Bounded rationality and asset pricing with intermediate consumption

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Bounded Rationality and Asset Pricing

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Abstract
A lot of attention has been given to behavioral models and their abilities to explain stylized facts in asset pricing. The recent market crashes and suspicions of gross overvaluation have also brought to the forefront worries about the impact of irrational traders. A question that arises when considering models with irrational agents is whether they can survive in the market when confronted to rational agents. In a recent paper Kogan, Ross, Wang and Westerfield (2002) show that when agents have utility over terminal consumption only, they can impact prices even when their wealth share is negligible. First, we show here that the introduction of intermediate consumption considerably alters this conclusion. Namely, irrational agents with low consumption share have a far smaller impact than in the no-intermediate consumption case. Second, we consider the impact of biased learning rules in a context of incomplete information, and study through simulation the distribution of irrational agents’ consumption shares. We find that over a reasonable horizon (50 years) under- and over reaction have little impact on an agent’s consumption share.

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1 Introduction

Anticipations are at the center of economic modelling in a stochastic environment. Agents construct their anticipations on the basis of their observations and make choices such as investment, saving, consumption by comparing their current situation with the anticipated future situation, which are contingent on the realization of a random event. In the standard Lucas (1978) pure exchange economy agents have rational anticipations, in the sense that the distributional structure of the underlying stochastic variables is known by all agents. Under incomplete information, the sophisticated agents are expected to learn from their observations of the economy and construct rational beliefs about the current and possible future states of the economy. In the early work on incomplete information and asset pricing (Detemple (1986), Genotte (1986) for continuous time, Bawa and Klein (1977) for discrete time) a representative agent is constructed, and his beliefs are updated using Bayes’ law. Later, as in Detemple and Murthy (1994), multiple agents are considered and heterogeneity in beliefs is shown to have a significant impact on equilibrium quantities, such as interest rate and Sharpe ratio.

In the psychology literature the rational construction of belief based on an optimal use of data is severely questioned. Numerous bias and anomalies have been documented at different level of the decision process. In a series of papers Kanheman and Tversky (1972,1974, 1979) have put forward a number of departures from the standard expected utility behavior. The fact that cognitive bias have been reported in experimental context is an important element that should be considered when modelling agents decisions in financial markets. Inclusion of cognitive bias in single agent model include, among others, Cechetti et al. (2000), Barberis et al. (2001), Bernatzi and Thaler (1995), Barberis et al. (1998), Daniel Hirshleifer and Subramanyam (1999). These papers show
that departure from rationality may help understanding some of the empirical puzzles in asset pricing related to the equity premium and risk free interest rate. Our concern in this paper is in the way agent process information to construct their beliefs and how this impact their survival and the market equilibrium. We consider the so called availability heuristic, when agents construct their beliefs while relying too heavily on recent data, and overconfidence when agents rely too heavily on their prior beliefs. A question that naturally arises in a multi-agent context, is whether natural selection will drive biased agent to bankruptcy and if yes how fast. Researchers have provided different answers based on different assumptions. Delong, Shleifer, Summers and Waldman (1990) present a model where some agents, the noise traders, behave randomly and interact with rational agents. Surprisingly, the noise traders, who non intentionally make very risky investments, end up dominating the market in some situations. In fact, randomly behaving agent may hold a portfolio that is closer to the log (growth optimal) portfolio, and as a result may dominate the market. In contrast, Blume and Easley (2001) demonstrate in a general equilibrium context that in a complete market finite state model, Bayesian learners eventually dominate the market when confronted to any other type of learner. In their model, as time passes, the agents who do not correctly update the state probabilities will assign zero probability to some possible event. So, they will therefore have a contingent consumption plan with zero consumption in some states with positive probability. This result is obtained under the assumption that the realization of the random states is observable, which is generally the case in a discrete time model. However, in a continuous time model with incomplete information, the true state is typically not observed, and therefore the survival of biased agents may differ. Kogan, Ross, Wang and Westerfield (2002) study the price impact and survival of an irrational trader in a continuous time economy. Their model
has 2 agents with the same constant relative risk aversion utility over terminal wealth. One agent has perfect knowledge of all parameters of the underlying stochastic process. The other, the irrational agent, uses parameters with a constant deviation from the true value. Then, they show that asymptotic survival of the irrational agent depends on the common risk aversion, and the value of the constant deviation. Furthermore, they show that price impact is not directly related to wealth share and that agents with small market share can have a large impact on equilibrium prices. Our approach is related to Kogan et al. (2002), but we differ in three key aspects. First, we are interested in survival in finite time, rather than asymptotic. Indeed, if we are to assess the viability of a particular biased behavior in a financial market, studying asymptotic properties is akin to assuming hereditary transmission of the biased behavior. In contrast, by considering finite time properties we ask the question of survival over the life cycle of an investor, which we will typically assume to be 50 years. Second, we do not make the extreme assumption that rational agents know the true value of the parameter. We only assume that they rationally update their beliefs, i.e. they use Bayes law. We also generalize the modeling of irrational agents. Namely, in our model, they also update their beliefs, but they display over or under reaction\(^1\). Third, we allow for intermediate consumption, an essential aspect of most investor’s situation. As a result we are able to discuss the equilibrium properties of the risk free market, which is not possible when intermediate consumption is not modeled. Kogan et al. (2002) have shown that in general, price impact is not related to an investor’s wealth share. We show in our model that when agent have logarithmic utility, when intermediate consumption is allowed, this is no longer the case, and it is the consumption share which summarizes the impact of an irrational agents. In fact, when agent

\(^1\text{We demonstrate that in the context of our model over / under reaction is equivalent to under / over confidence.}\)
have logarithmic utility, the stock price is not at all affected by the presence of irrational agents, but only the risk free interest rate is. When agent care only about terminal wealth, the state price density is a function of their conditional expectation of the terminal dividend, even when they have logarithmic utility. When intermediate consumption is introduced, the state price density for logarithmic utility depends on the current value of the dividend process and not on conditional expectations of future dividends. As a result, the stock price is unaffected by the divergence in beliefs. We also consider the price impact of irrational agents when risk aversion is greater than one. Again, it is their consumption shares which summarizes their impact on the equilibrium interest rate and the Sharpe ratio.

We develop these ideas by considering a pure exchange economy where the drift of aggregate consumption is unobserved. The economy is populated by agents with heterogeneous learning rules. Different types of biased agents, namely over and under reactive, interact simultaneously with a rational agent. We do not assume that irrational agents behave randomly, as in the noise trader literature. Rather, we model them as expected utility maximizers whose investment and consumption decisions are optimal conditional on publicly available information. However, their expectation are taken relative to a probability measure which does not result from an optimal use of the observations. We compute the competitive equilibrium in such a model and obtain, by simulation, the distribution of average and terminal consumption shares. We find that biased behavior has a limited impact on the consumption share of an agent when the horizon is set at 50 years. In a 2 agent economy the average consumption share of a rational agent ranges from 50 to 57 % percent depending on the bias coefficient. In a 3 agent economy, starting with equal endowment, all agents remain close to 33 % of consumption share, irrelevant of their learning rule. We find
however, that the standard deviation of the average consumption share across simulations is much larger for biased agents. The remaining of the paper is organized as follows. Section 2 introduces the model, discusses the structure of the economy, the agents preferences, the individual learning rules and ends with a characterization of the equilibrium. Section 3 discusses the price impact and the conditions for finite time survival of irrational agents. In section 4 we look at the distribution of average consumption for different levels of under or over reaction, using simulations. We also consider the effect of irrational agents on equilibrium volatility. We conclude in section 5.

2 Model

2.1 The economy

We consider a pure exchange economy with finite horizon $T$, where the uncertainty is described by probability space $(\Omega, F, \mathcal{F}, \mathbb{P}^0)$ on which two independent Brownian motions, $W_{1t}$ and $W_{2t}$ are defined. We let $\mathcal{F} = \mathcal{F}_t^{W_1, W_2}$ and $F = \mathcal{F}_T$. $\Omega$ is the canonical state space $\mathbb{C}^0$ and $\mathbb{P}^0$ is the Wiener measure. There is a single perishable good which serves as the numeraire. The financial market consists of two securities, a locally riskless money market account in zero net supply which pays an interest rate of $r_t$ and a risky stock with price $S_t$ representing a claim to an exogenous stream of dividend payment $D_t$. The dividend process is of the form

$$D_t = D_0 \exp \left[ \int_0^t \gamma_s - \frac{1}{2} \lambda^2 ds + \int_0^t \lambda dW_{1s} \right]$$  \hspace{1cm} (1)$$

where the volatility $\lambda$ is a positive constant and the drift process $\gamma_t$ evolves according to

$$d\gamma_t = \alpha (\beta - \gamma_t) dt + \delta_1 dW_{1t} + \delta_2 dW_{2t}.$$ \hspace{1cm} (2)
The initial value of the money market account is normalized to one, so in equilibrium the price process \( B_t \) is given by

\[
B_t = \exp \left( \int_0^t r_s ds \right).
\]

There are \( I \) agents in the economy endowed with initial number of shares \( \pi_{i0} > 0 \), \( i \in \{1, \ldots, I\} \). Units of money market account are denoted \( \pi_{it}^0 \). The flow of information available to the agents is restricted to the filtration generated by the dividend process, \( \mathcal{F}_D \). We assume that agents have heterogeneous gaussian initial beliefs about the drift process \( \gamma_t \), with mean \( m_{i0} \) and variance \( \varepsilon_{i0} \). Subjective probability measure are represented by \( \mathbb{P}^i \). Agents preference over consumption are described by the expected utility functional

\[
U_i(c) := E^i \left[ \int_0^T \rho_t \frac{e^{\int_0^t -R}}{1 - R} dt \right]
\]

where \( R \) refers to the common relative risk aversion, \( \rho_t := e^{-\psi t} \) is the common discount rate and \( \psi \) is a positive constant. The dividend process is represented in the subjective probability space as a function of the innovation process \( \chi_{is} := \int_0^t \lambda^{-1} \left( \frac{dP_t}{ds} - m_{is} ds \right) \), namely we have

\[
D_t = D_0 \exp \left[ t \int_0^t m_{is} ds - \frac{1}{2} \lambda^2 ds + \int_0^t \lambda d\chi_{is} \right].
\]

The cum dividend stock price process in the subjective measure is given by

\[
S_t + \int_0^t D_s ds = S_0 + \int_0^t S_t \mu_{is} ds + \int_0^t S_t \sigma_s d\chi_{it}
\]

The quantities \((\mu_{it}, \sigma_i, r_t)\) are obtained endogenously in equilibrium. Trading takes place continuously and there are no frictions. The wealth process \( X_{it} \)
associated to a trading strategy, $\pi_{it}$, takes the form

$$X_{it} = \pi_{it}S_t + \pi_{0it}B_t.$$  \hspace{1cm} (7)

A trading strategy is admissible if $X_{it}$ is uniformly bounded from below by a constant (this guarantees the absence of arbitrage, see Dybvig and Huang (1988)). A consumption plan $(c_i)$ is said to be feasible if there exists an admissible trading strategy which solves the agent’s dynamic budget constraint

$$dX_{it} = \pi_{0it}dB_t + \pi_{it}[dS_t + D_tdt] - c_itdt.$$  \hspace{1cm} (8)

### 2.2 Learning

We consider different types of agent: some are bayesian learners and some display learning bias. All agents continuously update their beliefs about $\gamma_t$ based on the observations of $D_t$. Each agent has a different subjective probability measure denoted $\mathbb{P}^i$, induced by her beliefs. Agents’ beliefs at time $t$, about the current value of $\gamma_t$, are then defined as $m_{it}$. It is well known (Lipster and Shiryaev (1978)), that for the Bayesian learner, the continuous updating equations for the conditional mean, $m_{it} = E^i[\gamma_t \mid \mathcal{F}_t^D]$, and the conditional variance $\varepsilon_{it} = E^i[(\gamma_t - m_{it})^2 \mid \mathcal{F}_t^D]$ are given by

$$dm_{it} = \alpha(\beta - m_{it})dt + \frac{\delta_2\lambda + \varepsilon_{it}}{\lambda^2} \left[ \frac{dD_t}{D_t} - m_{it}dt \right]$$  \hspace{1cm} (9)

$$\frac{d\varepsilon_{it}}{dt} = -2\alpha\varepsilon_{it} + \delta_1^2 + \delta_2^2 - \frac{(\delta_2\lambda + \varepsilon_{it})^2}{\lambda^2}.$$  

We can now describe how agent with learning bias update their beliefs. Note that this is not an optimal use of the observations, but simply an attempt at translating in an updating equation some of the cognitive anomalies reported
in the psychology literature. We will consider the following cases

- **Agent over/under reacts to new observations**

\[
\begin{align*}
dm_{it} &= \alpha(\beta - m_{it})dt + (1 - \vartheta)\frac{\delta_2\lambda + \varepsilon_{lt}}{\lambda^2} \left[ \frac{d\mathcal{D}_i}{D_i} - m_{it}dt \right] \\
\frac{d\varepsilon_{it}}{dt} &= -2\alpha\varepsilon_{it} + \delta_1^2 + \delta_2^2 - \frac{(\delta_2\lambda + \varepsilon_{lt})^2}{\lambda^2}.
\end{align*}
\]  

where \( \vartheta < 1 \). When \( \vartheta = 0 \), there is no under/over reaction and the observed information is optimally used to update the belief. When \( \vartheta < 0 \), the agent over-reacts, and when \( 0 < \vartheta < 1 \) the agent under-reacts.

- **Agent thinks he knows the true value of the unobserved parameter**

\[
\begin{align*}
m_{it} &= m_0, \forall t \in [0, T] \\
\varepsilon_{it} &= 0, \forall t \in [0, T]
\end{align*}
\]

in this case the agent never updates his initial belief.

- **Under reaction with \( \vartheta = 1 \), which is the extreme under reaction case where new information is ignored and the belief converges toward the long term mean \( \beta \).**

Figure (1) display path example of the under / over reaction bias, described above. As expected under (over) -reaction bias generates belief that are less (more) volatile than the rational benchmark. It should be noted that, under reaction, which consists in placing too much importance on prior beliefs, could also be interpreted as overconfidence. In fact there is a level of conditional
variance which corresponds exactly to a given under / over reaction level \( \vartheta \), precisely, one could arbitrarily set the conditional variance at a level of \( \varepsilon^* := \lambda^2 \left( \frac{1}{1+\lambda^2} \vartheta \right)^2 \) and generate similar effects.

2.3 Individual Optimality

Note that the economy is initially incomplete as the number of risky securities is inferior to the number of Brownian motions. However, every contingent claim adapted to \( \mathcal{F}_t^D \) is attainable, and therefore when information is incomplete and restricted to \( \mathcal{F}_t^D \), the market can be treated as complete for the purpose of individual consumption and investment decisions. We therefore use the standard results of Cox and Huang (1991) to solve the static problem equivalent to the dynamic consumption investment decision. In order to present optimal consumption we define the individual specific state price density process

\[
\xi_{it} := \exp \left[ - \int_0^t r_s ds - \int_0^t \theta_{is} d\chi_{is} - \frac{1}{2} \int_0^t \| \theta_{is} \|^2 ds \right]
\]

where \( \theta_{it} := \frac{\mu_t - r_t}{\sigma_t} \) is the subjective market price of risk (MPR). Individual state price densities are linked by the expression \( \xi_{it} = \eta_{it} \xi_{1t} \), where \( \eta_{it} \) satisfies

\[
\eta_{it} = \eta_{i0} \exp \left[ \frac{1}{2} \int_0^t \left( \frac{\Delta_{is}}{\lambda} \right)^2 ds + \int_0^t \frac{\Delta_{is}}{\lambda} d\chi_{is} \right]
\]

and \( \Delta_{it} = m_{it} - m_{i1} \). Notice here that \( \frac{1}{\eta_{i0}} = \frac{d\rho_i^t}{d\rho^t} \), is the change of probability measure from agent \( i \) to agent 1. The static budget set is defined as

\[
B(\pi_{i0} S_0) = \left\{ c_i : E^i \int_0^T \xi_{it} c_{it} dt \leq \pi_{i0} S_0 \right\}.
\]
The static optimization problem is given by

$$\max_{c_i} U(c_i)$$

$$c_i \in B(\pi_i S_0).$$ (14)

which yields the following optimal individual policies

$$c_{it} = \left( \frac{y_i \xi_{it}}{\rho_t} \right)^{-\frac{1}{\gamma}}$$ (15)

where $y_i$ is a Lagrange multiplier which solves agent $i$’s static budget constraint

$$x_{i0} = E^i \int_0^T \xi_{it} \left( \frac{y_i \xi_{it}}{\rho_t} \right)^{-\frac{1}{\gamma}} dt.$$ (16)

### 2.4 Equilibrium

An equilibrium is a combination $\{(c_{it}, \pi_{it}) ; (r_t, S_t)\}$, where $(c_{it}, \pi_{it})$ is an optimal admissible strategy for all $i$ taking $(r_t, S_t)$ as given and all markets clear $\sum_i c_{it} = D_t$, $\sum_i \pi_{it} = 1$, $\sum_i \pi_{it}^0 = 0$. In the goods market, the market clearing condition implies that all dividends are consumed. In the financial market, the market clearing conditions imply that the stock is held and the bond is in zero net supply. In order to derive the equilibrium, we choose a reference agent, which is arbitrarily called agent 1. This enables us to compute all expectations under the probability measure $P^1$. The equilibrium is expressed in terms of the reference agent subjective measure.

**Proposition 1** The competitive equilibrium for the economy described in this section is given by

$$\theta_{1t} = R \lambda + \sum_i \frac{c_{it}}{C_t} \lambda^{-1} \Delta_{it}$$ (17)
\[ S_t = E^1 \int_t^T \rho_t \frac{u'_t(c_{1v})}{u'_t(c_{1i})} D_v dv \]  

\[ r_t = \psi + R \sum_i \frac{c_{it}}{C_t} m_{it} - \frac{1}{2} R(R + 1) \lambda^2. \]

This result is classic in the literature (Detemple and Murthy (1996)), and the proof is omitted.

3 Survival and price impact

3.1 Survival in finite time

Blume and Easley (2001) propose to study the question of survival of irrational agent by looking at the ratio of marginal utilities of 2 agents in the economy, \( \frac{u'_t(c_{it})}{u'_t(c_{jt})} \). By virtue of the Inada condition this ratio diverges to \( \infty \) only if \( c_{it} \to 0 \) or \( c_{jt} \to \infty \), since in their model the aggregate endowment is bounded, the ratio of the marginal utilities diverges to \( \infty \) only when agent \( i \) disappears. Notice that the ratio of marginal utilities in our model can also be interpreted as a Radon-Nykodim derivative (or likelihood ratio) of the two subjective measures \( P^i \) and \( P^j \)

\[ \frac{u'_t(c_{it}(\omega))}{u'_t(c_{jt}(\omega))} = \frac{y_i \xi_{it}(\omega)}{y_j \xi_{jt}(\omega)} = \frac{y_i}{y_j} \frac{dP^j(\omega)}{dP^i(\omega)} \]  

(20)

In our model the measure of the rational agent is equivalent to the measure of the irrational agent, so in finite time, none will vanish. A similar result is stated in Blume and Easley (2001)

“..trader \( i \) survives on the set where the likelihood ratio of \( j \)’s forecasts to \( i \)’s forecasts remains bounded..”

In a finite state model, when an agent assigns a zero probability to a particular state, when the state is realized and observed by all agents, we will have divergence of the likelihood ratio. In our model the filtration is generated by
the dividend process, and the observable states are thus possible values of $D_t$. If an irrational learner assigns a zero probability to a particular value of the drift $\gamma$, it does not mean that she assigns a zero probability to an observable event, since irrelevant of the beliefs about $\gamma_t$ the dividend process still takes its value in $\mathbb{R}^+$. To verify the positive consumption of an irrational agent, it suffices to look at the process $\eta_{it}$. The ratio of marginal utilities diverges to $\infty$ only if $\eta_{it}$ diverges. Let us consider for example, the agent who does not incorporate new information in his filtering process and converges to the long term mean $\beta$ of the unobserved process. In this situation, $\eta_{2t}$ solves the following stochastic differential equation

$$d\eta_{2t} = \eta_{2t} \left[ \frac{(m_{1t} - \beta)}{\lambda^2} (\gamma - \beta)dt + \frac{m_{1t} - \beta}{\lambda^2} dW_{2t} \right]. \quad (21)$$

This process is almost surely finite if

$$\int_0^T \left| \frac{(m_{1t} - \beta)}{\lambda^2} (\gamma - \beta) \right| dt < \infty \quad \mathbb{P}^0 \text{ a.s.}$$

$$\int_0^T \left| \frac{m_{1t} - \beta}{\lambda^2} \right|^2 dt < \infty \quad \mathbb{P}^0 \text{ a.s.}$$

which for the mean reverting process $\gamma_t$ is verified. More generally the change of measure $\eta_{2t}$ gives the consumption ratio relative to the rational agent. Clearly when agent 2 displays no learning bias, the process $\eta_{2t}$ remains bounded in finite time, as eventually $m_{1t} = m_{2t}$ as we have assumed gaussian initial beliefs. Similar arguments demonstrate that, in finite time, none of the biases we have considered will lead an agent to disappear. A proof is given in appendix B. In general the following conditions must hold for an agent $i$ to survive relative to
agent 1

$$
\int_0^T \frac{1}{2} \frac{\Delta^2 t (1 + \gamma_t - m_1 t)}{\lambda^2} \, dt < \infty \quad \mathbb{P}^\circ \text{ a.s.}
$$

$$
\int_0^T \left| \frac{\Delta t}{\lambda^2} \right|^2 \, dt < \infty \quad \mathbb{P}^\circ \text{ a.s.}
$$

### 3.2 Price impact

We first discuss the case of logarithmic utility to illustrate the price impact of biased agents. From proposition 1 the stock price, in the log economy, is given by

$$
S_t = D_t \int_t^T \rho_{t,v} \, dv. \quad (22)
$$

It is of interest to note here that there is no direct impact on the stock price from the presence of irrational agents in the log economy with intermediate consumption. Irrational agents do affect the economy, but only through the risk free interest rate. When agents have utility over terminal wealth only, as in Kogan et al. (2002), the state price density depends on the conditional expectation of the terminal dividend, and as a result the stock price is affected by irrational traders’ beliefs. In the present context the state price density is given by

$$
\xi_{tt} = \frac{\rho_t + \sum_i \frac{\rho_t}{\eta_i}}{D_t} \quad (23)
$$

and depends only on the current dividend and likelihood ratios, which under agent 1’s probability measure are martingales. As a result, when calculating the stock price, we obtain

$$
S_t = E^1_t \int_t^T \frac{\rho_{t,v} + \sum_i \frac{\rho_t}{\eta_i}}{\rho_t + \sum_i \frac{\rho_t}{\eta_i}} \, D_v \, dv \quad (24)
$$
but as \( E_t^t \left( \frac{1}{\eta_t} \right) = \eta_t \), this simplifies to equation (22). The interest rate given in proposition 1 is equal to

\[
    r_t = \psi + \sum_i \frac{c_{it}}{C_t} m_{it} - \lambda^2
\]  

we see here that the beliefs of all agents are present and their relative consumption share summarizes their impact. The incorporation of intermediate consumption has a non trivial impact on the structure of the problem. In the setting of Kogan et al. (2002), it was shown that price impact and consumption shares (i.e. survival) were two different things. Here, when preferences are of the logarithmic type, irrational agents can only have an impact if their consumption share is large enough. This can be seen by looking at the evolution over time of the Sharpe ratio of stock return. The moments of the stock returns are not affected by the presence of irrational agents, indeed as the stock price is a deterministic function of the current dividend, the drift and volatility of the stock return are equal to the drift and volatility of the dividend. However, since the interest rate is affected, so will the Sharpe ratio. When considering a model with terminal consumption only, one can not assess the impact of irrational traders on the interest rate market, and the conclusion that irrational agent may affect prices even with low market share does not obtain when intermediate consumption is introduced. To illustrate this point we reproduce Figure (3) in Kogan et al (2002), where the Sharpe ratio is plotted along with the wealth share of the irrational agent. In their setting, the impact of the irrational agent on the Sharpe ratio persists when his wealth share decreases. For this exercise, we consider the following set of parameters to match theirs: \( \gamma = 0.05 \) and is constant through time, \( \lambda = 0.15 \), the Bayesian agent has perfect knowledge, i.e. \( m_{10} = \gamma_0 \) and \( \varepsilon_{10} = 0 \), the non-bayesian agent’s initial belief is given by \( m_{20} = m_{10} - 2\lambda^2 \) and the time horizon is 400 years. Clearly, we
can observe that the effect of the irrational agent is conditioned by his market share. The result is particularly strong when agents have logarithmic utility, but is similar if relative risk aversion is larger than 1. Figure (3) displays the average Sharpe ratio along with the consumption share of an irrational agent when $R = 4$. Here again we can observe the strong link between consumption share and irrational agent’s impact on the Sharpe ratio. However when $R > 1$ the consumption share and the market share are two separate quantities, and we are unable to obtain here closed form solution for the optimal individual portfolio, so it may well be the case that market shares do not follow irrational agent’s impact on the Sharpe ratio. When all agents have incomplete information about the underlying variables, we must look at the Sharpe ratio under the objective measure, which will incorporate all the individual errors. Using the definition of the innovation processes, and proposition 1, one can show that the Sharpe ratio under the objective (true) measure, which we denote $\theta_i^o$, is given by

$$\theta_i^o = R\lambda + \lambda^{-1} \left[ \gamma_t - \sum_i c_{it} m_{it} \right]$$  \hspace{1cm} (26)$$

And here again, the consumption share of each agent summarizes his contribution to the deviation from the full information value of the Sharpe ratio, given by $R\lambda$. The fact that irrational traders can only affect prices when their consumption is large enough holds also for assets with longer duration. Indeed, as the stock price pays a continuous dividend its duration will certainly be smaller than that of a long term bond. A long term bond, paying one dollar at time $T$, is valued as

$$l_{t,T} = E_t \left[ \frac{\bar{\xi}_{1T}}{\xi_{1t}} \right]$$  \hspace{1cm} (27)$$

For simplicity, we consider the 2 agent setting with logarithmic utility, where the Bayesian agent has perfect knowledge and the irrational agent deviates from
a constant term. The long term bond is given then by the following expression

\[ l_t = \left( \frac{1}{1 + \eta_{2t}} \right) \left( \exp \left[ (-\gamma + \lambda^2) (T - t) \right] + (\eta^{-1}_{2t} \exp \left[ (-\gamma + \lambda^2 + \Delta_2 \lambda) (T - t) \right] \right) \]  

(28)

In an economy populated by rational traders only we would obtain

\[ l'_t = \exp \left[ (-\gamma + \lambda^2) (T - t) \right]. \]  

(29)

Notice here that \( \eta_{2t}^{-1} \) is the relative consumption share of the irrational agent, and as such, he can only impact the long term bond price when his consumption share is sufficiently large relative to the rational agent. Figure (4) displays the evolution of the excess long term bond yield over time. We see again that the divergence follows the consumption share of the irrational agent.

4 Simulation

We study survival over finite time horizon using the following simulation methodology

1. Calibration of lagrange multipliers in the static budget constraint such that \( \pi_{i0} = \frac{1}{I}, \ i \in \{1, ..., I\} \), i.e. identical initial endowments.

2. Simulation of trajectories of the exogeneous dividend process

3. Updating of agent’s beliefs and consumption choices for each trajectory

Each experiment consists in the simulation of 100 000 trajectories with a time horizon of 50 years and a time step of one week. The state price density, which is the key element in determining the optimal consumptions is obtained by using the market clearing condition on the goods market

\[ \sum_i c_{it} = D_t \]
and individual optimality
\[ c_{it} = \left( \frac{y_i \xi_{it}}{\rho_t} \right)^{-\frac{1}{\pi}}. \]

To identify a single state price density we construct the experiment based on a reference agent as we did to compute the equilibrium. To do so we use the process \( \eta_{it} \) which is the ratio of state price density of agent \( i \) to the state price density of the reference agent
\[ \eta_{it} = \frac{\xi_{it}}{\xi_{1t}}. \]

We can reformulate the first order condition as
\[ c_{it} = \left( \frac{y_i \xi_{1t} \eta_{it}}{\rho_t} \right)^{-\frac{1}{\pi}}. \]

With this new definition it is easy to obtain \( \xi_{1t} \) from the market clearing condition
\[ \xi_{1t} = \left[ \frac{D_t}{\sum \left( \frac{y_i \eta_{it}}{\rho_t} \right)^{-\frac{1}{\pi}}} \right]^{-R}. \]

We can arbitrarily choose the reference agent as it has no impact on the consumption or wealth share. The equilibrium quantities that we discuss in the next section are also obtained from this representation, and when presenting distributional result we rely on the objective probability measure by using the link between the innovation process \( \nu_{1t} \) and the underlying brownian motion \( W_{2t} \). In all simulation, the following parameters are used
\[
\begin{align*}
\lambda &= 0.036 \\
\gamma_0 &= 0.018 \\
\alpha &= 1.16, \ psi = 0.01, \ delta_1 = 0.011, \ delta_2 = 0.011
\end{align*}
\]

These numbers are obtained from the consumption data in the US for the pe-
period 1870-2000. The important elements are mean reversion of the drift of the consumption growth rate, and low volatility $\lambda$. The prior of all agents are set equal to the long term mean of the unobservable process, which is also started at its long term mean. The conditional variances are set such that $\frac{d\mu}{dt} = 0$, i.e. we consider here the steady state, learning has occured long enough for conditional variances to reach their long term values.

4.1 Simulation with logarithmic preferences

We first focus on logarithmic preferences. When agents have logarithmic preferences, they try to hold the growth optimal portfolio, so any deviation from a rational belief construction should be detrimental. With higher risk aversion levels, the optimal portfolio is no longer the growth optimal portfolio, and in this case, as shown in Kogan et al. (2002), an irrational trader may, by mistake, hold a portfolio that his closer to the growth optimal portfolio, and as a result may accumulate wealth faster and hold a dominant position in the economy. Here, we want to assess whether an agent who irrationaly deviates from the growth optimal portfolio holds a negligible wealth in finite time, when parameters are chosen to mimick consumption data in the US over the last century.

4.1.1 2 agent economy

We simulate a 2 agent economy and report average consumption shares and their standard deviation for different values of the bias coefficient in table (1). It appears that over 50 years, biased agents, displaying over or under reaction, are not much penalized in terms of average consumption. Their average consumption share is decreasing with the size of the bias but remains close to 50 %. Even when $\theta = 1$, i.e. the biased agent does not incorporate any new information, his average consumption is 43 %.
4.1.2 3 agent economy

In a 3 agent economy, when the Bayesian agent is confronted with two biased learners, we can also observe, in table (2), that consumption shares are not very much affected by the biased behavior. However the standard deviation, across simulations, of the consumption shares is an order of magnitude larger for the biased agents, and increases with the level of the bias coefficient. This is even more apparent in figure (5) through (7) which display the distribution of pairs of consumption shares. The biased agents follow strategies which imply taking on non remunarated risk, and as a result the standard deviation of their consumption share increases. Over longer time horizon, the Bayesian learner clearly dominates, and average market share when the time horizon is 1000 years is over 95 percent. Again, we may wonder whether such a long horizon is suitable for studying the impact of a particular investor’s behavior.

4.2 Simulation with R = 4

When agents have risk aversion coefficient different from 1, the stock price is no longer a simple linear function of the dividend, and therefore, the presence of biased agent has an impact on both interest rate and stock price. We consider in this section the survival of biased agent when risk aversion is set at $R = 4$. Figure (8) through (10), display scatter plot diagram of average consumptions and table (3) report average consumption as well as standard deviation obtained from simulating the 3 agent economy 100 000 times. The results are very similar to the logarithmic utility experiment, average consumption shares remain approximately equal to one third for all agents, biased and unbiased. Here again, bayesian learner have a lower standard deviation of their average consumption share across simulations.
4.3 Trading volume and interest rate

With the logarithmic preference structure we can obtain in closed form the portfolio of each agent in the economy, and study the equilibrium trading volume. The optimal portfolio for a log investor is given by

$$\pi_{1t} = \theta_{1t} \sigma^{-1} X_{1t}$$  \hspace{1cm} (32)

In equilibrium we know that the stock volatility $\sigma$, is equal to the constant dividend volatility\(^2\) and we know from proposition 1 how to express the individual market price of risk as a function of exogeneous quantities. Using the proportional relationship between wealth and consumption, $X_{it} = c_{it} \int_t^T \rho_s ds$, we can obtain the equilibrium market share of agent 1

$$\frac{\pi_{1t}}{S_t} = \frac{c_{1t}}{C_t} + \frac{c_{1t}}{C_t} \lambda^{-2} \sum_i \frac{c_{it}}{C_t} \Delta_{it}.$$  \hspace{1cm} (33)

When beliefs are homogeneous, $\Delta_{it} = 0$ for all $i$, the market share of agent 1 is equal to his consumption share $\frac{c_{1t}}{C_t}$, furthermore since we are assuming identical preferences, it is a constant. Trading in this model is thus a direct consequence of divergence in beliefs. It is of interest to note that a slight divergence in beliefs generate important trading volume as illustrated in figure (11) and (12). The sample path observed here is obtained from a 2 agent model with equal prior, agent 1 is rational bayesian and agent 2 displays under reaction. The market share of agent 1 is highly volatile, even though the divergence in beliefs is of small magnitude.

We further study the effect of heterogeneity in learning rules by looking at the distributions of average market share in 2 agent economies, when agent 1 is rational bayesian and agent 2 displays the learning bias discussed earlier.

\(^2\)This is obtained by applying Röö’s lemma on equation (22).
As displayed in figure (13), the rational agent holds a larger share of the stock against the over-reactive agent than against the under-reactive agent. The over-reactive (under-reactive) agent increases (decreases) the volatility of the interest rate, making the bond market relatively less (more) attractive than the stock market since, in the log economy, there is no effect on the stock volatility due to divergence in beliefs. This effect is illustrated in figure (14). In a single agent incomplete information model the volatility of the interest rate, given our parametrization is 1.65 % on average, in the presence of under reactive agent the average volatility decreases to 1.4 % and in the presence of over reactive agent it increases to 1.84 %. This small difference has a significant effect on the portfolio allocation of the rational agent.

4.4 Volatility

To assess the effect of the bias we consider an economy populated by 2 agents with CRRA utility with $R \neq 1$. Using the Clark - Ocone formula and the martingale representation theorem (see appendix B), we are able to obtain a quasi analytical solution for the equilibrium volatility for the under / over reactive case,

$$\sigma_t = \lambda + \frac{\varepsilon_{2t}}{C_t} \lambda^{-1} \Delta_{2t} + (1 - R) \frac{\varepsilon_{1t} E_t^1 \int_t^T \xi_{1t, u}^1 D_u \frac{1 - e^{-\alpha(t - u)}}{\alpha} d \nu}{S_t}$$

$$E_t^1 \left[ \frac{\varepsilon_{1t} - (1 - \vartheta) \varepsilon_{2t} + \varepsilon_{2t} \lambda}{\lambda} \int_t^T \lambda^{-1} 2s \exp \left[ - \int_t^s (1 - \vartheta) \frac{\delta_1 \lambda + \varepsilon_{2u}}{\lambda^2} + \alpha d u \right] d s \right]$$

$$+ \frac{\lambda^{-1} \Delta_{2t} \left[ \frac{\varepsilon_{1t} - (1 - \vartheta) \varepsilon_{2t} + \varepsilon_{2t} \lambda}{\lambda} \int_t^T \lambda^{-1} 2s \exp \left[ - \int_t^s (1 - \vartheta) \frac{\delta_1 \lambda + \varepsilon_{2u}}{\lambda^2} + \alpha d u \right] d s \right]}{S_t}.$$

This expression depends on the value of $\vartheta$, the bias coefficient, and to assess whether the effect is positive or negative we rely on a numerical evaluation. Figure (15), displays the volatility as a function of the risk aversion and the bias.
coefficient. The presence of an under reactive agent in the economy decreases the volatility for all level of risk aversion, whereas the opposite effect is observed when considering an over reactive agent. The numerical results obtained here are of course associated with a set of paramaters, and a generalization of the previous argument can not be made. Take for example, a situation where \( \frac{\delta x + c u}{\lambda} \) is inferior to zero, then an increase in \( \theta \) (more under reaction) would actually increase the equilibrium volatility.

5 Conclusion

In this paper we have considered a continuous time pure exchange economy populated by agent with heterogeneous learning rules. In our framework, biased learners do not necessarily vanish when paired with a bayesian learner. This result crucially depends on the impossibility to assign a zero probability to an observable state. We have characterized the effect of a biased learner on the interest rate volatility and trading volume in an economy populated by logarithmic agent, and we have shown that market share can be sinificantly affected. Finally we have obtained an expression for the equilibrium stock volatility for agents with constant relative risk aversion superior to one an we have, numerically, shown that the presence of biased agent can have an important impact on the volatility, positive or negative, depending on the parametrization.

As far as price impact is concerned, the introduction of intermediate consumption in the structure of the model has an important effect. An agent’s effect (irrational or not) on equilibrium quantities is strongly related to his consumption share. In the case of logarithmic utility as consumption share and wealth share are identical, an irrational agent can have an impact on the Sharpe ratio only if his wealth share is large enough. As the wealth share of an irrational agent decreases in a log economy, his price impact will vanish at the exact same
rate. We have considered here only time separable utility function and further
research could be conducted in that direction. Also, while we allow biased agent
to learn about the unobserved process, we do not let them learn about their own
mistakes. An important improvement to the present model would be to describe
a learning rule which includes a bias about which the agent learns over time.
Appendix A

We show here that for the bias considered in section 2, an agent can never disappear in finite time. Technically, we want to guarantee the following

\[
\eta_0 \exp \left[ \frac{1}{2} \int_0^t \left( \frac{\Delta_is}{\lambda} \right)^2 ds + \int_0^t \frac{\Delta_is}{\lambda} d\chi_{1s} \right] < \infty, \quad \forall t \in [0, T]
\]

in \( P^o \) almost sure sense, where \( P^o \) refers to the objective probability measure. First using the definition of the innovation process \( \chi_{1t} := \int_0^t \lambda^{-1} \left( \frac{dD}{D} - m_{1s}ds \right), \) the condition can be rewritten as

\[
\eta_0 \exp \left[ \frac{1}{2} \int_0^t \frac{\Delta_i^2(1 + \gamma_t - m_{1t})}{\lambda^2} ds + \int_0^t \frac{\Delta_is}{\lambda^2} dW_{2s+} \right] < \infty, \quad \forall t \in [0, T]
\]

which is almost surely finite if and only if

\[
\int_0^T \frac{1}{2} \frac{\Delta_i^2(1 + \gamma_t - m_{1t})}{\lambda^2} dt < \infty \quad P^o \text{ a.s.}
\]

\[
\int_0^T \left( \frac{\Delta_is}{\lambda} \right)^2 dt < \infty \quad P^o \text{ a.s.}
\]

from the assumption of Gaussian prior beliefs and mean reverting properties of the process \( \gamma \), this is immediately verified, and thus agents do not disappear in finite time.
Appendix B

We show here how to obtain the equilibrium volatility. From the equilibrium stock price and first order condition, we write the stock price as

$$S_t = E^1 \left[ \int_t^T \frac{\xi_u}{\xi_t} D_v dv \mid \mathcal{F}_t^D \right].$$  \hspace{1cm} (35)

The innovation process $\chi_{1t} := \int_0^t \chi_s^{-1} \left( \frac{dD_s}{D_s} - m_{1s} ds \right)$ is a $(P^1, \mathcal{F}_t^D)$ - Brownian motion. Let $M_t$ be the following $(P^1, \mathcal{F}_t^D)$ - martingale

$$M_t := \xi_t S_t = E^1 \left[ \int_0^T \xi_s D_v dv \mid \mathcal{F}_t^D \right]$$  \hspace{1cm} (36)

it has a representation in term of $\chi_1$, namely

$$M_t = M_0 + \int_0^t \phi_s d\chi_{1s}$$  \hspace{1cm} (37)

for some unique square integrable process $\phi$. Applying Itô’s lemma to the product $\xi_t S_t$, we identify its diffusion term and obtain

$$\xi_t S_t [-\theta_t + \sigma_t] = \phi_t$$  \hspace{1cm} (38)

and therefore the volatility is given by

$$\sigma_t = \xi_t^{-1} S_t^{-1} \phi_t + \theta_t.$$  \hspace{1cm} (39)

To identify the process $\phi$ we need the following lemma.

**Lemma 2** *(Proposition 1.3.5 (Clark-Ocone) - Nualart (1995))*

Let $F \in D^{1,2}$, where $D^{1,2}$ is the closure of the class of smooth random variable
Suppose that $W$ is a one-dimensional Brownian motion. Then

$$ F = E(F) + \int_0^T E(\mathcal{D}_s F | \mathcal{F}_s) \, dW(s),  \quad (41) $$

taking conditional expectations

$$ E(F | \mathcal{F}_t) = E(F) + \int_0^t E(\mathcal{D}_s F | \mathcal{F}_s) \, dW(s).  \quad (42) $$

Proof see Nualart (1995) page 42.

The operator $\mathcal{D}$ in our setting is the Malliavin derivative with respect to the one-dimensional Brownian motion $\chi_1$, and $S$ is the class of smooth functionals of $\chi_1$. The process $\phi$ can be identified from Lemma 1

$$ \phi_t = E_t^1 \left[ \int_t^T \mathcal{D}_t (\xi_1 v) \, dv \right] \quad (43) $$

which we rewrite using the chain rule of Malliavin calculus as

$$ \phi_s = E_s^1 \int_t^T D_v D_t \xi_1 v \, dv + E_t^1 \int_t^T \xi_1 v D_v D_t \, dv. \quad (44) $$

Using this result we can write

$$ \xi_t S_t = E_0^1 \int_0^T \xi_1 v D_v dv + \int_0^t E_s^1 \left[ D_s \int_0^T \xi_1 v D_v dv \right] d\chi_1 s - \int_0^t \xi_1 v D_v dv. $$
Applying Itô’s lemma to the product $\xi_t S_t$ and equating stochastic terms yields the following equality

$$
\xi_{1t} S_t [-\theta_1^t + \sigma_t] = E_t^1 \left[ D_t \int_t^T \xi_{1v} D_v dv \right]
$$

which we rewrite using the chain rule of Malliavin calculus as

$$
\xi_{1t} S_t [-\theta_1^t + \sigma_t] = E_t^1 \int_t^T D_v D_t \xi_{1v} dv + E_t^1 \int_t^T \xi_{1v} D_t D_v dv.
$$

Explicit computation of the malliavin derivative can be found in Berrada (2003). We obtain the following representation for the volatility

$$
\xi_{1t} S_t \sigma_t = \xi_{1t} \theta_1^t S_t + E_t^1 \int_t^T D_v \left[ \frac{1}{\pi} \frac{1}{(y^T \eta^T \xi_1 v)} \xi_{1v} D_v \left[ \lambda + \int_t^v D_t m_{1v} dv \right] \right] dv
$$

$$
+ E_t^1 \int_t^T \sum_i \frac{\partial \xi_1}{\partial \eta_i} \left( \eta_{1v} \lambda^{-1} \left[ \int_t^v \lambda^{-1} D_t \Delta_{1s} ds - \int_t^v D_t \Delta_{1s} d\chi_{1s} + \Delta_{st} \right] \right) dv
$$

$$
+ E_t^1 \int_t^T \xi_{1u} D_v \left[ \lambda + \int_t^v D_t m_{1v} dv \right] dv
$$

The deviation introduced by the irrational agent is captured through the malliavin derivative of the process $\Delta_{2t}$ (the difference in beliefs), which evolves according to

$$
d\Delta_{2t} = (-1 - \varphi) \frac{\delta_2 \lambda + \varepsilon_{2t}}{\lambda^2} - \alpha) \Delta_{2t} dt + \frac{\varepsilon_{1t} - (1 - \varphi) \varepsilon_{2t} + \varphi \delta_2 \lambda}{\lambda} d\chi_{1t}
$$

as opposed to the usual case

$$
d\Delta_{2t} = (-\frac{\delta_2 \lambda + \varepsilon_{2t}}{\lambda^2} - \alpha) \Delta_{2t} dt + \frac{\varepsilon_{1t} - \varepsilon_{2t}}{\lambda} d\chi_{1t}
$$

We can see that even when conditional variances converges the difference in beliefs remain a stochastic process in the presence of a biased agent. We can
now compare the solution for the malliavin derivatives of $\Delta_{2t}$. In the normal case

$$
D_t (\Delta_{2v}) = \exp \left[ - \int_t^v \frac{\delta_2 \lambda + \varepsilon_2 s}{\lambda^2} + \alpha ds \right] \frac{\varepsilon_{1t} - \varepsilon_{2t}}{\lambda}
$$

in the presence of a biased agent

$$
D_t (\Delta_{2v}) = \exp \left[ - \int_t^v (1 - \vartheta) \frac{\delta_2 \lambda + \varepsilon_2 s}{\lambda^2} + \alpha ds \right] \frac{\varepsilon_{1t} - (1 - \vartheta)\varepsilon_{2t} + \vartheta \delta_2 \lambda}{\lambda}.
$$

Finally we obtain for the equilibrium volatility in the normal case

$$
\sigma_t = \lambda + \frac{c_{2t}}{C_t} \lambda^{-1} \Delta_{2t} + (1 - R) \frac{\delta_2 \lambda + \varepsilon_1 t E_t^1 \int_t^T \xi_{1t, v} D_v^{1 - e^\alpha(t-v)} dv}{S_t} \frac{\varepsilon_{1t} - \varepsilon_{2t}}{\lambda} \int_t^v \lambda^{-1} 2 \Delta_{2s} \exp \left[ - \int_s^t \frac{\delta_2 \lambda + \varepsilon_2 u}{\lambda^2} + \alpha du \right] ds \left[ \frac{\delta_2 \lambda + \varepsilon_2 u}{\lambda^2} + \alpha du \right] d\chi_{1u} + \Delta_{2t}
$$

and in the presence of a biased agent

$$
\sigma_t = \lambda + \frac{c_{2t}}{C_t} \lambda^{-1} \Delta_{2t} + (1 - R) \frac{\delta_2 \lambda + \varepsilon_1 t E_t^1 \int_t^T \xi_{1t, v} D_v^{1 - e^\alpha(t-v)} dv}{S_t} \frac{\varepsilon_{1t} - (1 - \vartheta)\varepsilon_{2t} + \vartheta \delta_2 \lambda}{\lambda} \int_t^v \lambda^{-1} 2 \Delta_{2s} \exp \left[ - \int_s^t (1 - \vartheta) \frac{\delta_2 \lambda + \varepsilon_2 u}{\lambda^2} + \alpha du \right] ds \left[ \frac{\delta_2 \lambda + \varepsilon_2 u}{\lambda^2} + \alpha du \right] d\chi_{1u} + \Delta_{2t}
$$

and

$$
\frac{\partial \xi_{1v}}{\partial \eta_{2v}} = - \left( \frac{y_1}{\rho_1} \right)^{\frac{1}{\pi}} \left( \frac{y_2}{\rho_2} \right)^{-\frac{1}{\pi}} \left( \frac{\eta_{2v}}{\eta_{2v}} \right)^{-\frac{1}{\pi}} \left( \frac{\eta_{2v}}{\rho_2} \right)^{-\frac{1}{\pi}}.
$$
References


### Tables

#### Table 1: Empirical mean and standard deviation of average consumption share of the Bayesian agent in a 2 agent economy when confronted to an over reactive agent or an under reactive agent. The parameters used are $\alpha=1.16$, $\beta=0.018$, $\lambda=0.036$, $\delta_1=0.11$, $\delta_2=0.11$, $\psi=0.01$. Agents have log utility. The time horizon is equal to 50 years. We use 0.25, 0.5, 0.75 and 1 for the bias coefficient. Average consumption shares are obtained by simulating the economy 100 000 times.

<table>
<thead>
<tr>
<th>$\vartheta$</th>
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<th>Std</th>
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<tr>
<td></td>
<td>0.25</td>
<td>0.5</td>
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<tr>
<td>Vs over reactive</td>
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<tr>
<td>Vs under reactive</td>
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<td>51.43</td>
</tr>
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</table>

#### Table 2: Empirical mean and standard deviation of average consumption shares in a 3 agent economy populated by a Bayesian agent, an Under Reactive agent and an Over Reactive agent. The parameters used are $\alpha=1.16$, $\beta=0.018$, $\lambda=0.036$, $\delta_1=0.11$, $\delta_2=0.11$, $\psi=0.01$. Agents have logarithmic utility. The time horizon is equal to 50 years. We use 0.25, 0.5 and 0.75 for the bias coefficient. Average consumption shares are obtained by simulating the economy 100 000 times.

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<thead>
<tr>
<th>$\vartheta$</th>
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<th>Std</th>
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<tr>
<td></td>
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<tr>
<td>Bayesian</td>
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<td>33.51</td>
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<tr>
<td>Under reactive</td>
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<td>34.96</td>
</tr>
<tr>
<td>Over reactive</td>
<td>32.42</td>
<td>31.54</td>
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</table>
Table 3: Empirical mean and standard deviation of average consumption shares in a 3 agent economy populated by a Bayesian agent, an Under Reactive agent and an Over Reactive agent. The parameters used are $\alpha=1.16$, $\beta=0.018$, $\lambda=0.036$, $\delta_1=0.11$, $\delta_2=0.11$, $\psi=0.01$. Agents have CRRA utility with $R = 4$. The time horizon is equal to 50 years. We use 0.25, 0.5 and 0.75 for the bias coefficient. Average consumption shares are obtained by simulating the economy 100,000 times.

<table>
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<tr>
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<th>Std</th>
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</thead>
<tbody>
<tr>
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<tr>
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<td>Over reactive</td>
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<td>32.70</td>
</tr>
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</table>

The time horizon is equal to 50 years. We use 0.25, 0.5 and 0.75 for the bias coefficient. Average consumption shares are obtained by simulating the economy 100,000 times.
Figure 1: Sample path of the unobserved process and filters. The top panel shows the true unobserved process $\gamma_t$. The top right panel shows the Bayesian filter. The bottom left panel displays the Under reaction bias and the bottom right panel the over reaction bias. The parameters used are $\alpha = 1.16$, $\beta = 0.018$, $\lambda = 0.036$, $\delta_1 = 0.11$, $\delta_2 = 0.11$, $\psi = 0.01$. The time horizon is equal to 50 years. The bias coefficient is set at $\vartheta = 0.5$. Conditional variances for all filters are set at the steady state, and initial beliefs are all equal to $\beta$. 
Figure 2: Average path of the irrational agent’s consumption share (or wealth share equivalently) and Sharpe ratio $\frac{\mu_t - r_t}{\sigma_t}$. The parameters are set at $\gamma = 0.05$ and constant through time, $\lambda = 0.15$, $m_{10} = \gamma$, $\varepsilon_{10} = \varepsilon_{20} = 0$, $m_{20} = m_{10} - 2\lambda^2$, $R = 1$ and $T = 400$. 
Figure 3: Average path of the irrational agent’s consumption share (or wealth share equivalently) and Sharpe ratio $\frac{\mu - \mu_t}{\sigma_t}$. The parameters are set at $\gamma = 0.05$ and constant through time, $\lambda = 0.15$, $m_{10} = \gamma$, $\varepsilon_{10} = \varepsilon_{20} = 0$, $m_{20} = m_{10} - 2\lambda^2$, $R = 4$ and $T = 400$. 
Figure 4: Average path of the irrational agent’s consumption share (or wealth share equivalently) and excess yield on long term bond. The parameters are set at $\gamma = 0.05$ and constant through time, $\lambda = 0.15$, $m_{10} = \gamma$, $\varepsilon_{10} = \varepsilon_{20} = 0$, $m_{20} = m_{10} - 2\lambda^2$, $R = 1$ and $T = 400$. 
Figure 5: Scatter plot of the average consumption shares in a 3 agent economy, populated by a Bayesian agent, an Under Reactive agent and an Over Reactive agent. Bayesian agent is shown on X-axis and Under Reactive agent on Y-axis. The parameters used are $\alpha = 1.16$, $\beta = 0.018$, $\lambda = 0.036$, $\delta_1 = 0.11$, $\delta_2 = 0.11$, $\psi = 0.01$. Agents have logarithmic utility. The time horizon is equal to 50 years. In the top left panel the bias coefficient is set at $\vartheta = 0.25$, in the top right panel $\vartheta = 0.5$, and in the bottom left panel $\vartheta = 0.75$. Average consumption shares are obtained by simulating the economy 100 000 times.
Figure 6: Scatter plot of the average consumption shares in a 3 agent economy, populated by a Bayesian agent, an Under Reactive agent and an Over Reactive agent. Bayesian agent is shown on X-axis and Over Reactive agent on Y-axis. The parameters used are $\alpha = 1.16$, $\beta = 0.018$, $\lambda = 0.036$, $\delta_1 = 0.11$, $\delta_2 = 0.11$, $\psi = 0.01$. Agents have logarithmic utility. The time horizon is equal to 50 years. In the top left panel the bias coefficient is set at $\vartheta = 0.25$, in the top right panel $\vartheta = 0.5$, and in the bottom left panel $\vartheta = 0.75$. Average consumption shares are obtained by simulating the economy 100,000 times.
Figure 7: Scatter plot of the average consumption shares in a 3 agent economy, populated by a Bayesian agent, an Under Reactive agent and an Over Reactive agent. Over Reactive agent is shown on X-axis and Under Reactive agent on Y-axis. The parameters used are $\alpha = 1.16$, $\beta = 0.018$, $\lambda = 0.036$, $\delta_1 = 0.11$, $\delta_2 = 0.11$, $\psi = 0.01$. Agents have logarithmic utility. The time horizon is equal to 50 years. In the top left panel the bias coefficient is set at $\vartheta = 0.25$, in the top right panel $\vartheta = 0.5$, and in the bottom left panel $\vartheta = 0.75$. Average consumption shares are obtained by simulating the economy 100 000 times.
Figure 8: Scatter plot of the average consumption shares in a 3 agent economy, populated by a Bayesian agent, an Under Reactive agent and an Over Reactive agent. Bayesian agent is shown on X-axis and Under Reactive agent on Y-axis. The parameters used are α = 1.16, β = 0.018, λ = 0.036, δ₁ = 0.11, δ₂ = 0.11, ψ = 0.01. Agents have CRRA utility with $R = 4$. The time horizon is equal to 50 years. In the top left panel the bias coefficient is set at $\vartheta = 0.25$, in the top right panel $\vartheta = 0.5$, and in the bottom left panel $\vartheta = 0.75$. Average consumption shares are obtained by simulating the economy 100 000 times.
Figure 9: Scatter plot of the average consumption shares in a 3 agent economy, populated by a Bayesian agent, an Under Reactive agent and an Over Reactive agent. Bayesian agent is shown on X-axis and Over Reactive agent on Y-axis. The parameters used are $\alpha = 1.16$, $\beta = 0.018$, $\lambda = 0.036$, $\delta_1 = 0.11$, $\delta_2 = 0.11$, $\psi = 0.01$. Agents have CRRA utility with $R = 4$. The time horizon is equal to 50 years. In the top left panel the bias coefficient is set at $\vartheta = 0.25$, in the top right panel $\vartheta = 0.5$, and in the bottom left panel $\vartheta = 0.75$. Average consumption shares are obtained by simulating the economy 100,000 times.
Figure 10: Scatter plot of the average consumption shares in a 3 agent economy, populated by a Bayesian agent, an Under Reactive agent and an Over Reactive agent. Bayesian agent is shown on X-axis and Over Reactive agent on Y-axis. The parameters used are $\alpha = 1.16$, $\beta = 0.018$, $\lambda = 0.036$, $\delta_1 = 0.11$, $\delta_2 = 0.11$, $\psi = 0.01$. Agents have CRRA utility with $R = 4$. The time horizon is equal to 50 years. In the top left panel the bias coefficient is set at $\vartheta = 0.25$, in the top right panel $\vartheta = 0.5$, and in the bottom left panel $\vartheta = 0.75$. Average consumption shares are obtained by simulating the economy 100 000 times.
Figure 11: Divergence in beliefs in a 2 agents economy populated by a Bayesian learner and an under reactive agent. We use the following parameters \( \lambda=0.036, \delta_1=0.11, \delta_2=0.11, \psi=0.01 \). Agents have logarithmic utility. The time horizon is equal to 50 years. We use 0.5 for the bias coefficient.
Figure 12: Market share of the Bayesian learner in a 2 agent economy populated by a Bayesian learner and an under reactive agent. We use the following parameters $\lambda=0.036$, $\delta_1=0.11$, $\delta_2=0.11$, $\psi=0.01$. Agents have logarithmic utility. The time horizon is equal to 50 years. We use 0.5 for the bias coefficient.
Figure 13: Kernel density of the market share of the Bayesian agent in a 2 agent economy. We use the following parameters $\lambda=0.036$, $\delta_1=0.11$, $\delta_2=0.11$, $\psi=0.01$. Agents have logarithmic utility. The time horizon is equal to 50 years. We use 0.5 for the bias coefficient.
Figure 14: Kernel density of the standard deviation of interest rate in a 2 agent economy. We use the following parameters $\lambda=0.036$, $\delta_1=0.11$, $\delta_2=0.11$, $\psi=0.01$. Agents have logarithmic utility. The time horizon is equal to 50 years. We use 0.5 for the bias coefficient.
Figure 15: Equilibrium volatility at time 0, as a function of relative risk aversion, $R$, and bias coefficient. NB: Negative values of the bias coefficient correspond to over reaction and positive value to under reaction. We use the following parameters $\lambda=0.036$, $\delta_1=0.11$, $\delta_2=0.11$, $\psi=0.01$. The time horizon is equal to 50 years.