On ill-posedness of nonparametric instrumental variable regression with convexity constraints

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Reference


DOI: 10.1111/ectj.12071
ON ILL-POSEDNESS OF NONPARAMETRIC INSTRUMENTAL VARIABLE REGRESSION WITH CONVEXITY CONSTRAINTS

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First draft: January 2016.

Abstract

This note shows that adding monotonicity or convexity constraints on the regression function does not restore well-posedness in nonparametric instrumental variable regression. The minimum distance problem without regularisation is still locally ill-posed.

JEL Classification: C13, C14, C26.

Keywords: Nonparametric Estimation, Instrumental Variable, Ill-Posed Inverse Problems.

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*Acknowledgements: We thank Xiaohong Chen and Daniel Wilhelm for helpful comments.
1 Introduction

We consider estimation of the regression model $Y = \varphi_0(X) + U$, where the error $U$ is independent of the instrument $Z$. The variable $X$ has compact support $\mathcal{X} = [0, 1]$ and is potentially endogenous. The instrument $Z$ has compact support $\mathcal{Z} = [0, 1]$. The parameter of interest is the function $\varphi_0$ defined on $\mathcal{X}$ which satisfies the nonparametric instrumental variable regression (NIVR):

$$E[Y - \varphi_0(X)|Z] = 0.$$  \hfill (1)

NIVR estimation has received considerable attention in the recent years building on a series of fundamental papers on ill-posed endogenous mean regressions (Ai and Chen (2003), Newey and Powell (2003), Hall and Horowitz (2005), Blundell, Chen, and Kristensen (2007), Darolles, Fan, Florens, and Renault (2011), Horowitz (2011)), and the review papers by Carrasco, Florens, and Renault (2007). The main issue in nonparametric estimation with endogeneity is overcoming ill-posedness of the associated inverse problem. It occurs since the mapping of the reduced form parameter (that is, the distribution of the data) into the structural parameter (that is, the instrumental regression function) is not continuous. We need a regularization of the estimation to recover consistency. For example, Gagliardini and Scaillet (GS, 2012a) study a Tikhonov Regularized (TiR) estimator (Tikhonov (1963a,b), Groetsch (1984), Kress (1999)). They achieve regularization by adding a compactness-inducing penalty term, the Sobolev norm, to a functional minimum distance criterion. As discussed in Horowitz and Lee (2015), economic theory often provides shape restrictions on functions of interest in applications, such as monotonicity, convexity, non-increasing (non-decreasing) returns to scale but economic theory does not provide finite-dimensional parametric models. This motivates nonparametric estimation under shape restrictions. Since nonparametric estimates are often noisy, shape restrictions helps to stabilize nonparametric estimates without imposing arbitrary restrictions (Blundell, Horowitz, and Parey (2012)). Following that line of thought, we could hope that adding monotonicity or convexity constraints on the regression function would help to restore well-posedness in nonparametric instrumental variable regression. The next section shows that this is unfortunately not the case since the minimum distance problem without regularisation is still locally ill-posed. Chetverikov and Wilhelm (2015) look at imposing two monotonicity conditions: (i) monotonicity of the regression function $\varphi_0$ and (ii) monotonicity of the reduced form relationship between the endogenous regressor $X$ and the instrument $Z$ in the
sense that the conditional distribution of $X$ given $Z$ corresponding to higher values of $Z$ first-order stochastically dominates the same conditional distribution corresponding to lower values of $Z$ (the monotone IV assumption). They show that these two monotonicity conditions together significantly change the structure of the NPIV model, and weaken its ill-posedness. In particular they point out that, even if well-posedness is not restored, those two monotonicity constraints improve the rate of convergence and can have a significant impact on the estimator finite sample behavior. Chen and Christensen (2013) show that imposing shape restrictions only is not enough to improve convergence rates.

2 Ill-posedness with convexity constraints

The functional parameter $\varphi_0$ belongs to a subset $\Theta$ of the Sobolev space $H^2(\mathcal{X})$ of order 2, that is the completion of $\{\varphi \in C^2(\mathcal{X}) : \|\varphi\|_H < \infty\}$ w.r.t the Sobolev norm $\|\varphi\|_H := \langle \varphi, \varphi \rangle_H^{1/2}$, where $\langle \varphi, \psi \rangle_H := \sum_{s=0}^{2} \langle \nabla^s \varphi, \nabla^s \psi \rangle$, is the Sobolev scalar product, and $\langle \varphi, \psi \rangle = \int \varphi(x)\psi(x)dx$.

We assume the following identification condition.

Assumption 1: $\varphi_0$ is the unique function $\varphi \in H^2(\mathcal{X})$ that satisfies the conditional moment restriction (1).

We refer to Newey and Powell (2003), Theorems 2.2-2.4, for sufficient conditions ensuring Assumption 1.

Let us a nonparametric minimum distance approach for $\varphi_0$. This relies on $\varphi_0$ minimizing

$$Q_\infty(\varphi) := E \left[ m(\varphi, Z)^2 \right], \ \varphi \in H^2(\mathcal{X}),$$

where $m(\varphi, Z) = E[Y - \varphi(X)|Z]$. We can write the conditional moment function $m(\varphi, z)$ as:

$$m(\varphi, z) = (A\varphi)(z) - r(z) = (A\Delta\varphi)(z),$$

with $\Delta\varphi := \varphi - \varphi_0$, and where the linear operator $A$ is defined by $(A\varphi)(z) := \int \varphi(x)f_{X|Z}(x|z)dx$ and $r(z) := \int yf_{Y|Z}(y|z)dy$, where $f_{X|Z}$ and $f_{Y|Z}$ are the conditional densities of $X$ given $Z$, and $Y$ given $Z$. Assumption 1 on identification of $\varphi_0$ holds if and only if operator $A$ is injective. Further, we assume that $A$ is a bounded operator from $L^2(\mathcal{X})$ to $L^2(Z)$, where $L^2(Z)$ denotes the $L^2$ space of square integrable functions of $Z$ defined by scalar product $\langle \psi_1, \psi_2 \rangle_{L^2(Z)} = E[\psi_1(Z)\psi_2(Z)]$. 

The limit criterion (2) becomes

\[ Q_\infty(\varphi) = \langle A\Delta \varphi, A\Delta \varphi \rangle_{L^2(Z)}, \]

(4)

**Assumption 2:** The linear operator \( A \) from \( L^2(X) \) to \( L^2(Z) \) is compact.

Assumption 2 on compactness of operator \( A \) holds under mild conditions on the conditional density \( f_{X|Z} \) (see e.g. GS).

The following proposition shows that the minimum distance problem above is locally ill-posed (see e.g. Definition 1.1 in Hofmann and Scherzer (1998)) even if we consider monotonicity, monotonicity nonnegativity, or convexity constraints. There are sequences of increasingly oscillatory functions arbitrarily close to \( \varphi_0 \) that approximately minimize \( Q_\infty \) while not converging to \( \varphi_0 \). In other words, function \( \varphi_0 \) is not identified in \( \Theta \) as an isolated minimum of \( Q_\infty \). Therefore, ill-posedness can lead to inconsistency of the naive analog estimators based on the empirical analog of \( Q_\infty \). In order to rule out these explosive solutions, we can use penalization as in GS (see Gagliardini and Scaillet (2012b) for the quantile regression case).

**Proposition 1** Under Assumptions 1 and 2, even if we consider monotonicity, monotonicity nonnegativity, or convexity constraints, the minimum distance problem is locally ill-posed, namely for any \( r > 0 \) small enough, there exist \( \varepsilon \in (0, r) \) and a sequence \( (\varphi_n) \subset B_r(\varphi_0) := \{ \varphi \in L^2(X) : \| \varphi - \varphi_0 \| < r \} \) such that \( \| \varphi_n - \varphi_0 \| \geq \varepsilon \) and \( Q_\infty(\varphi_n) \to Q_\infty(\varphi_0) = 0 \).

**Proof:** The proof of Proposition 1 gives explicit sequences \( (\varphi_n) \) generating ill-posedness when \( \varphi_0 \) satisfies monotonicity, monotonicity nonnegativity, or convexity constraints.

Let us build \( \varphi_n = \varphi_0 + \varepsilon \psi_n, \varepsilon > 0 \), where \( \psi_n(x) := -(2n + 1)^{1/2}(1 - x)^n \) and \( \varphi_0 \) is monotone and increasing. Then \( \varphi_n \in H^2(X) \) and \( \nabla \varphi_n \geq 0 \). Since \( \| \psi_n \| = 1 \), when we choose \( \varepsilon > 0 \) sufficiently small, we have \( (\varphi_n) \subset B_r(\varphi_0) \), and \( \varphi_n \to \varphi_0 \). Since \( A \) is compact and \( (\varphi_n) \) is bounded, the sequence \( A\varphi_n \) admits a convergent subsequence \( A\varphi_m(n) \to \xi \). Since the weak limit is unique, we have \( \xi = A\varphi_0 \). Thus \( A\varphi_m(n) \to A\varphi_0 \) and \( Q_\infty(\varphi_m(n)) \to 0 \) but \( \| \varphi_m(n) - \varphi_0 \| \geq \varepsilon \), hence the stated result follows.

The above argument works also with the function \( \psi_n(x) := \frac{(2n + 1)^{1/2}}{(2n+1)^{1/2} - 1}(1 + x)^n \), which yields a monotone nonnegative function \( \varphi_n \in H^2(X) \) if \( \varphi_0 \geq 0 \) and is monotone. This shows that the positivity constraint does not help here either.
Since $\nabla^m \psi \geq 0$, $m \geq 1$, this example also shows that positivity constraints on higher-order derivatives $\nabla^m \varphi_0 \geq 0$, such as a convexity constraint $\nabla^2 \varphi_0 \geq 0$, does not restore well-posedness of the estimation problem in the nonparametric IV regression setting.

References


