frac{\lambda d \lambda}{\sqrt{1 + x} \sqrt{1 + x'}} \frac{\lambda d \lambda}{\sqrt{1 + x} \sqrt{1 + x'}} u_L(\lambda k_L, y \mu_L) \]

(5.19)

where \( \lambda = |k + k'| / k_L \). For \( Q_r \) the same parametrization may be used.

It turns out that, in spite of the existence of an acoustic plasmon mode, the pole-approximation of Eqs. (4.16-18) is not good enough, giving mainly positive values for \( Q_r \) that is, decreasing \( T_c \). An evaluation taking into account the explicit form (4.8) in Eq. (4.15), however, goes beyond the scope of this paper and will be considered elsewhere.

References