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Received October 2006; received in revised form March 2007; accepted 12 March 2007

Abstract

In the standard model for insurance demand, the risk is totally exogenous and the insurance premium is paid for out of riskless wealth. This model yields results that are mostly in contradiction to everyday observation and have been used to question the pertinence of expected utility theory on which the model is based. For some years now, several papers have made attempts to provide foundations for a theory of insurance demand that leads to less provocative comparative statics results. In these papers, the risk for which coverage is sought becomes endogenous, and the decision to purchase insurance is made simultaneously with the decision on how much to invest in insurable assets. All these papers use a standard financial investment framework. This paper offers a contribution to this literature by using a slightly different framework: the case of a firm exposed to an insurable risk affecting return on a real investment project. The model is kept simple by using a two-state environment. It yields results that are both more complete and more general than results in previous work with the same motivation.

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\textit{JEL classification:} D21; D80; G22

\textit{Keywords:} Insurance coverage; Risk aversion; Normality; Giffen good; Actuarially fair premium

1. Introduction

In the economic theory of insurance, optimal insurance purchasing is often considered from the point of view of an individual faced with the possibility of losing part of his wealth due to an external event. The composition of wealth (its allocation among safe and risky assets) is already given, and the possible loss amount is therefore totally exogenous. The insurance premium is paid out of sure wealth, which is assumed available in a sufficient quantity; see Mossin (1968) for the standard model, and Schlesinger (2000) for a review of insurance demand theory. In this setting, as is well known, an expected utility maximizer purchases full insurance if the premium is “fair”, that is, equal to the actuarial value of the loss, or if only a fixed loading is added to the actuarial premium; he prefers partial coverage if the premium incorporates a proportional loading factor, which is the most general case.\textsuperscript{1} This simple model, based on a strong assumption, is also known to yield some provocative comparative statics results. In particular, it turns out that insurance is an inferior good under decreasing absolute risk aversion (DARA) — the most commonly accepted preference assumption.\textsuperscript{2} Being inferior, insurance may also be Giffen (see Hoy and Robson (1981), Briys et al. (1989)). Other comparative statics analyses yield ambiguous results. For example, it is not possible to sign the effect of an increase in the loss probability on insurance demand when the individual has DARA preferences and the insurer

\textsuperscript{1} The optimal behavior of a non-expected utility maximizer is considered in Schlesinger (1997). Doherty and Schlesinger (1983) analyze the demand for insurance by an expected utility maximizer in presence of an uninsurable background risk.

\textsuperscript{2} The result is weakened if changes are introduced in the basic model, for example if saving and insurance decisions are made jointly (see Dionne and Eeckhoudt (1984)).
adjusts the premium to take the increased loss probability into account (see Jang and Hadar (1995)).

This paper departs from the standard setting by considering a case where the insured risk and insurance coverage are determined simultaneously. Thus, the risk is endogenous, with the amount at risk being influenced by the presence of insurance and the conditions of its availability (price, contract form, etc.). The case considered hereafter is specific but quite simple, and it illustrates nicely the issues under discussion. It leads to results that depart significantly from results obtained in the traditional framework. In this case, a firm (or a department in a firm) is endowed with a real investment project which is risky in the sense that part of the project income may be lost due to an external insurable event. The firm is budget constrained: the amount available to perform the investment, which includes the firm’s borrowing capacity, is given at the outset. For this reason, more insurance coverage means less funds available for the investment itself. The insurance decision and the decision about the size of the project are made simultaneously.

In the last few years, several papers have also attempted to depart from the traditional insurance model by considering endogenous risks: see Meyer and Ormiston (1995), Eeckhoudt et al. (1997), Meyer and Meyer (2004). In the first of these papers, the individual is endowed with a certain amount of risk-free wealth and with a risky asset, and insurance can only be paid for by selling a part of the risky asset. Hence, the model provides the opposite pole to Mossin’s model, in which the individual is also endowed with risk free wealth and a risky asset, but where insurance is only paid for out of risk free wealth. Thus, in contrast to Mossin, insurance coverage and the amount at risk are determined simultaneously. Meyer and Ormiston use a continuous state environment to avoid corner solutions, but it turns out that the change in the source of payment for insurance does not affect the results significantly: in particular, it is still true that insurance is an inferior good under DARA.  

Eeckhoudt et al. (1997) study the case of an individual who simultaneously allocates some non-random wealth to the purchase of a safe asset, a risky asset and an insurance product on the latter asset. Again, in this model, the investment in the risky asset is endogenous, and the risk is determined by a continuous probability density function over finite limits. Both purchases of the risky asset and the insurance premium are paid for from riskfree wealth, but at the optimal portfolio it is not necessarily true that all wealth is spent on the risky asset and insurance coverage. If the insurance indemnity is not linear, it may be optimal to keep some wealth in the risk free asset while purchasing insurance. This is in contrast to the model presented in this paper, where investment in the risk free asset is dominated by the insured real investment: here, all the available budget is allocated to the risky investment project, whether some insurance is purchased simultaneously or whether no insurance is optimal. The generality of the model considered in Eeckhoudt et al. does not allow for clear comparative statics results to be derived, in most cases. In particular, concerning the effect of an increase in wealth on optimal decisions, and assuming DARA preferences, the only case that can be ruled out is a simultaneous decrease in the amount of the risky asset held and increase in optimal insurance coverage: insurance cannot be normal when the risky asset is inferior. But all other cases are possible.

Recently, Meyer and Meyer (2004) reconsidered the demand for insurance by studying the case where the individual is endowed with a composite portfolio of risky and riskless assets in fixed proportion. The amount allocated to this composite portfolio is allowed to change, but the relative proportions of the two assets held in the portfolio are given. The risky asset is assumed to be insurable. Under these peculiar circumstances, the comparative statics of the demand for insurance become more reasonable. In particular, insurance becomes normal whenever relative risk aversion is nondecreasing, a characteristic that is satisfied by a wide range of often used utility functions, including many that are decreasingly absolute risk averse. In addition, Meyer and Meyer prove that insurance is ordinary (not Giffen) if relative risk aversion is less than or equal to one, a condition already derived in previous work based on the standard model (Hoy and Robson, 1981) as well as in Eeckhoudt et al. (1997). Combined with their result about normality, this implies that relative risk aversion must be both decreasing and larger than one to allow for insurance demand to increase in reaction to an increase in the insurance price.

Note that the condition “relative risk aversion less than or equal to one” is also pivotal in Meyer and Meyer (2004) – as well as in Meyer and Ormiston (1995) and in Eeckhoudt et al. (1997) – to determine whether insurance demand will increase or not in reaction to an increase in the size of loss (first order stochastic dominance shift). This condition is often met in the economics of uncertainty, but it is somewhat restrictive. The model presented below yields comparative statics results that are independent of this condition. Compared to previous work on insurance demand with endogenous risk, it is also distinctively simpler as it is derived in a two-state framework.

The model is presented in Section 2, where the optimality conditions for the degree of insurance coverage are derived and discussed. Section 3 focuses on the comparative statics concerning the normality and ordinance of insurance demand. Then, in Section 4, additional comparative statics results are obtained, with an emphasis on the reaction of insurance demand to an increase in the probability of loss when the insurer reacts

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3 See also an earlier attempt by Eeckhoudt and Venezuelan (1990). These authors consider a two-state model in which an individual invests in the riskless asset and purchases simultaneously a risky asset and a co-insurance contract on this asset. As the riskless asset and the insured risky asset are substitutes, the model returns mostly corner solutions. For this reason, it is not well suited to the study of comparative statics.

4 This result from the implicit assumption about the allocation of wealth increases: they are fully allocated to the riskless asset, without any impact on the amount available to invest in the risky asset (and insurance): see in particular footnote 4 in the paper.

5 Most of the empirical work about the value of relative risk aversion considers that a coefficient value larger than one is more acceptable: see Friend and Blume (1975) for an early work and Levy (1994) for more recent results.
simultaneously by increasing the insurance price. Section 5 concludes.

2. The model

Consider a firm endowed with a real investment project. The size of the project is \( x \). This is chosen by the firm. The project is not risky from a financial point of view. For a project of size \( x \), the revenue is known and equals \( x \). However, the project is risky in the sense that it may fail partially due to some external event: accident, explosion, natural catastrophe, theft, sabotage, etc. For a project of size \( x \), the proportional loss is \( ax, a < 1 \). It occurs with probability \( p \).

The external event is insurable, and it will be assumed that the firm is willing to insure, provided the cost is not prohibitive. The willingness to insure can be justified either by the size of the firm (a closely-held corporation), or by other considerations such as agency and bankruptcy costs (see Mayers and Smith (1982), financing costs (Froot et al., 1993), and non-linear taxes (McMinn and Garven, 2000).

To keep the model simple and comparable to models of insurance demand by an individual, we will abstract from taxes and transaction costs. We will assume a small closely-held corporation displaying risk aversion.\(^6\)

The firm insures for an amount \( c \), under the usual institutional constraints of non-negative insurance and no over-insurance: \( 0 \leq c \leq ax \). If the optimal value of \( c \) is \( c^* < ax \), insurance is partial. The revenue from the investment is then \( x \) with probability \( (1-p) \), and \( (1-a)x + c < x \) with probability \( p \). Under full insurance, it is \( x \) with probability 1.

The firm is also endowed with initial funds \( w \). This includes initial wealth and the borrowing capacity from banks, which is limited. For the sake of simplicity, the interest rate is equal to zero, for lending and borrowing. The funds are used to implement the project, at a cost \( vx + \pi c \), and to pay for the insurance premium \( \pi c \), \( \pi < 1 \). If insurance is full, \( c^* = ax \), the total cost from the project is thus \( (v + a\pi)x \), and the riskless revenue is \( x \). For the firm to get an interest in full insurance, we must have \( v + a\pi \leq 1 \), which will be assumed throughout. Otherwise, the fully insured project would be dominated by lending the funds at the zero interest rate.

Note, that if \( v + a\pi < 1 \) is observed, a money machine cannot be created due to limited borrowing.\(^7\) Note, further, that the rate of return on the investment project need not exceed a hurdle rate larger than the riskless rate, since the only risk faced by the project is the possible loss from the occurrence of the external event, which is insurable (the risk is idiosyncratic).

Formally, we have:

\[
\text{Max } U(x, c) = (1-p)u(x) + pu((1-a)x + c)
\]
\[
s.t. w \geq vx + \pi c \quad \text{and} \quad 0 \leq c \leq ax.
\]

Before proceeding with the optimality conditions, let us make sure that the project will indeed be undertaken, given the above conditions.

**Lemma 1.** As \( v + a\pi \leq 1, x^* = 0 \) is excluded.

**Proof.** The condition for \( x^* = 0 \) (no investment, where an asterisk denotes an optimal value) is: \( U(x, c) < u(w) \) for all \( x > 0, 0 \leq c \leq ax \), with \( vx + \pi c \leq w \).

This must hold, in particular, for \( c = ax \), yielding \( u(x) < u(w) \), i.e., \( x < w \) for all \( x > 0 \), with \( x(v + a\pi) \leq w \).

As strictly positive values of \( x \) exist such that \( x \geq w \) and \( x(v + a\pi) \leq w \) as long as \( v + a\pi \leq 1 \), a contradiction arises and the lemma is proved.  

The Lagrangian is:

\[
L(x, c, \delta_1, \delta_2) = (1-p)u(x) + pu[(1-a)x + c] \\
+ \delta_1(w - vx - \pi c) + \delta_2(ax - c)
\]

with first order Kuhn–Tucker conditions:

\[
\frac{\partial L}{\partial x} = (1-p)u'(x^*) + p(1-a)u'[(1-a)x^* + c^*] \\
- \delta_1^* v + a\delta_2^* = 0
\]

\[
\frac{\partial L}{\partial c} = pu'[1 - (a)x^* + c^*] - \delta_1^* \pi - \delta_2^* \leq 0
\]

\[
\frac{\partial L}{\partial \delta_1} = w - vx^* - \pi c^* \geq 0
\]

\[
\frac{\partial L}{\partial \delta_2} = ax^* - c^* \geq 0
\]

and complementary slackness conditions:

\[
c^* \frac{\partial L}{\partial c} = 0, \quad \delta_1^*(w - vx^* - \pi c^*) = 0 \quad \text{and} \quad \delta_2^*(ax^* - c^*) = 0
\]

where the \( \delta_1^* \)'s are Lagrange multipliers and asterisks denote optimal values. Condition (1a) sets the optimal \( x > 0 \). Condition (1b) reflects the fact that \( c \) is constrained from below, as stated by (1e). Either (1b) is null for \( c^* \geq 0 \), or it is negative for all \( c \geq 0 \). In this latter case, \( c \) is decreased (to increase \( L \)) until it reaches the value 0. For \( c = 0 \), \( \frac{\partial L}{\partial \delta_1} < 0 \) and \( c \) is at a constrained optimum. Conditions (1c) and (1d) guarantee that the remaining constraints are satisfied in equilibrium. Conditions (1f) and (1g) and the slackness conditions are the usual Kuhn–Tucker constraints.

From (1a), we observe that \( \delta_1^* > 0 \), necessarily. Hence, the budget constraint is binding:

\[
w = vx^* + \pi c^*.
\]

The constraints concerning optimal insurance purchasing, \( c^* \), require more comments.

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\(^6\)The case of a department with some autonomy in a large firm could also be considered. The department is entrusted with the project and assumes responsibility for the end-result.

\(^7\)If full insurance was optimal in this case, we could expect a long run adjustment of \( v \) to satisfy \( v + a\pi = 1 \), due to many firms trying to implement the same non-risky and profitable investment. But, as shown below, full insurance is not optimal in this case.
2.1. Constrained full insurance

From (1a) and (1b), the condition for a full insurance corner solution, where \( c^* = ax^* \), \( \delta^*_2 > 0 \) and \( (\partial L/\partial c) = 0 \), is:

\[
\pi < \frac{vp}{1-ap}.
\]

(2)

This can be rewritten \( \pi < p (v + a\pi) \). As \( v + a\pi \leq 1 \), this is impossible unless insurance is more than fair (\( \pi < p \)), which will be assumed away from now on.

2.2. Partial and full insurance

The condition for an interior solution, where \( 0 \leq c^* \leq ax \), \( \delta^*_2 = 0 \) and \( (\partial L/\partial c) = 0 \), is:

\[
\pi \geq \frac{vp}{1-ap}.
\]

(3)

This corresponds to partial insurance, with full insurance and no insurance as non-constrained limiting cases. The condition leads to \( \pi \geq p(v + a\pi) \), which is necessarily observed under fair or loaded insurance (\( \pi \geq p \)).

8 Non-constrained full insurance is only observed under the two simultaneous conditions: \( v + a\pi = 1 \) and \( \pi = p \). Indeed, four cases may be considered:

(1) Fair insurance and \( v + a\pi = 1 \). In this case, full insurance is optimal, and the firm earns the riskless rate of return on its investment.

(2) Fair insurance and \( v + a\pi < 1 \). In this case, partial insurance is optimal. The project is risky in equilibrium.

(3) Loaded insurance and \( v + a\pi = 1 \). Partial insurance is optimal. The project is risky in equilibrium.

(4) Loaded insurance and \( v + a\pi < 1 \). Partial insurance is optimal. The project is risky in equilibrium.

From there, we get our first proposition:

**Proposition 1.** (i) Partial insurance is optimal if the premium is loaded. It may also be optimal with no loading (“fair premium”).

(ii) If full insurance is optimal, the premium is fair and the firm earns the riskless rate of return in the riskless equilibrium.

As this will be useful below, note also that at any interior solution we observe the equilibrium condition:

\[
\frac{u'(x^*)}{u'(1-a)x^* + c^*} = \frac{p[v - (1-a)p]}{\pi(1-p)}.
\]

(4)

The condition implies: \( v > (1-a)\pi \). As \( v + a\pi \leq 1 \), this leads further to: \( \pi < v + a\pi \leq 1 \).

2.3. No insurance

The case of constrained zero insurance corresponds to a situation where the firm would prefer to sell insurance, if this was possible. The conditions for such a case are: \( 0 = c^* < ax^* \), \( \delta^*_2 = 0 \) and \( (\partial L/\partial c) < 0 \). Using these conditions into (1a) and (1b) yields:

\[
\frac{u'(x^*)}{u'(1-a)x^* + c^*} > \frac{p[v - (1-a)\pi]}{\pi(1-p)}.
\]

(4')

where \( x^* = w/v \), as implied by the budget constraint when \( c^* = 0 \).

Considering (4') and comparing it with (4) yields the following observations.

- Firstly, note that the LHS of these conditions are independent of \( \pi \), while the RHS is decreasing in \( \pi \). Thus, for any \( a < 1 \), we will observe a limit value of \( \pi \) — say \( \pi^* \) — such that \( c^* = 0 \) for \( \pi \geq \pi^* \). As the LHS of (4') is less than 1, the RHS should be less than 1 at \( \pi^* \), which rearranges to \( \pi^* > \frac{vp}{1-ap} \).

- Secondly, note that the LHS of (4') is decreasing in \( a \), everywhere non-negative, and valued at 1 when \( a = 0 \). On the other hand, the RHS is increasing and linear in \( a \), valued at \( \frac{v}{1-p} \) (which could be positive or negative, but is less than 1) at \( a = 0 \), and valued at \( \frac{vp}{1-p} \leq 1 \) from (3) when \( a = 1 \). Hence, we cannot get that the graph of the RHS is everywhere above the graph of the LHS, which from (4') would be a necessary and sufficient condition for \( c^*(a) > 0 \) at all levels of \( a \). So we get the following proposition:

**Proposition 2.** When insurance is demanded for some proportional loss \( a \), there always exists one value \( \bar{a} > 0 \) such that \( c^*(a) = 0 \) for all \( a \leq \bar{a} \).

Stated differently, for any value of \( a \) there is a corresponding value \( \tilde{\pi}(a) \), and (4') will hold whenever \( \pi > \tilde{\pi}(a) \) for this value of \( a \).

To summarize, partial insurance or no insurance will obtain in general, depending on the values observed for \( \pi \) and \( a \). Full insurance is a limit case where the premium is fair (necessary but not sufficient), the insured has no risk and earns the riskless rate of return.

3. Comparative statics: Normality and ordinarity

In the rest of the paper, we consider the comparative statics of the solution vector as the parameters of the system are altered. Perhaps the most interesting of these comparative statics exercises are the effects of marginal changes in \( w \) and \( \pi \) on the optimal values of \( x \) and \( c \). For this reason, we consider them first. The effects of other parameter changes will be considered in the next section.

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8 Note that the implications of condition (3) are the same, independently of the loading characteristics: proportional or lump sum.

9 Defining the RHS as \( f(\pi) \), we get: \( f'(\pi) = -\frac{p}{\pi(1-p)} < 0 \).

10 It will take the value 0 at \( a = 1 \) only if marginal utility at \( w = 0 \) approaches infinity (for example power utility).
Of course, for these comparative statics exercises, we assume an interior solution. Hence (3) holds as an inequality, and insurance is partial: \( c^* < ax^* \). The tangency condition (4) is also observed.

3.1. Normality

We begin by considering the normality of the demands for insurance and for the investment project, by analyzing the effect of a change in the firm’s budget \( w \) on the optimal values \( c^* \) and \( x^* \). Recall that a good is said to be “normal”/“inferior” if the demand for this good increases/decreases when the consumer’s income (or wealth) increases. Empirically, most goods are normal. Inferior goods are considered as exceptions, with examples primarily in the domain of food (e.g., potatoes).

In the traditional Mossin’s model (where an individual purchases insurance with the amount of the risky good being held constant), insurance is inferior under decreasing absolute risk aversion (DARA).\(^{11}\) Insurance is also inferior under DARA in the model of Meyer and Ormiston (1995), and inferior in most cases in Eeckhoudt et al. (1997). Meyer and Meyer (2004) find that insurance is normal, even under DARA, if relative risk aversion is non-decreasing. In our model of a firm dealing with a risky project, denoting absolute risk aversion by \( R_a \), we get:

**Proposition 3.** \( \frac{\partial c^*}{\partial w} \geq 0 \) as \( R_a(x^*) \geq (1-a)R_a[x^*(1-a)+c^*] \), and \( \frac{\partial x^*}{\partial w} > 0 \) always.

**Proof.** Differentiating the budget constraint with respect to \( w \) conditional on being at a solution, gives:

\[
\frac{\partial x^*}{\partial w} + \frac{\partial c^*}{\partial w} = 1.
\]

Then, differentiating the tangency condition (4) with respect to \( w \) conditional on being at a solution, defining \( y \equiv (1-a)x + c \), and simplifying, gives:

\[
\frac{\partial x^*}{\partial w} = R_a(y^*) - (1-a)R_a(y^*)\frac{\partial c^*}{\partial w}.
\]

Combining these two equations yields:

\[
\frac{\partial x^*}{\partial w} = \frac{R_a(y^*)}{\pi R_a(x^*) + [v - (1-a)\pi] R_a(y^*)} > 0
\]

where the denominator of these two expressions is positive by condition (4).

Sufficient, but not necessary, conditions for normality of insurance can be obtained from **Proposition 3**.

**Corollary 1.** Insurance is normal if \( a > \frac{R_a(y^*) - R_a(x^*)}{R_a(x^*)} \), which is less than 1 under DARA. If \( a = 1 \), then insurance is always normal.

This last result is expected. When the loss is total, income in the loss state is simply the insurance indemnity, and it is well known that under separable utility, all state contingent consumptions are normal.

**Corollary 2.** Insurance is normal if the utility function displays non-decreasing absolute risk aversion.

This result is also expected. It holds in all models of insurance demand.

**Corollary 3.** Assuming some insurance is purchased, insurance is normal if the utility function displays DARA and non-decreasing relative risk aversion.

**Proof.** From the first result in **Proposition 3**, multiplying both sides of the condition by \( x^* \) and adding \( c^* R_a(y^*) \) to each side, we get the condition:

\[
\frac{\partial c^*}{\partial w} \geq 0 \quad \text{as} \quad c^* R_a(y^*) \geq R_a(y^*) - R_a(x^*)
\]

where \( R_a \) denotes relative risk aversion. The RHS of the condition in (5) is non-positive under non-decreasing relative risk aversion. In this case, if \( c^* > 0 \), the condition is satisfied with the LHS greater than the RHS. Insurance is normal.

This third corollary mirrors the result of Meyer and Meyer (2004). However, normality of insurance occurs here for a wider range of utility types than those displaying non-decreasing relative risk aversion. From condition (5), if absolute risk aversion is high enough, the condition may be satisfied even under decreasing relative risk aversion.

3.2. Ordinarity of insurance

In the theory of consumption, a good is defined as “ordinary” if the quantity demanded decreases (increases) when the price of the good increases (decreases). If this is not the case, the good is called “Giffen”. As any normal good is ordinary, insurance is ordinary if the condition for normality of insurance in **Proposition 3** is satisfied. For example, from **Corollary 3**, it turns out that insurance is an ordinary good if relative risk aversion is non-decreasing, whatever its value. However, even if insurance is inferior, it may still turn out to be ordinary.

**Proposition 4.** (a) \( \frac{\partial c^*}{\partial \pi} \geq 0 \) as \( c^*[x^*(1-a) + c^*] - R_a(x^*) \geq 0 \), \( (a) \frac{\partial c^*}{\partial \pi} < 0 \).

\( ^{11} \)In the theory of risk aversion, based on expected utility theory, the individual’s aversion to risks of given size is measured by the coefficient of absolute risk aversion, \( R_a(y) = -\frac{u''(y)}{u(y)} \), where \( u(\cdot) \) is the utility function and \( y \) the variable providing utility — wealth in most cases. Marginal utility, \( u'(y) \), is assumed positive — in particular if \( y \) represents wealth. Thus, \( R_a \) is positive provided the decision maker exhibits risk aversion, which obtains if \( u''(y) < 0 \). Empirically, \( R_a \) is found to be decreasing (DARA), \( R_a(y) < 0 \). For example, in the theory of finance, the DARA assumption is in conformity with the general observation that risky assets are not inferior “goods”. When his/her wealth increases, the decision maker is less risk averse to risks of a given size, and his/her demand for risky assets increases. **Relative risk aversion**, \( R_a(y) = yR_a(y) \), measures the aversion to proportional risks — in fact, risks affecting a given proportion of the decision maker’s wealth. It is still a matter of debate whether relative risk aversion should be increasing, decreasing or constant. The matter was not convincingly settled by empirical observations. For this reason, the convenient assumption of constant relative risk aversion (CRRA) is most often met in the economic and financial literature.
**Proof.** See Appendix.

Proposition 4(a) provides a link between the necessary and sufficient conditions for the inferiority of insurance (Proposition 3), and the necessary and sufficient conditions to establish insurance as a Giffen good. The proposition confirms that the value of \((1 - a) R_a(x(1 - a) + c^*)\) is critical. If it is larger than \(R_a(x^*)\), then insurance is inferior. (The first term in bracket in Proposition 4(a) is positive in this case.) If it is even larger (larger than \(R_a(x^*) + \frac{c^2}{\sigma^2 + \pi^2} (1 + \alpha a - \alpha \pi)\)), then insurance is a Giffen good.

Proposition 4(b) implies that an increase in the price of insurance will always reduce the net amount \(x^*\) invested in the risky project (if some insurance is purchased). If insurance is ordinary, \(c^*\) and \(x^*\) are reduced simultaneously and appear as “complements”. However, if insurance is Giffen, an increase in the price of insurance will simultaneously reduce the amount invested in the project and increase the amount of insurance purchased for this project: oddly enough, in this particular case, insurance and the risky project will appear as “substitutes”.

4. Additional comparative statics results

We start with two intuitive results.\(^{12}\) Firstly, if the proportion of the risky investment that may be lost increases, we get the result that less of it will be implemented, and it will be more heavily insured:

**Proposition 5.** \(\frac{\partial x^*}{\partial a} < 0 < \frac{\partial x^*}{\partial \pi}\).

Secondly, we consider how the optimal solution is affected by an increase in risk aversion. Following Pratt (1964), an increase in risk aversion can be studied by substituting the utility function \(u(\bullet)\) by \(h(u(\bullet))\), where \(h\) is a strictly increasing and strictly concave function from \(R^1\) to \(R^1\).

**Proposition 6.** An increase in risk aversion will increase the amount of coverage and decrease the size of the investment.

The fact that an increase in risk aversion leads to both less risk being undertaken and more insurance being purchased is, of course, entirely natural and in accordance with logic.

Next, we can consider how a change in the probability of loss will affect the optimal choice. To begin with, we do this holding the premium constant. In the traditional model of wealth lotteries with fully endowed risk, an increase in the probability of loss which is not accompanied by an increase in the premium, and where partial insurance is optional, will always lead to an increase in the amount of coverage purchased (see Mossin (1968)). As the next proposition shows, the same is true in the current model. But, here we are also able to show that the amount allocated to the risky investment project will decrease (due to the budget constraint):

**Proposition 7.** \(\frac{\partial c^*}{\partial p} \bigg|_{\pi} > 0 > \frac{\partial x^*}{\partial p} \bigg|_{\pi}\).

\(^{12}\) The proofs are provided in the Appendix.

**Proof.** See Appendix.

However, since the insurance premium will normally be calculated as an increasing function of the loss probability, and since asymmetric information between the insurer and the firm has not been assumed in this model, it is natural to consider the premium rate \(\pi\) as an increasing function of \(p\). Thus an effect that is very closely related to the ordinarity of insurance appears when the probability of loss is changed. If insurance is ordinary, and if the demand for insurance increases with \(p\) when the premium is constant, then the overall effect will be indeterminate, since insurance will increase as the insured attempts to compensate for the increase in risk, but then it will decrease as the price of insurance is increased. Formally, with \(\pi = \pi(p)\), and \(\pi'(p) > 0\), we get:

\[
\frac{\partial c^*}{\partial p} = \frac{\partial c^*}{\partial \pi} + \frac{\partial c^*}{\partial \pi} \pi'(p). \quad (6)
\]

Since the first summand is positive from Proposition 7, the overall effect is positive if insurance is Giffen (the second summand is also positive in this case): an increase in the probability of loss increases the demand for insurance. However, if insurance is ordinary (the most general case), the second summand is negative and the overall effect is indeterminate.

The case of how the demand for insurance is affected by an increase in the probability of loss when the premium also increases was studied by Jang and Hadar (1995) – hereafter cited as JH – in a model that is very close to Mossin’s (1968) model. JH consider the effect of an increase in the probability of loss in a two-state model, retaining that the amount of insurable risk is exogenous. In this case, JH show that the effect of an increase in the probability of loss is indeterminate if the utility function displays DARA, and that the demand for insurance decreases if the utility function displays either CARA or IARA.

In order to reconsider their result when the amount at risk is endogenous, we now assume that insurance is priced as a linear function of the probability. Specifically,

\[\pi(p) = kp, \quad k \geq 1.\]

If \(k = 1\), the premium is “fair”. If \(k > 1\), the premium is loaded.

We have shown with Proposition 2 that insurance will not be purchased by the firm if \(a < \bar{a}\). From this result, we get directly our next proposition:

**Proposition 8.** \(\frac{\partial c^*(a)}{\partial p} = 0 \forall a < \bar{a}\).

In order to find out what happens in the case where insurance is positive, we need to evaluate the general effect of an increase in \(p\).

- Firstly, using \(\pi = kp\), the effect of an increase in \(p\) on the budget constraint is:

\[
0 = \frac{\partial x^*}{\partial p} + kc^* + kp \frac{\partial c^*}{\partial p}. \quad (7)
\]

From (7), it is clear that \(\frac{\partial x^*}{\partial p}\) and \(\frac{\partial c^*}{\partial p}\) cannot be simultaneously positive. They are either both negative, or one is positive and the other is negative.
- Secondly, differentiating (4) with respect to \( p \), introducing absolute risk aversion \( R_a \), and defining \( y \equiv (1 - a)x + c \), the effect of an increase in \( p \) on the tangency condition is:

\[
\frac{\partial x^*}{\partial p} \left[ (1 - a)R_a(y^*) - R_a(x^*) \right] + \frac{\partial c^*}{\partial p}R_a(y^*) = \frac{v - k(1 - a)}{(1 - p)[v - kp(1 - a)]}.
\]  

(8)

In order to say more, it is necessary to solve the two Eqs. (7) and (8) in their unknowns. Since we are primarily interested in \( \frac{\partial^2 c^*}{\partial p^2} \), we shall concentrate on that unknown.

Solving (7) for \( \frac{\partial c^*}{\partial p} \), substituting into (8), and then simplifying yields the following result:

\[
\frac{\partial c^*}{\partial p} \left[ R_a(y^*)(v - kp(1 - a)) + kpR_a(x^*) \right] = \frac{v - k(1 - a)}{(1 - p)[v - kp(1 - a)]} + kc^* \left[ (1 - a)R_a(y^*) - R_a(x^*) \right].
\]  

(9)

Since \( v - kp(1 - a) = v - \pi(1 - a) > 0 \) from (4) at an internal solution, the term in brackets on the RHS of (9) is positive. Thus, the sign of \( \frac{\partial c^*}{\partial p} \) is the same as the sign of the RHS of (9). In the RHS of (9), the sign of the second term is positive (negative) whenever insurance is inferior (normal) (see Proposition 3). Concerning the first term, its denominator is positive at an internal solution, but the numerator may be positive, negative, or equal to zero.

If we define:

\[
g(a) \equiv \frac{v[k - k(1 - a)]}{k(1 - p)[v - kp(1 - a)]}
\]

\[
h(a) \equiv -c^* \left[ (1 - a)R_a(y^*) - R_a(x^*) \right]
\]

then we have:

\[
\frac{\partial^2 c^*}{\partial p^2} \geq 0 \iff g(a) \geq 0 \iff h(a).
\]  

(10)

Let us consider the two functions \( g(a) \) and \( h(a) \) in more detail. Beginning with \( g(a) \), we have:

1. \( g(1) = \frac{v}{k(1-p)} \). As we are assuming an interior solution, (3) holds as an inequality: \( \pi > \frac{vp}{1-\alpha} \). With \( \pi = kp \), this reads \( 1 > \frac{v}{k(1-p)} \), and so at \( a = 1 \) we have \( g(1) < 1 \).

2. \( g'(a) = \left( \frac{v}{k(1-p)} \right)^2 \). Thus, we have \( g'(a) > 1 \) at all \( a < 1 \) and \( g'(a) = 1 \) at \( a = 1 \).

Therefore, the graph of \( g(a) \) is an increasing function, with slope greater than 1 at all internal points. It cuts the axis from below at the point \( a = \frac{v}{k} \) and reaches height \( \frac{v}{k(1-p)} < 1 \) at \( a = 1 \). It can be shown, in addition, that the second derivative of \( g(a) \) is negative, so that it is a strictly concave function. Such a graph is shown on Fig. 1.

Now consider the function \( h(a) \). Since this function depends upon the optimal choice, it will normally be kinked at the level of \( a \) for which the optimal insurance goes to zero. Given that this function is far more complex, we will focus on the most important constant relative risk aversion (CRRA) case. The study of more general preferences is left for future research.

To start with, we write \( h(a) \) as a function of the relative risk aversion \( R_r \). With constant relative risk aversion, \( R_r(y) = R_r(x) \equiv R \), and so we have:

\[
h(a) = -c^*R \left[ \frac{(1 - a)y^* - 1}{x^*} \right].
\]

Recalling that \( y = (1 - a)x + c \), this can be written as:

\[
h(a) = R \left[ \frac{c^*}{x^*} - \frac{(1 - a)c^*}{(1 - a)x^* + c^*} \right].
\]  

(11)

This function is flat and equal to zero for all \( c^* = 0 \), i.e. for \( a \neq \bar{a} \). For \( c^* > 0 \), it is increasing in \( a \) and convex. Moreover, the function is increasing in risk aversion \( R \) (see Appendix).

We begin with the polar case: \( R = 1 \).

A-Case \( R = 1 \)

In this case, we have logarithmic utility, and at the tangency condition (4), we get:

\[
\frac{(1 - a)x^* + c^*}{x^*} = \frac{v - pk(1 - a)}{k(1 - p)}.
\]  

(12)

First inverting this equation, then multiplying on both sides by \( (1 - a)c^*/x^* \) and using the result into (11) with \( R = 1 \), we get:

\[
h(a) = \frac{c^*}{x^*} \left( \frac{v - k(1 - a)}{v - kp(1 - a)} \right).
\]  

(11')

Then, obtaining \( c^*/x^* \) from (12) and substituting into (11') yields:

\[
h(a) = \frac{(v - k(1 - a))^2}{k(1 - p)(v - kp(1 - a))}.
\]  

(11'')

From (11''), one gets that \( h(1) = \frac{v}{k(1-p)} = g(1) \) and \( h(a) = 0 \) at \( a = \frac{v}{k} = \bar{a} \).

Thus, for the case \( R = 1 \), the function \( h(a) \) coincides with the function \( g(a) \) at the two extreme points \( a = \bar{a} \) and \( a = 1 \). It must lie below \( g(a) \) at all points where \( \bar{a} < a < 1 \) (which are all the points at which the demand for insurance is strictly positive) since it is increasing and convex while \( g(a) \) is increasing and concave. Hence the proposition:

\( \bar{a} \) It can be checked that we have indeed the critical value \( \bar{a} \) noted in Proposition 2 such that \( \bar{a} = \frac{v}{k} \) when \( R = 1 \). Noting that, in this case, \( (1 - a) = v/k \), and using this value in (12), we get \( c^* = 0 \).
Proposition 9. With relative risk aversion constant and equal to 1, if the demand for insurance is positive, then \( \frac{\partial c^*}{\partial p} > 0 \).

Further, \( \frac{\partial c^*}{\partial p} < 0 \).

The second part of the proposition is implied by (7): as already noted, \( \frac{\partial x^*}{\partial p} \) and \( \frac{\partial c^*}{\partial p} \) cannot be simultaneously positive. When one is positive, the other must be negative.

Proposition 9 is shown graphically in Fig. 2.

B-Case \( R < 1 \)

As was noted above and proved in the Appendix, \( h(a) \) is increasing in relative risk aversion \( R \). For this reason, when \( R \) decreases from 1, \( h(a) \) is shifted down in Fig. 2. The values of \( a \) for which \( c \) is positive are reduced, and \( h(a) \) lies entirely below \( g(a) \). Given (10) and (7), this leads to the following proposition:

Proposition 10. With the relative risk aversion constant and less than 1, if the demand for insurance is positive, then \( \frac{\partial c^*}{\partial p} > 0 \) and \( \frac{\partial x^*}{\partial p} < 0 \).

C-Case \( R > 1 \)

As \( R \) increases from 1, the \( h(a) \) schedule is shifted up from its position in Fig. 2. We get Fig. 3.

The values of \( a \) for which the insurance demand is positive are increased, but the effect of an increase in \( p \) on insurance demand depends on the particular value of \( a \). For low and high values of \( a \) — below a limit \( a_1 \) and above a limit \( a_2 \) — \( h(a) \) is above \( g(a) \), which means that the impact is negative: \( \frac{\partial c^*}{\partial p} < 0 \). For intermediate values of \( a \) (\( a_1 < a < a_2 \)), \( h(a) \) is below \( g(a) \), which means that the impact is positive: \( \frac{\partial c^*}{\partial p} > 0 \). However, if the relative risk aversion \( R \) is high enough — above a threshold determined by the parameters of the problem such as \( k \) and \( v \) — the \( h(a) \) schedule is entirely above the \( g(a) \) schedule, and we get that \( \frac{\partial c^*}{\partial p} < 0 \) for all \( a \).

These results are summarized in the following proposition:

**Proposition 11.** With relative risk aversion being constant and greater than 1, \( \frac{\partial c^*}{\partial p} < 0 \) in general if insurance is purchased; except for low values of risk aversion and intermediate values of \( a \), where \( \frac{\partial c^*}{\partial p} > 0 \) obtains.

For example, with \( p = 0.5, k = 1.1, v = 1 - 0.6a \) (which satisfies \( v \leq 1 - kpa = 1 - 0.55a \)) and with \( R = 2 \), we get that:

- insurance is purchased only if \( a \geq 0.155 \),
- the marginal impact of an increase in \( p \) on an insurance purchase is positive if \( a \) lies between \( a_1 = 0.18 \) and \( a_2 = 0.74 \),
- and the same marginal impact is negative if \( a \) lies outside of these values (and is nevertheless greater than 0.155).

With the same parameter values and \( R = 4 \),

- insurance is purchased if \( a \geq 0.0788 \),
- \( \frac{\partial c^*}{\partial p} > 0 \) if \( a \) lies between \( a_1 = 0.23 \) and \( a_2 = 0.35 \),
- and \( \frac{\partial c^*}{\partial p} < 0 \) for \( a \) outside of these values (and larger than 0.0788).

With these parameters, we get \( \frac{\partial c^*}{\partial p} < 0 \) for all \( a \) if \( R \geq 4.28 \).

We expect to find that \( c^* \) = 0 for a more restricted set of \( a \) values when \( R \) increases. However, it may appear counter-intuitive to find that the effect of an increase in the loss probability \( p \) is to decrease insurance purchasing when risk aversion \( R \) is high enough. The result may be explained as follows: when \( R \) increases, for a given \( p \), insurance purchasing \( c^* \) increases (Proposition 6); therefore, if the premium rate \( \pi \) increases with an increase in \( p \), the impact on the insurance bill will be more sensible; the price effect will dominate and the overall effect of an increase in \( p \) will be to reduce insurance purchasing.

5. Conclusions

In this paper, we focussed on insurance purchasing when the insured risk is endogenous by considering the case of a risk averse firm endowed with a risky investment project. The project may fail partially due to some exogenous state of nature, but the resulting loss is insurable. The firm is financially constrained by a fixed budget — including borrowed funds — to be allocated to the project. Increasing insurance coverage means having less funds available for the investment itself.

Using a very simple two-state model and mild assumptions, we obtain several interesting results. Firstly, partial insurance is optimal not only when the insurance premium is loaded, but also in general with a “fair” (unloaded) premium. In the limit case where full insurance is optimal with a fair premium, the firm earns the riskless rate of return on its insured investment. Secondly, the fraction of the investment subject to a loss plays an important role in the model. If this fraction is low enough, no insurance is optimal. With a larger fraction (and positive

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14 Starting from Eq. (A.3) in the Appendix, and assuming CRRA, it may be checked that the effect of an increase in \( \pi \) on \( c^* \) (the negative price effect) is stronger, the larger insurance purchasing \( c^* \).
insurance purchasing), the fraction value is often critical in comparative statics analysis — along with the degree of risk aversion. Thirdly, insurance is normal (and thus ordinary) for a much wider set of preferences than in the traditional insurance model. Insurance is normal if the relative risk aversion is non-decreasing (instead of the non-decreasing absolute risk aversion in the traditional model). But even with decreasing relative risk aversion, insurance is still normal if risk aversion is high enough. In addition, insurance is also normal if the proportion of the investment subject to loss is higher than the rate at which absolute risk aversion decreases. Fourthly, the effects of increases in risk were considered by analyzing how the optimal demand for insurance reacts when the proportion of the investment subject to loss increases (or when the loss probability increases). An increase in the percentage loss unambiguously increases the demand for insurance and reduces the investment — given the budget constraint. When the loss probability increases and the insurer simultaneously adjusts the premium rate to take this change into account, it turns out that the demand for insurance increases if the relative risk aversion is constant and less than or equal to 1. With the relative risk aversion being constant and larger than 1, insurance demand may still increase for some values of the percentage loss if the risk aversion is low enough. With higher risk aversion coefficients, an increase in the loss probability induces a drop in the demand for insurance. That is, the indirect effect of the premium increase outweighs the direct effect of the increase in probability. These last results extend significantly previous results obtained by Jang and Hadar (1995) on the same topic using the traditional model of insurance demand.

Obviously, the model can be extended by considering more states of nature (full loss in one state, partial loss in other states) in a discrete or continuous setting. But the tractability of the two-state model is useful and sufficient for our purpose. It allows us to set aside the unsatisfactory assumption of a purely exogenous risk and to reconsider the results of the traditional insurance demand model in the realistic case where insurance purchasing and the risk amount are simultaneously determined.

Appendix

Proof of Proposition 4. The proposition can be proved using the Slutsky equation. The proofs starting from the budget constraint and the tangency condition are quicker. Differentiating the budget constraint with respect to $\pi$, conditional on being at an interior solution, gives:

$$\frac{\partial x^*}{\partial \pi} = -c^* - \pi \frac{\partial c^*}{\partial \pi}. \quad (A.1)$$

Differentiating the tangency condition (4) with respect to $\pi$, conditional on being at an interior solution, using the Arrow–Pratt coefficient of absolute risk aversion $R_a$, and simplifying gives:

$$\frac{\partial x^*}{\partial \pi} \left[ (1 - a) R_a(y^*) - R_a(x^*) \right] + R_a(y^*) \frac{\partial c^*}{\partial \pi} = \frac{-v}{\pi [v - (1 - a)\pi]} \quad (A.2)$$

Combining the two equations yields:

$$\frac{\partial c^*}{\partial \pi} = \frac{1}{D} \left\{ c^* \left[ (1 - a) R_a(y^*) - R_a(x^*) \right] \right\}
- \frac{\pi}{\pi} \left[ \frac{v^2}{v - (1 - a)\pi} \right] \quad (A.3)$$

where $D = \pi R_a(x^*) + [v - (1 - a)\pi] R_a(y^*)$.

As $D > 0$, part (a) in Proposition 3 follows from (A.3). Then, using (A.3) in (A.1) proves part (b):

$$\frac{\partial x^*}{\partial \pi} = -\frac{1}{D} \left\{ c^* R_a(y^*) \right\} < 0. \quad (A.4)$$

Proof of Proposition 5. Differentiating with respect to $a$ from the tangency condition (4) and simplifying gives:

$$-R_a(x^*) \frac{\partial x^*}{\partial a} + R_a(y^*) \left[ (1 - a) \frac{\partial x^*}{\partial a} + \frac{\partial c^*}{\partial a} - x^* \right] = \frac{\pi}{v - \pi(1 - a)}.$$

Then, differentiating the budget constraint with respect to $a$ gives:

$$\frac{\partial x^*}{\partial a} = -\frac{\pi}{v} \frac{\partial c^*}{\partial a}.$$

Thus $\frac{\partial x^*}{\partial a}$ and $\frac{\partial c^*}{\partial a}$ are opposite in sign.

Combining the two equations leads to:

$$\frac{\partial c^*}{\partial a} = \frac{1}{D} \left\{ vx R_a(y^*) + \frac{\pi v}{v - \pi(1 - a)} \right\} > 0,$$

where $D$ is given in (A.3) above. As $(\partial c^*/\partial a) > 0$, $(\partial x^*/\partial a) < 0.$

Proof of Proposition 6. Using the utility function $\psi(x) = h[u(x)]$, with $h' > 0$ and $h'' < 0$, to represent an increase in risk aversion, the tangency condition is now:

$$\psi'(\hat{x}) = \frac{p \left[ v - \pi (1 - a) \right]}{\pi (1 - p)},$$

where $\hat{x}$ and $\hat{y}$ are the optimal values of $x$ and $y$ under utility function $\psi$.

Combining (A.5) with (4) and using the definition of $\psi$ leads to:

$$h' \left[ u(\hat{x}) \right] u'(\hat{x}) \psi'(\hat{x}) = \frac{u'(x^*)}{h' \left[ u(y^*) \right] u'(y^*)}.$$ 

As insurance is partial, $\hat{x} > \hat{y}$, implying $h' \left[ u(\hat{x}) \right] < h' \left[ u(\hat{y}) \right]$, and thus:

$$u'(\hat{x}) > u'(x^*) \quad (A.6)$$

Therefore, we can rule out that $x^* = \hat{x}$ and $y^* = \hat{y}$.

Now rewrite the budget constraint as:

$$w = vx + \pi c
= [v - \pi (1 - a)] x + \pi c + \pi (1 - a) x
= [v - \pi (1 - a)] x + \pi y
= q x + \pi y$$

where $q \equiv v - (1 - a) \pi$. 

Using this reformulated budget constraint to substitute for \( y \) in (A.6) gives:

\[
\frac{u'(\hat{x})}{u'(x^*)} > \frac{u'(y^*)}{u'(x^*)}.
\]

(A.6')

The only possibility for this inequality to hold is to have \( \hat{x} < x^* \), i.e., \( \hat{y} > y^* \) by the budget constraint. This implies \( \hat{c} > c^* \), by the definition of \( y \).

Proof of Proposition 7. Again, starting with the tangency condition (4), differentiating with respect to \( p \) and simplifying gives:

\[
R_a(y^*)\frac{\partial y^*}{\partial p} - R_a(x^*)\frac{\partial x^*}{\partial p} = \frac{1}{p(1-p)} > 0.
\]

Hence we know that \( R_a(y^*)\frac{\partial y^*}{\partial p} > R_a(x^*)\frac{\partial x^*}{\partial p} \).

However, from the budget constraint, we also know that the two partial derivatives must have opposite sign. The only possibility is \( \frac{\partial c^*}{\partial p} > 0 > \frac{\partial c^*}{\partial p} \). This leads, further, to \( \frac{\partial c^*}{\partial p} > 0 \), by the definition of \( y \).

The impact of risk aversion on the function \( h(a) \)

With constant relative risk aversion \( R \), we have: \( h(a) = R \left[ \frac{c^*}{x^* - (1-a)c^*} \right] \).

Giving the two terms in the square brackets a common denominator leads to:

\[
h(a) = R \left[ \frac{(c^*)^2}{x^*[(1-a)x^* + c^*]} \right].
\]

From there, differentiating with respect to \( R \), we have:

\[
\frac{\partial h(a)}{\partial R} = R \left[ \frac{(c^*)^2}{x^*[(1-a)x^* + c^*]} \right] + R \frac{\partial}{\partial R} \left[ \frac{(c^*)^2}{x^*[(1-a)x^* + c^*]} \right].
\]

The first term in the square brackets is positive. The second term includes a derivative which determines its sign. Using \( y \equiv (1-a)x + c \), the value of this derivative is:

\[
2c^* x^* y^* - c^* x^* y^* = c^* y^2 (x^* y^* + x^* y^*).
\]

The sign of the derivative is the same as the sign of its numerator. Dividing this numerator by \( c^* \) (which does not change the sign), expanding \( y^* \) and simplifying, the numerator can be expressed as:

\[
2(1-a)x^* + c^* \left[ x^* \frac{\partial e^*}{\partial R} - c^* \frac{\partial x^*}{\partial R} \right].
\]

As \( c^* \) is increasing in risk aversion and \( x^* \) is decreasing in risk aversion (see Proposition 6), it turns out that the sign of the numerator is positive. Thus we can conclude:

\[
\frac{\partial h(a)}{\partial R} > 0.
\]

References


Eeckhoudt, L., Venezian, E., 1990. Insurance purchases when the level of investment in a risky asset is endogenously determined (unpublished manuscript).


