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Romana Rytsar and Thierry Pun

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Keywords — EEG inverse problem, global optimization, real head model, Finite-element method.

The electrical activity inside the human brain consists of the currents generated by biochemical sources at the cellular level. This activity can be measured by an electroencephalograph (EEG) at the electrode positions. Reconstruction of the sources from a given scalp measurement can be achieved by solving the so-called EEG inverse problem. The current dipole can accurately model neural activity localized on one site, representing the coherent activation of a large number of individual neurons [Scherg et al. 1985, Snyder 1991, Fender 1987]. Each dipole is characterized by three location parameters and three values specifying the dipole moment magnitude and orientation. The inverse problem is then defined as the estimation of the location and moment parameters of the dipoles whose modeled potentials best fit the actual measurements in a least squares sense. The least-square fitting implies minimizing a cost function, which is the residual variance between the measured and modeled scalp potentials [Snyder 1991, Fender 1987, Mosher et al. 1992]. For source reconstruction in a shallow tank containing saline the current-density minimization approach has been developed using the advantages of the natural framework associated with finite-elements [Miga et al. 2002]. In this article we propose a new objective function obtained from 3-D finite-element method (FEM) discretization scheme of the Poisson’s equation with the Neumann boundary condition. This function is based on the current density boundary integral as a basis for parameter optimization. Mathematically, the minimization of the objective function is a difficult nonlinear optimization problem since the objective function is very complex. It may contain multiple equivalent solutions [Koles 1998] and it always has many local minima, especially in the case of large number of the dipole sources. Therefore, the source localization accuracy greatly depends on the optimization technique. For example, the gradient-based methods are hardly suitable for this specific problem since they easily can be trapped by the local minima [Goldberg 1989, Marquardt 1963]. The key requirement to any global optimization approach is its ability to escape from local minima and to continue the search until the optimal solution is reached. The gradient–free methods have shown good promise in solving such nonlinear estimation problems with many local minima. The simulated annealing (SA) and the genetic algorithm (GA) have been adopted for the solution of MEG inverse problem [Jiang et al. 2003].
Moreover, the simulated annealing method has been used for single-dipole estimation [Gerson et al. 1994] in modeling wave of the brain-stem auditory evoked potential and multiple-dipole estimation in the case of EEG brain source reconstruction [Khosla et al. 1997]. The SA success rate for three well-separated dipoles was indicated in the range of 95%-99% with the different noise levels (0%-10%) [Khosla et al. 1997]. However, this previous work has been limited by a spherical head model. In the present paper we provide the comparative study of the SA and the GA methods for EEG inverse problem in the case of real head model. The proposed methods were evaluated and their performances were compared using a range of computer simulations.

The EEG localization accuracy is highly sensitive to the errors in the forward problem. The forward problem is defined as modelling of the electrical potentials on the human scalp due to the dipole in the brain [Rytsar et al. 2007, Ramon et al. 2006, Awada et al. 1997]. The accuracy of the forward problem is critically determined by the assumptions concerning the shape and conductivity of the volume conductor [Rytsar et al. 2007, Ramon et al. 2006]. The human head is often approximated by three or four spherical layers representing the brain, the cerebrospinal fluid (CSF), the skull, and the scalp [Awada et al. 1997, Zhang 1995]. However, such simplification induces significant differences in the current source locations between patient-specific head models and spherical models [Roth et al. 1997, Yvert et al. 1997]. Realistic head models can improve the accuracy of inverse source localizations. More recent studies indicate the necessity of highly heterogeneous models of the head for accurate source localization [Rytsar et al. 2007, Ramon et al. 2006]. Thus, the realistic head model with five different tissues such as the scalp, the skull, the CSF, the white matter, and the gray matter has been used in our simulations.

The paper focuses on the accuracy and computational time aspects of the EEG inverse problem solution. The plan of the article is the following. Section 1 briefly presents the 3-D finite element modeling of the human head. Section 2 defines the EEG forward problem solution using FEM. In Section 3, the source reconstruction procedure is considered as an optimal solution search using the SA and the GA approaches. The computer simulations are described in Section 4 and the dipole localization results are discussed in Section 5. The final section presents the summary of our conclusions.

1. CONSTRUCTION OF THE REALISTIC FEM HEAD MODEL

The generation of the realistic FEM head model implies the segmentation of the MRI and the construction of the mesh for an arbitrary complexity head shape. The segmentation of the head tissues has been performed interactively using the 3D Doctor software package [3-D Doctor software]. The real head shape is represented as a set of 180 coronal slices. The morphological information about an individual head shape is extracted by segmentation of all images. Figure 1 shows a single slice and a segmented slice with five head compartments: the scalp, the skull, the CSF, the gray matter, and the white matter.

![Fig. 1. A single coronal slice and a segmented slice with five head compartments.](image)
From the boundary data the 3-D volume model was reconstructed using the volume-rendering. The surfaces of three segmented tissues are shown in Figure 2.

Fig. 2. Examples of surfaces extracted from MR images: scalp, gray matter and white matter surfaces (from left to right).

The second essential prerequisite of the FEM modeling is the construction of a mesh that adequately represents the geometric and electric properties of the head volume conductor. The surface geometry of the head has been depicted by 14964 triangles and 7484 nodes (see Figure 3). A mesh with first-order tetrahedral elements has been created using the HyperMesh software package [HyperMesh software].

Fig. 3. Surface-meshed and volume-meshed models.

The medium is assumed to be isotropic for simplicity. The generated realistic finite element head model includes 212439 tetrahedrons altogether, and after being labeled, 46040 tetrahedrons belong to the scalp, 37679 to the skull, 40531 to the cerebrospinal fluid, 41214 to the gray matter, and 27270 to the white matter. The total number of the nodes is 39575. Appropriate conductivity of each voxel was specified knowing the exact coordinates of all mesh nodes and the type of tissue at each of these points. Isotropic conductivities were assigned to the scalp (0.33 S/m), the skull (0.0042 S/m), the CSF (1.79 S/m), the gray matter (0.33 S/m), and the white matter (0.14 S/m) [Wolters et al. 2005].

2. THE EEG FORWARD PROBLEM

The EEG forward problem is defined as the determination of the electrical potential on the scalp of human head from a given configuration of the source in the brain, the geometry and the distribution of the electrical conductivity \( \sigma(\vec{r}) \) within the head. The above-mentioned finite element head model has been used for modeling of the scalp potentials. The source is assumed to be a dipole of amplitude \( I_0 \) in the direction \( \hat{p} \). An ideal current dipole can be described as two-point sources of opposite polarity with an infinitely large current \( I_0 \) and an infinitely small separation \( \vec{d} \). Mathematically, the current source density in the element volume \( (A/m^1) \) is defined by [Awada et al. 1997]:

\[ \sigma(\vec{r}) \frac{dI_0}{d\vec{r}} \]
\[
\rho(\vec{r}) = \lim_{|\vec{r}| \to \infty} \left[ \delta \left( \vec{r} - \vec{r}_0 - \frac{\vec{d}}{2} \right) - \delta \left( \vec{r} - \vec{r}_0 + \frac{\vec{d}}{2} \right) \right],
\]

where \( \delta(\cdot) \) denotes 3-D Dirac delta function.

Theoretically, the forward problem is governed by the Poisson’s equation for the electric potential \( U(\vec{r}) \) in the volume \( \Omega \) of the head:

\[
\nabla \cdot (\sigma(\vec{r}) \nabla U(\vec{r})) = \rho(\vec{r}) \text{ in } \Omega.
\]

At the outer boundary of the medium there exists a homogeneous Neumann boundary condition:

\[
\frac{\partial U(\vec{r})}{\partial n} = 0 \text{ on } \partial \Omega,
\]

where \( n \) is the distance measured normal to the boundary.

The numerical solution of the problem is calculated at the nodes using the FEM. Thus, the numerical approximation \( \tilde{U} \) of the solution \( U \) for a mesh consisting of \( N \) nodes is given by:

\[
\tilde{U} = \sum_{i=1}^{N} U_i H_i,
\]

where \( U_i \) is the solution at node \( i \) and \( H_i \) is the linear shape function that describes the contribution of the value at node \( i \). The potential values at the nodes \( U_i \) are obtained by solving the standard FEM system of equations [Awada et al. 1997, Bertrand et al. 1991]:

\[
[K_{ij}][U_i] = [\rho],
\]

where \([K_{ij}]\) is an \( N \times N \) matrix that depends only on the shape functions and conductivities distribution; \([\rho]\) is an \( N \)-dimensional vector in which the source function is incorporated. The elements of the stiffness matrix \([K_{ij}]\) are defined as:

\[
[K_{ij}] = \int_{\Omega^{(e)}} \sigma(\vec{r}) \left( \nabla H_i^{(e)} \right) \cdot \nabla H_j^{(e)} d\Omega^{(e)},
\]

and the \( i \)-th component \( \rho_i \) of this vector is given by:

\[
\rho_i = \int_{\Omega^{(e)}} \rho(\vec{r}) H_i^{(e)} d\Omega^{(e)} = I_o \lim_{|\vec{r}| \to \infty} \left[ H_i^{(e)} \left( \vec{r}_0 + \frac{\vec{d}}{2} \right) - H_i^{(e)} \left( \vec{r}_0 - \frac{\vec{d}}{2} \right) \right] = \vec{M} \cdot \nabla H_i^{(e)} \cdot \hat{n},
\]

where \((e)\) signifies an element subdomain, and \( \vec{M} = I_o \vec{d} \) is the dipole moment.

The accuracy and computational time of the scalp potential modeling due to the dipole has been investigated [Rytsar et al 2007, Rytsar et al 2006]. Figure 4 shows the potential values calculated at 1814 nodes on the scalp for the particular dipole when a single dipole is located in the cortex at the distance of 35.8 mm from the scalp along the z-axis.
In the case of linear shape functions equation (5) has an important implication [Awada et al. 1997, Rytsar et al 2006]. The gradients of shape functions are constant within each tetrahedral element. For a fixed dipole orientation the elements $\rho_i$ of right hand-side (7) do not change when the dipole location is varied within a tetrahedron element. As a result it becomes impossible to find the location of the dipole within an element. Thus, the forward solution is completely insensitive to dipole location within an element. This implication seriously limits the resolution in the inverse problem solution. The quadratic shape functions can improve the resolution. However, it involves adding new nodes into the mesh. Thus, the computational time of the inverse problem solution will be increased. A tradeoff should thus be accepted between computational time and location error caused by the resolution.

3. THE EEG INVERSE PROBLEM

The EEG inverse problem is defined as the reconstruction of the EEG sources from scalp potential measurements. The electrical potentials were assumed to be measured by $N_{el}$ ($N_{el}=123$) electrodes at a single time instant placed on the scalp (see Fig.5). The electrode positions were obtained from the HUG (Hospital Universitaire de Genève) for standard 123-electrodes cap.

The measured potentials can be represented by the vector $U_{mes}$ ($U_{mes} \in \mathbb{R}^{N_{el}}$). The problem is then to find assumed sources, which generate the same electrical potential values $U_{dip}$ ($U_{dip} \in \mathbb{R}^{N_{el}}$) as $U_{mes}$. The localization problem can be solved by varying the dipole parameters until the global minimum for residual energy ($RE$)
\[ RE = \| U_{\text{mes}} - U_{\text{true}} \|^2 \]  

(8)

is found. For each equivalent dipole the six parameters, namely 3 location coordinates \((x, y, z)\), 2 degrees of orientation \((\theta, \phi)\) and the dipole strength \(\vec{M}\) have to be determined by a minimization procedure.

In the framework of the FEM the optimization search can be specified as a minimization of the following objective function based on current-density boundary integral (see Appendix 1 for details):

\[ F_{\text{obj}}(x, y, z, \theta, \phi, \vec{M}) = \sum_{l=1}^{N_{\text{el}}} \left( \int_{\partial \Omega} \hat{n} \cdot \vec{J}_l \, ds \right)^2, \]

(9)

where \(\vec{J}_l\) and \(\vec{H}_l\) are a current density and a shape function, respectively, associated with \(l\)-th node at the boundary. The optimization objective is to vary the modeled dipole such that the Neumann boundary condition to be satisfied. Thus, the EEG source localization can be considered as a nonlinear optimization problem. Mathematically, this is a very difficult nonlinear optimization problem due to dimension and complexity of the objective function. It was shown [Koles 1998] that this function may contain multiple equivalent solutions. It always has many local minima, especially in the case of linear interpolation of the electrical potentials over the tetrahedral elements and when the number of dipole sources is large.

The cost function (9) depends nonlinearly on the dipole location while linearly on the dipole parameters. In the case of residual energy (8) the computational complexity of the EEG inverse problem can be greatly reduced by separating the linear and nonlinear parameters [Mosher et al. 1992, Jiang et al. 2003]. Using the gain matrix \(G\) the cost function (8) can be expressed as:

\[ RE_{\text{sep}} = \| U_{\text{mes}}^T (I_m - GG^+) U_{\text{mes}} \| \]

(10)

where \(G^+\) is Moore-Penrose pseudo-inverse of matrix \(G\). The cost function (10) does not depend on the dipole moment parameters. Thus, the total number of searching parameters for one dipole is reduced from six to three location parameters. In this paper we are focused on the accurate estimation of dipole location parameters.

The localization accuracy of the nonlinear dipole parameters greatly depends on the optimization technique. The SA and the GA are applied as global optimization techniques for brain source reconstruction.

Simulated annealing is a numerical optimization technique based on the principles of thermodynamics. SA exploits an analogy between the search for the minimum and annealing in solids. The value of the cost function takes the role of the energy of the system. We refer to [Aarts et al. 1989] for details about the SA. Because of its ability to avoid becoming trapped in local minima, the SA has been used in a variety of different ways for the MEG/EEG source localization problem [Jiang et al. 2003, Khosla et al. 1997]. The selection of the initial temperature parameter \(T_0\) is very important. It defines the trade-off between the possibility of the global minimum convergence and the amount of computation. For actual implementation \(T_0\) was chosen as \(\Delta E / \lg (\beta^{-1})\), where \(\Delta E\) is an average increase in cost function for some times (e.g., 1000) and a ratio \(\beta = 0.85\). Such determination of the control parameter ensures that virtually all replacement points generated by the random thermal fluctuation were accepted. We adopt the following temperature decrement rule: the new temperature is determined by multiplying the previous one by a constant value \(\alpha \approx 0.95\).

A genetic algorithm imports the mechanism of natural adaptation into numerical optimization. In a GA,
a candidate solution is typically called a chromosome, and the evolutionary viability of each chromosome is given by a fitness function. Typically, genetic algorithms are defined by different parameters: selection, crossover, mutation and elitist strategy. We refer to [Goldberg 1989] for details about the GA. For the different values of population size we analyzed the average number of generations which were necessary for reaching the global minimum (as shown on Fig. 6). The “optimal” parameters were chosen as those requiring the minimum number of objective function calculations. Thus, a population of size 60 was used. Figure 6 illustrates the example of the best and mean scores of the population of size 60 at every generation obtained by minimizing the objective function (9).

![Fig. 6. Best and mean scores of the population at every generation.](image)

In our algorithm, we adopt the roulette wheel selection mechanism of the parents choosing for the next generation. Two individuals, or parents, form a crossover child for the next generation by taking an average of the parents. The elitist strategy [Goldberg 1989] was implemented in order to improve the mathematical convergence of the GA to a global optimum. Computational time and localization accuracy of the GA depend greatly on the population size parameter. The linear approximation of the electrical potential over the tetrahedral element leads the constant value of the objective function over each element. Thus, the objective function has a lot of local minima. In order to obtain an accurate enough solution the population size has to be increased. This ensures searching the solution space more thoroughly thereby reducing the chance that the algorithm will return a local minimum. However, a large population size also causes the algorithm to run more slowly.

4. COMPUTER SIMULATIONS

There is no well established gold standard that would allow judging the accuracy of the result of the different inverse solutions. Commonly, source localization algorithms are evaluated using simulations with artificial data. Our aim is to evaluate the effectiveness of the SA and the GA approaches to EEG brain source reconstruction. Therefore, we assume the correct modeling of potential. Thus, we avoid the translation of the EEG forward problem solution errors into errors in source localization.

The non-uniqueness of the EEG inverse problem implies that assumptions on the source model as well as anatomical and physiological a priori knowledge about the source region should be taken into account to obtain a unique solution. In our simulations we restrict the search space to the limited area of the brain volume that corresponds to the cortical surface. Following a 3D segmentation of the brain, the dipolar source is constrained perpendicularly to the cortical surface. This constraint follows a
model of the organization of macro-column assemblies of neurons in the grey matter which are very likely to be responsible for most of EEG/MEG signals [Nunez 1981].

In our experiments we generate EEG data with different noise intensity levels. We assume the noise to be Gaussian and the power of noise to signal are 0%, 5% and 10%, respectively. For all inverse solutions the starting guess was the center of the head, i.e. [0, 0, 0] cm. The performance of the SA and the GA approaches as global optimization techniques was assessed and compared via 100 computational simulations of EEG inverse problem at single-time-point for each noise level and each optimization strategy. Each simulation consists of the following steps:

1. Computation of the scalp electrical potential using the forward model and dipole parameters with following addition of the noise to the obtained forward data;
2. Estimation of the source location using the SA and the GA approaches by minimizing the fitness function (9);
3. Comparison of the SA and the GA performance.

The performance is evaluated using the localization error \( LE \), which is determined as the 2-norm of the difference between the simulated and estimated source location:

\[
LE = \sqrt{(x - x_{\text{mes}})^2 + (y - y_{\text{mes}})^2 + (z - z_{\text{mes}})^2},
\]

where \( (x_{\text{mes}}, y_{\text{mes}}, z_{\text{mes}}) \) are coordinates of the simulated dipole and \( (x, y, z) \) are coordinates of the estimated dipole.

In the case of linear interpolation of the electrical potential over the tetrahedral element we deem the estimation of the dipole to be “exact” when the tetrahedron, where the dipole was placed, is found since the objective function is constant over each tetrahedral element. In our case of the mesh density it is within the range of 0.033 cm. Thus, the correct localization is defined by the resolution of the EEG inverse problem.

5. RESULTS

The SA and the GA implementation assume a choice of method parameters that have influences on the result accuracy and the computational time. In order to compare the efficiency of both optimization methods, we computed run time for all inverse solutions. Both methods were tested on a Pentium 4 CPU at 3 GHz, with 2 GB of RAM. In case of the noiseless data the CPU time for one-dipole parameters search is about 8 min for the SA method. The average run time for the GA (98% of success) is about 2 times higher than of the SA.

Table 1 presents the rate of successful reaching the global minimum by two proposed optimization techniques. Genetic algorithm and simulated annealing found the global minimum in 98 and 94 of the 100 runs, respectively, when the data were noise-free. Their success rates decrease to 92% (GA) and 87% (SA) with the increasing of the noise to 10%. The presented results suggest that the GA outperforms the SA in reaching the global minimum for the proposed method parameters. The success rates did not change significantly across 0%, 5% and 10% noise levels for both approaches. Thus, the results demonstrate the robustness of the using approaches as an optimization methodology.

<table>
<thead>
<tr>
<th>Method/Noise</th>
<th>0%</th>
<th>5%</th>
<th>10%</th>
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<tr>
<td>SA</td>
<td>94%</td>
<td>91%</td>
<td>87%</td>
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<tr>
<td>GA</td>
<td>98%</td>
<td>96%</td>
<td>92%</td>
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The performance of the SA can be improved, for example, by using a slower cooling schedule and increasing the number of the iterations at a fixed annealing temperature. The success rates of the GA can be increased, for example, by increasing the population size. However, it obviously leads to increasing of the computational time.

For both cost functions the residual energy (10) and the objective function based on current-density boundary integral (9), the inverse matrix is calculated only once. The advantage of the objective function (9) in comparison with residual energy (10) is the lower size of the inverse matrix $[K_{ij}]^{-1}$. Moreover, the GA converges faster to the global minimum. On average this takes two generations.

Figure 7 illustrates the single dipole current-density integral reconstruction (the solid line corresponds to the initial configuration of the source while the asterisks line corresponds to the current-density boundary integral values at the electrodes in the case of the reconstructed dipole).

We investigated also the influence of the assumed additional constraints concerning the source model on the source localization accuracy. For this purpose, we removed the constraint of the perpendicular direction of dipole orientation to the cortical surface. The dipole was placed in the cortex at the distance of 35.8 mm from the scalp along the z-axis (as used for Fig.4). The dipole direction was chosen randomly for each simulation. In the noiseless case the success rates of the SA and the GA decreased to 66% and 68%, respectively. The search of the nonlinear dipole parameters was often trapped by the local minimum when the deviation of dipole orientation from the cortical tangential surface is small.

6. CONCLUSIONS

The SA - Simulated Annealing and GA - Genetic Algorithm methods have been implemented for EEG source localization using a finite-element real head model. The nonlinear search space was reduced to only the dipole location parameters. In a Finite Element Method framework the current-density boundary integral has been proposed as a basis for parameter optimization. The source reconstruction results obtained by both optimization methods do not depend on the initial estimates. The effectiveness of the proposed approaches as nonlinear optimization techniques in finding the global minimum and accurately estimating the source parameters were demonstrated. The computer simulation results show that the GA to be 4-6% more effective than the SA at converging to the true global minimum for source reconstruction problems that were simulated in this work. However, the computational cost of the GA is higher than for the SA.
We consider the FE discretization scheme for Poisson’s equation (2) with Dirichlet boundary condition (3). In order to create a weak formulation we multiply governing equation (2) by a weighting function $\mathbf{H}_i$ and we integrate over the whole domain $\Omega$

$$\int_{\Omega} \nabla \cdot (\mathbf{\sigma}(\mathbf{r}) \nabla U(\mathbf{r})) \cdot \mathbf{H}_i \, d\Omega = \int_{\Omega} \mathbf{\rho}(\mathbf{r}) \cdot \mathbf{H}_i \, d\Omega; \quad \mathbf{H}_i \in C^0. \quad (12)$$

Applying interpolation by parts to the left-hand side of the equation we yield to a weak form

$$\int_{\Omega} \mathbf{\sigma}(\mathbf{r}) \nabla U(\mathbf{r}) \cdot \nabla \mathbf{H}_i \, d\Omega = -\int_{\partial \Omega} \mathbf{n} \cdot \mathbf{\sigma}(\mathbf{r}) \nabla U(\mathbf{r}) \cdot \mathbf{H}_i \, dS + \int_{\Omega} \mathbf{\rho}(\mathbf{r}) \cdot \mathbf{H}_i \, d\Omega \quad (13)$$

Substituting the expansion $U(\mathbf{r}) = \sum_{i=1}^{N} U_i \cdot \mathbf{H}_i(\mathbf{r})$ into (2) we yield to the spatial discretization of the Poisson’s equation:

$$\sum_{i=1}^{N} \int_{\Omega} \mathbf{\sigma}(\mathbf{r}) \nabla U_i \cdot \nabla \mathbf{H}_i \, d\Omega = -\int_{\partial \Omega} \mathbf{n} \cdot \mathbf{J} \cdot \mathbf{H}_i \, dS + \int_{\Omega} \mathbf{\rho}(\mathbf{r}) \cdot \mathbf{H}_i \, d\Omega, \quad (14)$$

where $\mathbf{J}$ is a current density $(-\mathbf{\sigma} \nabla U(\mathbf{r}))$. In order to determine the objective function we change the boundary condition in the following way: we specify the measured potentials at the electrodes as Dirichlet boundary conditions in forward problem. Practically, we substitute $U_{mes} = (U_{mes} \in R^{N_e \times 1})$ into (3) at the electrodes. The equations (3) are used to calculate the magnitude of the current-density boundary integral. Since each current density $\mathbf{J}$ must approach the limit of zero as dipole estimation improves, the boundary integral in equation (3) must also approach zero. This leads to the minimization of

$$F_{obj}(x, y, z, \theta, \phi, M) = \sum_{i=1}^{N_e} \left( \int_{\partial \Omega} \mathbf{n} \cdot \mathbf{J}_i \, dS \right)^2, \quad (15)$$

where $F_{obj}(x, y, z, \theta, \phi, M)$ is an objective function of $(x, y, z, \theta, \phi, M)$ dipole parameters; $\mathbf{J}_i$ is a current density associated with $l$-th node at the boundary.

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