Incomplete information, heterogeneity, and asset pricing

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ABSTRACT
We consider a pure exchange economy where the drift of aggregate consumption is unobservable. Agents with heterogeneous beliefs and preferences act competitively on financial and goods markets. We discuss how equilibrium market prices of risk differ across agents, and in particular we discuss the properties of the market price of risk under the physical (objective) probability measure. We propose a number of specifications of risk aversions and beliefs where the market price of risk is much higher, and the riskless rate of return lower, than in the equivalent full information economy (homogeneous and heterogeneous preferences) and thus can provide an(other) answer to the equity premium and risk-free rate puzzles. We also derive a representation of the equilibrium volatility and numerically assess the role of heterogeneity in beliefs. We show that a high level of stock volatility can be obtained with a low level of aggregate consumption volatility when beliefs are heterogeneous. Finally, we discuss how incomplete information may explain the apparent predictability in stock returns and show that in-sample predictability cannot be exploited by the agents, as it is in fact a result of their learning processes.

KEYWORDS: equity premium, heterogeneous beliefs and preferences

The determination of the equilibrium excess return of stock prices has been a central question in the asset pricing literature. The traditional approach, from Lucas (1978), links variations in stock price to variations of aggregate consumption in an elegantly simple manner. However, the consumption capital asset

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pricing model (CAPM) lacks empirical support. Observed stock returns are on average too high relative to the riskless rate. As Mehra and Prescott (1985) pointed out, in order to calibrate the model to consumption and stock price data, one needs to assume a very high level of risk aversion for the representative agent. This in turn would yield an unreasonably high riskless rate, as discussed in Weil (1989). Furthermore, the second moment of the stock return distribution is too high relative to the volatility of aggregate consumption growth, as initially mentioned in Grossman and Shiller (1981). In this article we show how incomplete information and heterogeneity in beliefs and preferences can help one understand the behavior of the stock price return and riskless rate and the failure of the consumption CAPM.

Several authors have studied the impact of incomplete information on portfolio consumption decisions [Detemple (1986), Genotte (1986), Karatzas and Xue (1991), Lakner (1995, 1998), Rogers (2001)] and equilibrium [Detemple and Murthy (1994, 1997), Veronesi (2000), Brennan and Xia (2001), Cecchetti, Lam, and Mark (2000), Riedel (2001), Jouini and Napp (2003)]. They have shown that problems under incomplete information can be transformed into problems under complete information, where unknown parameters and state variables are projected on the information set of agents. One can show that under this reformulation it is possible to write a representative agent model where the consumption CAPM holds. Most of this literature has focused on the equilibrium behavior of stock prices as it is perceived by the agents in the economy, and has not discussed the properties of the empirically measured returns and volatilities. In an incomplete information environment, random events are perceived differently by agents with heterogeneous beliefs. What appears as a negative surprise to an agent can be interpreted as a positive surprise by another agent. The econometrician who estimates the mean return and volatility does not observe the beliefs of the agents, and his measurements are affected by the true (objective) underlying evolution of the stochastic processes. Therefore, in order to understand the statistical properties of asset prices, one needs to identify the corresponding evolution of the stock prices in the objective probability measure. This is what we propose to do in this article.

We consider a pure exchange economy with incomplete information and heterogeneous agents. In our model, the expected growth rate of consumption is unobservable and follows a mean-reverting process. We depart from the existing literature by assuming that the long-term mean of the unobserved process is unknown and must be filtered as well. Agents have heterogeneous beliefs and are endowed with isoelastic utility with different levels of relative risk aversion. We develop the model by focusing on three problems.

Our first contribution is to consider how incomplete information and heterogeneity affect the market price of risk (MPR) and the riskless rate. Brennan and Xia (2001) and Cecchetti, Lam, and Mark (2000) also study the return dynamics under the objective probability measure, but in a homogeneous agent economy. The equilibrium MPR in an economy populated by agents differing in their
beliefs, but not in their preferences, was first established, for the objective probability measure, in Berrada (2001), to the best of our knowledge. In their discussion of the properties of the MPR and riskless rate in the objective probability measure, Jouini and Napp (2003) allow the agents to differ also in terms of preferences\(^1\). We consider here agents who differ in beliefs and preferences. In a homogeneous economy, high excess returns are obtained by assuming a pessimistic representative agent. In our setting, perceived MPRs differ across agents and incorporate the differences in beliefs and preferences. In the objective probability measure, the market price of risk is equal to its complete information equivalent plus a consumption-weighted average of the errors made by the agents. We show that a number of configurations of risk aversions and beliefs yield a high market price of risk and a low riskless rate. For instance, when initial beliefs are Gaussian and symmetric in mean around the true value of the unobserved parameter (no average pessimism or optimism), but the pessimistic agents have lower risk aversion and conditional variances, we obtain market prices of risk and a riskless rate of comparable magnitude to the empirical values. When multiple agents are considered, the representative agent may be pessimistic without having to assume that a majority of the agents are pessimistic.

Our second contribution is to provide an analytical formula for the volatility of the stock price. It is shown that it incorporates the immediate effect of the heterogeneity in beliefs, the expected impact of the future evolution of these beliefs, and the volatility of the aggregate consumption. Furthermore, the volatility is perceived identically by all agents and does not differ under the objective probability measure. The effects of incomplete information and heterogeneity in beliefs can be either positive or negative. Therefore the volatility of the stock price can be either higher or lower than the volatility of aggregate consumption. The idea that incomplete information can help one understand the high volatility of stock prices was first expressed in Bulkley and Tonks (1989) for the case of homogeneous beliefs. Timmermann (1993, 1996) formalized the argument in a discrete-time model with a partial equilibrium approach. Lewellen and Shanken (2002) extended the approach to a discrete-time equilibrium model, with short-lived agents endowed with exponential utility function. Zapatero (1998) considered a setting close to ours, but with logarithmic agents, and focused on the volatility of the interest rate induced by the heterogeneity in beliefs. Brennan and Xia (2001) and Veronesi (2000) provide an analysis of the volatility when agents have homogeneous beliefs. In a concurrent article, Gallmeyer (2000) obtains an analytical representation of the equilibrium volatility when beliefs are heterogeneous. His approach and the one developed in this article rely on the same mathematical tools and differ essentially in the choice of the underlying unobservable process. Gallmeyer (2000) does not discuss the impact of

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\(^1\) The MPR in the objective probability measure equals the homogeneous beliefs MPR plus a weighted sum of the individual errors. This result holds whether preferences are identical or differ. Differences in preferences only affect the weight of each error. A model with heterogeneous constant relative risk aversions is also considered in an appendix in Berrada (2001: section 5.7.3).
heterogeneous beliefs and preferences on the MPR and the riskless rate in the objective measure, which is our central result.

Our last contribution relates to the predictability of asset returns. We show how past dividend variations affect future stock returns. Timmermann (1993, 1996) and Lewellen and Shanken (2002) have also studied the impact of incomplete information on the predictability of asset return and we validate their results in a more general framework. We simulate the economy and regress the stock returns against past values of the dividend yield and dividend growth and obtain statistically significant coefficients. We explain this result by looking at the long-term equilibrium value of the excess return. A term related to the average error made by the agents in the economy is shown to be related to past dividends. However, that term is by definition unpredictable, conditional on the information available to the agents. They rationally update their beliefs by taking into account the observable variations of the dividend, and this creates an appearance of predictability.

This article is organized as follows. Section 1 describes the economy, along with the optimal consumption and equilibrium. Section 2 discusses the connection between subjective and objective measures. Section 3 provides the numerical results for the MPR, the volatility, and the predictability. We conclude in Section 4.

1 ECONOMY

We consider a pure exchange economy with finite horizon $T$, where the uncertainty is described by a probability space $(\Omega, \mathcal{F}, \mathbb{P}^0)$ on which two independent Brownian motions, $W_1$ and $W_2$, are defined. We let $\mathcal{F} = \mathcal{F}_W^{W_1, W_2}$ and $F = \mathcal{F}_T$. $\Omega$ is the canonical state space $C^0$ and $\mathbb{P}^0$ is the Wiener measure. There is a single, perishable good that serves as the numeraire. The financial market consists of two securities, a locally riskless money market account in zero net supply that pays an interest rate of $r_t$ and a risky stock with price $S_t$ representing a claim to an exogenous stream of dividend payments $D_t$. The dividend process is of the form

$$D_t = D_0 \exp \left[ \int_0^t \left( \gamma_s - \frac{1}{2} \lambda_s^2 \right) ds + \int_0^t \lambda_s dW_1_s \right],$$

where the volatility $\lambda_t$ is a square integrable process adapted to the natural filtration of $W_1$ and the drift process $\gamma_t$ is a square integrable process adapted to $\mathcal{F}$. The initial value of the money market account is normalized to one, so in equilibrium, the price process $B_t$ is given by

$$B_t = \exp \left[ \int_0^t r_s ds \right].$$

There are two agents in the economy endowed with an initial number of shares $\pi_{i0} > 0$. Units of the money market account are denoted $\pi_{i0}$. The flow of information available to the agents is restricted to the filtration generated by the
dividend process $\mathcal{F}^D$. We assume that agents have heterogeneous Gaussian initial beliefs about the drift process $\gamma_t$, with mean $m_{i0}$ and variance $\varepsilon_{i0}$. Subjective probability measures are represented by $\mathbb{P}^i$. Agents’ preferences over consumption are described by the expected utility function

$$U_i(c) := \mathbb{E}^i \left[ \int_0^T \rho_t \frac{c_t^{1-R_t}}{1 - R_t} dt \right],$$

where $R_t$ refers to the individual specific relative risk aversion, $\rho_t := e^{-\psi t}$ is the common discount rate, and $\psi$ is a positive constant. The dividend process is represented in the subjective probability space as a function of the innovation process:

$$D_t = D_0 \exp \left[ \int_0^t \left( m_{is} - \frac{1}{2} \lambda_t^2 \right) ds + \int_0^t \lambda_s d\chi_{is} \right].$$

The cum dividend stock price process in the subjective measure is given by

$$S_t + \int_0^t S_s \mu_{is} ds = S_0 + \int_0^t \int_0^t S_s \sigma_s d\chi_{is} + \int_0^t S_s \sigma_s d\chi_{is}.$$

The quantities $(\mu_{it}, \sigma_{it}, \rho_t)$ are obtained endogenously in equilibrium.

Trading takes place continuously and there is no friction. The wealth process $X_{it}$ associated to a trading strategy $\pi_{it}$ takes the form

$$X_{it} = \pi_{it} S_t + \pi_{it}^0 B_t.$$

A trading strategy is admissible if $X_{it}$ is uniformly bounded from below by a constant [this guarantees the absence of arbitrage; see Dybvig and Huang (1988)]. A consumption plan $(c_t)$ is said to be feasible if there exists an admissible trading strategy that solves the agent’s dynamic budget constraint

$$dX_{it} = \pi_{it}^0 dB_t + \pi_{it} [dS_t + D_t dt] - c_{it} dt.$$

### 1.1 Individual Choices

Note that the economy is initially incomplete, as the number of risky securities is inferior to the number of Brownian motions. However, every contingent claim adapted to $\mathcal{F}^D_t$ is attainable, and therefore, when information is incomplete and restricted to $\mathcal{F}^D_t$, the market can be treated as complete for the purpose of individual consumption and investment decision. We therefore use the standard result of Cox and Huang (1989) to solve the static problem equivalent to the dynamic consumption-investment decision. In order to present optimal consumption, we define the individual specific state price density process

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2 For a general discussion of continuous-time filtering and innovation processes, see Lipster and Shiryaev (1978).
\[
\xi_{it} := \exp \left[ - \int_0^t r_s ds - \int_0^t \theta_{is} d\chi_{is} - \frac{1}{2} \int_0^t \|\theta_{is}\|^2 ds \right],
\]
where \( \theta_{it} := \frac{\mu_{i} - \bar{\mu}_{i}}{\sigma_{i}} \) is the subjective MPR. Individual state price densities are linked by the expression \( \xi_{2t} = \eta_{2t} \xi_{1T} \), where \( \eta_{2t} \) satisfies
\[
\eta_{2t} = \eta_{20} \exp \left[ \frac{1}{2} \int_0^t \left( \frac{\Delta_{2s}}{\lambda_t} \right)^2 ds + \int_0^t \frac{\Delta_{2s}}{\lambda_t} d\chi_{1s} \right]
\]
and \( \Delta_{2t} = m_{1t} - m_{2t} \). The static budget set is defined as
\[
B(x_0) = \left( c_i : E^t \left[ \int_0^T \xi_{it} c_{it} dt \right] \leq \pi_{i0} S_0 \right).
\]
The static optimization problem is given by
\[
\max_{c_i} U(c_i) \quad c_i \in B(\pi_{i0} S_0),
\]
which yields the following optimal individual policies
\[
c_{it} = \left( y_i \xi_{it} \rho_t \right)^{-\frac{1}{\rho_t}},
\]
where \( y_i \) is a Lagrange multiplier that solves agent \( i \)'s static budget constraint
\[
x_{i0} = E^t \left[ \int_0^T \xi_{it} \left( y_i \xi_{it} \rho_t \right)^{-\frac{1}{\rho_t}} dt \right].
\]

### 1.2 Equilibrium

An equilibrium is a combination \( \{(c_{it}, \pi_{it}); (r_t, S_t)\} \), where \( (c_{it}, \pi_{it}) \) is an optimal admissible strategy for all \( i \) taking \( (r_t, S_t) \) as given and all markets clear, \( \sum_i c_{it} = D_t, \sum_i \pi_{it} = 1, \sum_i \pi_{it}^0 = 0 \). In the goods market, the market clearing condition implies that all dividends are consumed. In the financial market, the market clearing conditions imply that the stock is held and the bond is in zero net supply. In order to derive the equilibrium, we choose a reference agent, which is arbitrarily called agent 1. This enables us to compute all expectations under the probability measure \( \mathbb{P}^1 \). The equilibrium is expressed in terms of the reference agent subjective measure.

**Proposition 1.** When it exists, the competitive equilibrium for the economy described in Section 2 is given by
\[
\theta_{1t} = R_{st} \lambda_t + \frac{c_{2t} R_{sf}}{C_f R_2 \lambda_f^{-1}} \Delta_{2t}
\]
\[ \theta_{2t} = \theta_{1t} - \lambda_t^{-1} \Delta_{2t} \]  

(15)

\[ r_t = \psi + R_{at} m_{1t} - \frac{c_{2t}}{C_t} R_{at} \lambda_t^{-1} \Delta_{2t}, \theta_{2t} - \sum_i \frac{c_{it}}{C_t} \frac{R_{at} (1 + R_i)}{R^2_i} \theta_{it}^2 \]  

(16)

\[ S_t = E^1 \left[ \int_t^T \rho_{t,v} \frac{u'_1(c_{1v})}{u'_1(c_{1v})} D_v d\nu \right] \]  

(17)

where \( R_{at} = \left[ \sum_i \frac{c_i}{C_i} \frac{1}{R} \right]^{-1} \) and \( C_t = c_{1t} + c_{2t} \). [Proof see Detemple and Murthy (1997).]

The existence and uniqueness of this equilibrium can be established as in Riedel (2001), provided that the volatility of the stock remains invertible and other coefficients satisfy integrability conditions. We will assume this is the case.

The perceived MPRs, \( \theta_{it} \), differ across agents, and the difference is due to the heterogeneity in beliefs. The stock price is equal to the expected discounted sum of future dividends, and the discount factor is the intertemporal marginal rate of substitution. The equilibrium is stated relative to agent 1, but there are no disagreements about the price of the risky security, that is, \( E^1_t \left[ \int_t^T \rho_{t,v} \frac{u'_1(c_{1v})}{u'_1(c_{1v})} D_v d\nu \right] = E^2_t \left[ \int_t^T \rho_{t,v} \frac{u'_2(c_{2v})}{u'_2(c_{2v})} D_v d\nu \right] \). The full information value of the market price of risk is given here as a benchmark. In the full information case, \( m_{it} = \gamma_t \) for both agents, and therefore \( \Delta_{2t} = 0 \), which yields

\[ \theta_t = R_{at} \lambda_t. \]  

(18)

This result leads to the well-known consumption CAPM.

2 SUBJECTIVE AND OBJECTIVE MEASURES

Under the objective measure, the stock price may be written as

\[ \frac{dS_t + D_t dt}{S_t} = \mu^o dt + \sigma^o dW_{2t} + \lambda^{-1}_t (\gamma_t - m_{it}) dt \]  

(19)

\[ = \mu^o dt + \sigma^o dW_{2t}. \]  

(20)

\( \mu^o \) and \( \sigma^o \) denotes the instantaneous stock return and volatility under the objective measure. An econometrician considering the time series of stock prices will get an estimate of \( \mu^o \), and not \( \mu_{it} \). Extending our definition of the subjective MPR, we define \( \theta^o_t \equiv \frac{\mu^o_t - \gamma_t}{\sigma^o_t} \), the MPR in the objective probability measure.

2.1 Market Price of Risk

The perceived MPR for any agent differs from the market price of risk under the objective probability measure. The latter is given by the formula
\[ \theta_t^o = R_{at}\lambda_t + \lambda_t^{-1}\left[\gamma_t - \sum_i c_{it} \frac{R_{at}}{R_i} m_{it}\right]. \]  

(21)

Note that this construction is equivalent to the consensus characteristic construction as in Jouini and Napp (2003). As they point out, the difference between the homogeneous beliefs MPR and the heterogeneous beliefs MPR for the case of power utility is given by a weighted average of the individual subjective beliefs. Let us define the aggregate belief \( m_{at} = \frac{\sum_i \frac{m_{it}}{c_{it}}}{\sum_i \frac{1}{c_{it}}}. \) The MPR is then given by

\[ \theta_t^o = R_{at}\lambda_t + \lambda_t^{-1}[\gamma_t - m_{at}]. \]  

(22)

We recover here a well-known result that a representative agent must be pessimistic, that is, \( m_{at} < \gamma_t, \) in order to obtain values of \( \theta_t^o, \) which are above the full information result [see Cecchetti, Lam, and Mark (2000) and Abel (2002)]. It is difficult, however, to justify, in the single-agent framework, why the beliefs, if formed rationally, should systematically remain below the true value. Heterogeneity seems to provide a justification, as the process \( m_{at} \) does not represent a Bayesian updating, but rather a weighted sum of beliefs. The restriction to consider in order to observe high values for the market price of risk becomes

\[ m_{at} < \gamma_t. \]

Apparent pessimism at the aggregate level, \( m_{at}, \) can thus be driven by any combination of three effects: (i) the wealth effect, if the agent with lowest beliefs has larger wealth; (ii) the risk aversion effect, when agents with the lowest beliefs also have the lowest risk aversion; and (iii) the beliefs effect, when risk aversion and wealth are identical, aggregate pessimism is obtained by assuming that the dispersion of beliefs is no longer symmetric on average. This is obtained by assuming, for example, symmetric initial beliefs and lower conditional variance for the pessimistic agents.

2.2 Volatility

In this section we discuss the equilibrium volatility and provide an analytical representation that emphasizes the role of incomplete information and heterogeneity in beliefs.

**Proposition 2.** In the economy described in Section 1, the volatility of the stock price is identical under the subjective and objective probability measures. For the case of \( \lambda_t = \lambda, \) a constant, it is given by the formula

\[ \sigma_t = \lambda + \Omega_t + \Pi_t + \Xi_t, \]  

(23)

where \( \Omega_t \) is the immediate impact of heterogeneity in beliefs and preferences, \( \Pi_t \) is the expected impact of the future evolution of the beliefs of agent 1, and \( \Xi_t \) is the correction for heterogeneity in beliefs and preferences on the future impact of incomplete information. \( \Omega_t, \Pi_t, \) and \( \Xi_t \) are adapted to the filtration \( \mathcal{F}^T \) and are derived in the appendix.
The volatility is obtained by considering the equilibrium stock price.\(^3\)

\[ S_t = E_{t}^{1} \left[ \int_{t}^{T} \xi_{t, v}^1 D_v d\nu \right], \]  

(24)

which can also be written as

\[ \xi_t S_t = E_{t}^{1} \left[ \int_{0}^{T} \xi_{v}^1 D_v d\nu \right] - \int_{0}^{t} \xi_{v}^1 D_v d\nu. \]  

(25)

We use the fact that \( E_{t}^{1} \left[ \int_{0}^{T} \xi_{v}^1 D_v d\nu \right] \) is a \((F_{t}^{0}, F_{1})\)-martingale and admits a representation as an integral of the innovation process,

\[ E_{t}^{1} \left[ \int_{0}^{T} \xi_{v}^1 D_v d\nu \right] = E_{0}^{1} \left[ \int_{0}^{T} \xi_{v}^1 D_v d\nu \right] + \int_{0}^{t} \phi_s d\chi_1. \]  

(26)

As \( \chi_1 \) is an \((F_{t}^{0}, F_{1})\)-Brownian motion, the Clark-Ocone formula [Clark (1970)] applies, and the integrand \( \phi_s \) can be expressed as a conditional expectation of the Malliavin derivative\(^4\) of \( \int_{0}^{T} \xi_{v}^1 D_v d\nu \). The integrand \( \phi_s \) plays an important role in the volatility formula presented in Proposition 2. Malliavin derivatives and the generalized version of Clark's theorem have been used in Karatzas and Ocone (1991) and recently in Detemple, Garcia, and Rindisbacher (2003) for the representation of optimal portfolios. For a detailed discussion on Malliavin calculus, refer to Nualart (1995).

3 NUMERICAL ANALYSIS

3.1 Market Price of Risk and Riskless Rate

Equation (21) provides a closed-form solution for the market price of risk under the objective probability measure, which we denote as \( \theta_{t}^o \). An econometrician observing the time series of stock returns will obtain an estimate of this quantity. Ideally we would like to obtain an expression for the average of the market price of risk over a given time interval, that is, \( E_{t} \left[ \frac{1}{T-t} \int_{t}^{T} \theta_{t}^o dt \right] \). Due to the complex path dependency of the quantities appearing in \( \theta_{t}^o \), it is not possible to obtain a closed-form solution to the latter expression. To overcome this difficulty, we will simulate the economy and approximate this expectation. Furthermore, we restrict the preferences to the case \( R_2 = pR_1 \), with \( p \) an integer. In this case, the state price density is the root of a polynomial of order \( p \) which admits a unique real positive solution for \( p \in \{2, 3, 4, 5\} \).\(^5\) We proceed by describing the evolution of the underlying stochastic processes.

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\(^3\) The term \( \xi_{t, v} \) is to be understood as \( \xi_t \).

\(^4\) Notice here that the Malliavin operator acts on the innovation process and not on the Brownian motion.

\(^5\) This result was first pointed out in Wang (1996).
then we discuss the filtering procedure and simulation methodology, and present and explain the numerical results.

3.1.1 Filtering. We assume that the unobservable dividend growth rate $\gamma_t$ follows a mean-reverting process of the form

$$d\gamma_t = \alpha(\beta - \gamma_t)dt + \delta_1 dW_{1t} + \delta_2 dW_{2t}.$$  \hfill (27)

The speed of convergence $\alpha$ and the volatility coefficients $\delta_1$ and $\delta_2$ are common knowledge. Unlike previous studies [Brennan and Xia (2001)], we do not assume that the long-term mean of the dividend growth rate is known. The agents must construct their beliefs based on the observations of the dividend $D_t$. Two conditional means and a conditional covariance matrix must be constructed. Let us define the individual specific covariance matrix and conditional mean vectors

$$\sum^i_t = \begin{bmatrix} \varepsilon_{i,11}^t & \varepsilon_{i,12}^t \\ \varepsilon_{i,12}^t & \varepsilon_{i,22}^t \end{bmatrix}$$

$$M_{it} = \begin{bmatrix} m_{it} \\ l_{it} \end{bmatrix},$$ \hfill (28)

where $l_{it} = E^i[\beta | \mathcal{F}_t]$ and $m_{it} = E^i[\gamma_t | \mathcal{F}_t]$. The filtering equations are obtained using Theorem 12.6 in Lipster and Shiryaev (1978). The conditional expectations evolve according to

$$dm_{it} = \alpha(l_{it} - m_{it})dt + \frac{\delta_2 \lambda + \varepsilon_{i,11}^t}{\lambda^2} \left[ \frac{dD_t}{D_t} - m_{it}dt \right]$$

$$dl_{it} = \frac{\varepsilon_{i,12}^t}{\lambda^2} \left[ \frac{dD_t}{D_t} - m_{it}dt \right],$$ \hfill (29)

while the covariance matrix evolution is given by

$$\frac{d\varepsilon_{i,11}^t}{dt} = 2\alpha(\varepsilon_{i,12}^t - \varepsilon_{i,11}^t) + \delta_2^2 + \delta_2^2 - \frac{[\delta_2 \lambda + \varepsilon_{i,11}^t]^2}{\lambda}$$

$$\frac{d\varepsilon_{i,12}^t}{dt} = \alpha(\varepsilon_{i,22}^t - \varepsilon_{i,11}^t) - \frac{\delta_2 \lambda + \varepsilon_{i,11}^t \varepsilon_{i,12}^t}{\lambda^2}$$

$$\frac{d\varepsilon_{i,22}^t}{dt} = -\left(\frac{\varepsilon_{i,12}^t}{\lambda}\right)^2.$$ \hfill (30)

Under the agents’ subjective probability measures, the conditional mean $m_{it}$ follows a mean-reverting process, with the speed of convergence equal to the unobservable dividend growth rate and long-term mean equal to the conditional expectation of the true long-term mean.

All expectations given in the following results are given relative to the objective probability measure $\mathbb{P}^0$. 

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3.1.2 Simulation Methodology. We construct the simulation as follows. First, we simulate the path of the exogenous variables \( \gamma_t, D_t \). Then, using the individual prior beliefs and the filtering equations, we construct \( (M_{it}, \Sigma_{it}, \chi_{it}) \) as well as the divergence process \( \Delta_{2t} \) and the process \( \eta_{2t} \). Finally, as we want to initially endow each agent with half of the stock, we calibrate the Lagrange multiplier of the static budget constraints, introduced in Equation (12), to guaranty that initial levels of wealth are equal and that markets clear at all times. Namely, we choose \((y_1, y_2)\) such that

\[
E_t^1 \int_0^T \xi_{1t} \left( \frac{y_1 \xi_{1t}}{\rho_t} \right)^{-\frac{1}{\alpha_1}} dt = E_t^1 \int_0^T \xi_{1t} \eta_{2t} \left( \frac{y_2 \xi_{1t} \eta_{2t}}{\rho_t} \right)^{-\frac{1}{\alpha_2}} dt
\]

and

\[
\left( \frac{y_1 \xi_{1t}}{\rho_t} \right)^{-\frac{1}{\alpha_1}} + \left( \frac{y_2 \xi_{1t} \eta_{2t}}{\rho_t} \right)^{-\frac{1}{\alpha_2}} = D_t
\]

holds for all \( t \in [0, T] \). Note that all expectations in the following sections are given relative to the objective probability measure \( \mathbb{P}^0 \).

3.1.3 Results. In the simulation exercise, the horizon is set at 20 years and it is assumed that the subjective discount rate \( \psi \) equals 0.02. Parameter values for the consumption process are obtained from the Shiller dataset (www.econ.yale.edu/~shiller/data.htm), which provides per capita consumption in the United States over the period 1871–2003. Parameters are listed in Table 1; the salient feature is the low volatility of consumption, which forces a high level of risk aversion to fit the full information consumption CAPM to the stock return data.

We want to assess the level of heterogeneity necessary to obtain a high level of MPR (0.3) and a low level of interest rate (0.01), and more importantly, whether the two issues can be addressed simultaneously. For our set of parameters and a risk aversion level of 4.5, the full information and homogeneous preference levels for the MPR and the riskless rate are, respectively, 0.16 and 0.0749. We compute average values of the MPR and the riskless rate when varying the average divergence, defined as \( \frac{1}{T} \int_0^T (m_{1t} - m_{2t}) dt \), from 0.5% to 2%. The computation is repeated with \( R_2 = pR_1 \) and \( p \in \{2, 3, 4, 5\} \). In all computations we assign the lowest initial conditional standard deviation to the pessimistic agent\(^6\) and set initial beliefs symmetric around the true long-term mean. Results are displayed in Figure 1. As expected, the MPR increases and the riskless rate decreases as a function of the divergence in beliefs. When preferences are close \( (p = 2) \), the decrease in interest rate is fast and a level of MPR of 0.3 is obtained only by allowing the interest rate to drop to \(-3\%\). In order to maintain an acceptable level of interest rate we must consider important differences in risk aversion. Indeed,

\( ^6 \) This assumption ensures that the pessimistic belief has a slower convergence toward the long-term mean and therefore lowers \( m_{at} \). An opposite configuration would increase \( m_{at} \) and lower the MPR in the objective probability measure.
Table 1 Parameters used in the simulation for the computation of average MPR and riskless rate.

<table>
<thead>
<tr>
<th>Consumption growth rate coefficients</th>
<th></th>
</tr>
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<tbody>
<tr>
<td>( \alpha )</td>
<td>1.16</td>
</tr>
<tr>
<td>( \delta_1 )</td>
<td>0.011</td>
</tr>
<tr>
<td>( \delta_2 )</td>
<td>0.011</td>
</tr>
<tr>
<td>( \lambda )</td>
<td>0.036</td>
</tr>
<tr>
<td>( \beta )</td>
<td>0.018</td>
</tr>
</tbody>
</table>

Prior conditional covariance (pessimistic)

| \( \varepsilon_{11}^{1} \) | 0.001 |
| \( \varepsilon_{12}^{1} \) | -0.0003 |
| \( \varepsilon_{22}^{1} \) | 0.0005 |

Prior conditional covariance (optimistic)

| \( \varepsilon_{11}^{2} \) | 0.005 |
| \( \varepsilon_{12}^{2} \) | 0.0001 |
| \( \varepsilon_{22}^{2} \) | 0.001 |

Consumption coefficients are obtained from per capita consumption over the period 1871–2003 in the United-States by maximum-likelihood estimation of the exact discretization of the Ornstein-Uhlenbeck process postulated for the growth rate of aggregate consumption.

Figure 1 Pairs of MPRs and riskless rates for the level of average divergence ranging from 0.55% to 2.14%. \( R_2 = pR_1 \); dotted line, \( p = 2 \); dashed line, \( p = 3 \); dashed-dotted line, \( p = 4 \); solid line, \( p = 5 \).
when $p = 5$, a level of MPR of 0.3 is obtained with an interest rate of 1.22%. This is achieved with an average divergence of 1.68%, which is less than half of a standard deviation of the growth rate of aggregate consumption, and therefore perfectly plausible. Figures 2 and 3 display values obtained for the MPR and the riskless rate by varying the level of disagreement. Note that the divergence in beliefs about the growth rate of aggregate consumption implies a divergence in beliefs about the market return. Indeed, from the equilibrium market price of risk we obtain

$$\mu_{it} - \mu_{2t} = \frac{\sigma_t}{\lambda} (m_{1t} - m_{2t}).$$  \hfill (31)

It has been argued in a number of articles [Anderson, Ghysels, and Juergens (2005), Ajinkya and Gift (1985)] that one may use dispersion among analysts’ forecasts as a measure of heterogeneous beliefs. As we are using a reduced-form model with only two agents, the standard deviation of beliefs is equal to the divergence in beliefs, and therefore, to support our results, the standard deviation of market return forecasts must be in the range $\sqrt{\frac{\sigma_t}{N}} \times [0.5\%, 2\%]$. While we assume a constant value for $\lambda$ ($\lambda = 0.036$, see Table 1), we have shown in Section 3.2 that $\sigma_t$ is a stochastic process, and therefore the corresponding range of divergence for market forecast is also

Figure 2 MPR as a function of average divergence in beliefs when $R_2 = 7.5$ and $R_1 = 1.5$. 
stochastic. If we take an approximate value for $\sigma_t$ of 20%, we obtain a range of approximately [1.18%, 4.7%] for the standard deviation of market return forecasts. Anderson, Ghysels, and Juergens (2005) report analysts’ forecast dispersion around 4% for the return of the S&P 500 between 1991 and 1997. From the Livingston Survey\(^7\) of the Federal Reserve Bank of Philadelphia, we compute the standard deviation of the 12-month-ahead market return forecasts made by a panel of economists from industry, government, banking, and academia between December 1994 and December 2003, and we obtain values ranging from 3.7% to 11.2% (see Table 2). Based on this simple verification, the level of divergence required to generate a high market price of risk along with a low risk-free rate does not seem unreasonably high. It could be argued, however, that the level of heterogeneity in preferences ($p = 5$) required to obtain a sufficiently low risk-free interest rate is too large. This could be explained by the fact that our model does not incorporate frictions such as restrictions on short sales or other forms of incompleteness. To obtain a large MPR, we give a larger weight to the pessimistic belief by endowing the pessimistic agent with a lower risk aversion and therefore, on average, a larger

---

\(^7\) Data from the Livingston Survey may be freely downloaded from http://www.phil.frb.org/econ/liv/index.html. The data used for the computation is constructed from all the available forecasts (approximately 50 economists) given in June and December of each year.
consumption share. An opposite configuration would of course generate an MPR inferior to the full information case. A combination of time-varying risk aversion and heterogeneous beliefs could therefore support episodes of large as well as low MPR.

3.2 Volatility

It is computationally infeasible to simulate dynamic properties of the stock price volatility, as each point in time requires extensive simulation to obtain the value of conditional expectation of stochastic integrals. However, we may consider comparative statics at a given point in time. In this section we simplify the filtering procedure and consider the case where the long-term mean is known and agents differ in anticipations, but not in preferences. The volatility equation can then be expressed as (see Appendix A)

\[ \sigma_t = \lambda + \Omega_t + \Pi_t + \Xi_t, \]

where

<table>
<thead>
<tr>
<th>Date</th>
<th>No. of forecasts</th>
<th>Mean</th>
<th>Standard deviation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Dec-94</td>
<td>59</td>
<td>2.36%</td>
<td>3.88%</td>
</tr>
<tr>
<td>Jun-95</td>
<td>49</td>
<td>2.01%</td>
<td>4.89%</td>
</tr>
<tr>
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<td>59</td>
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<td>6.67%</td>
</tr>
<tr>
<td>Jun-96</td>
<td>55</td>
<td>3.39%</td>
<td>6.44%</td>
</tr>
<tr>
<td>Dec-96</td>
<td>64</td>
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<td>7.24%</td>
</tr>
<tr>
<td>Jun-97</td>
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<td>1.18%</td>
<td>7.57%</td>
</tr>
<tr>
<td>Dec-97</td>
<td>50</td>
<td>-0.43%</td>
<td>8.54%</td>
</tr>
<tr>
<td>Jun-98</td>
<td>53</td>
<td>3.17%</td>
<td>9.17%</td>
</tr>
<tr>
<td>Dec-98</td>
<td>56</td>
<td>0.06%</td>
<td>7.05%</td>
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<tr>
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</tr>
<tr>
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<td>11.17%</td>
</tr>
<tr>
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<td>38</td>
<td>3.90%</td>
<td>8.84%</td>
</tr>
<tr>
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<td>8.36%</td>
</tr>
<tr>
<td>Jun-01</td>
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<td>8.65%</td>
<td>5.02%</td>
</tr>
<tr>
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<td>8.65%</td>
<td>3.71%</td>
</tr>
<tr>
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<td>12.47%</td>
<td>7.05%</td>
</tr>
<tr>
<td>Dec-02</td>
<td>27</td>
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<td>4.49%</td>
</tr>
<tr>
<td>Jun-03</td>
<td>28</td>
<td>15.33%</td>
<td>7.93%</td>
</tr>
<tr>
<td>Dec-03</td>
<td>30</td>
<td>10.31%</td>
<td>4.90%</td>
</tr>
</tbody>
</table>

The stock volatility is equal to the full information volatility, that is, the volatility of the dividend, plus three extra terms. The first extra term is due to the immediate effect of the heterogeneity in beliefs on the perceived market prices of risk. The second extra term is the effect of incomplete information only, and the last term is the long-term effect of heterogeneity. It is important to note that the last term vanishes when \( \eta = \frac{1}{2} \). Heterogeneity has a long-term effect only through the divergence in precision. When the conditional variances are equal, the evolution of the divergence in beliefs, \( \Delta_t \), is deterministic. The evolution of \( \Delta_t \) is given by

\[
d\Delta_t = \left( \frac{\partial \lambda}{\partial \eta} - \frac{\partial \lambda}{\partial \eta} \right) \Delta_t dt + \frac{\partial \lambda}{\partial \eta} d\chi_t
\]

and has therefore a volatility equal to \( \frac{\partial \lambda}{\partial \eta} \).

Figure 4 displays the volatility as a function of risk aversion and divergence in beliefs. Parameter values are identical to the previous section and conditional standard deviations for the pessimistic and optimistic agents are set equal to 2% and 2.5%, respectively. The model is able to generate a high level of volatility (15–20%) with a relatively low level of risk aversion (4–5) and low dividend volatility (3.6%) when beliefs are not identical. Notice also that, for some parameter values, the stock volatility could be lower than the dividend volatility, which in some sense further restricts the class of plausible parameters.

### 3.3 Predictability of Asset Returns

Empirical studies such as Fama and Schwert (1977) and Kothari and Shanken (1997), to name a few, have found that variables such as interest rates, dividend yields, and default premium have the ability to forecast stock returns. This troubling fact might raise doubts about either market efficiency or investors’ rationality. Since past dividend yields and interest rates are available to investors,
any informational content should be included in the price and the variations in returns should not be predictable based on this information. In this section we look at the implication of incomplete information on the issue of the predictability of stock returns. Our model displays no predictability when considering the perspective of the investor (and therefore his subjective probability measure), but when we look at the statistical properties of the excess return, we indeed find that there is a link between past dividends and future expected stock returns.

We will first consider the case of homogeneous beliefs, which simplifies the exposition and yet conveys the general idea. When beliefs are homogeneous, the excess return in the objective probability measure is given by

$$r_t - R = \mu_t - \frac{\sigma_t}{\lambda_t} (\gamma_t - m_t),$$

which is a particular case of Equation (21) when $m_i = m$ for all $i$. Considering the mean-reverting assumption for the drift of the dividend process, and assuming that $\alpha$ and $\beta$ are known, the current belief $m_t$ is given by

$$m_t = m_0 + \int_0^t \alpha [\beta - m_s] \, ds + \int_0^t \frac{\delta_2 \lambda_t + \epsilon_s}{\lambda_t^2} \left[ \frac{dD_s}{D_s} - m_s \, ds \right],$$

which we can alternatively write for any $v \in [0, T]$

**Figure 4** Equilibrium volatility as a function of risk aversion and divergence in beliefs. Beliefs are symmetric around the true unobserved parameter. $T = 20, \psi = 0.02$, $(\varepsilon_1)^{1/2} = 2\%$, $(\varepsilon_2)^{1/2} = 2.5\%$, $\delta_1 = 0.011$, $\delta_2 = 0.011$, $\beta = 0.018$, $\lambda = 0.036$. 


\[ m_t = m_v + \int_v^t \alpha[\beta - m_s] \, ds + \int_v^t \frac{\delta_2 \lambda_t + \varepsilon_s}{\lambda_t^2} \left[ \frac{dD_s}{D_s} - m_s \right] ds. \]  \hspace{1cm} (36)

Let us consider an interval \([v, t]\) where \(\int_v^t \frac{\delta_2 \lambda_t + \varepsilon_s}{\lambda_t^2} \left[ \frac{dD_s}{D_s} - m_s \right] ds > 0\), implying realized dividend growth exceeds the growth anticipated by the agent. If the time interval is sufficiently short (e.g., 1 month), the mean reversion effect based on empirical values of the coefficient \(\alpha\) is very small. Therefore, the main impact on variation of \(m_t\) is due to the innovation effect. So in this case, a high dividend growth period implies an increase in \(m_t\), but what matters for the predictability is how \(m_t\) compares to \(\gamma_t\). If we take \(m_v = \gamma_v\), then we have a negative relation between dividend growth and stock return; high dividend growth will be followed by lower expected excess return. But depending on the relation between \(m_v\) and \(\gamma_v\), it may well be the case that high dividend growth periods are followed by higher expected excess return.

Since in the long run \(m_t\) will fluctuate around \(\gamma_t\), the negative relation should prevail. Note that from the perspective of the investor, the expected excess return is always equal to \(R \lambda \sigma_t\), since under his probability measure \(\mathbb{E}[\gamma_t - m_t] = 0\) by definition. Predictability in this model appears only to the researcher studying a sufficiently long sample to observe the true distributional properties of the returns, and it is therefore possible to have perfectly rational agents operating in an efficient market and still observe predictability.

The discussion for heterogeneous beliefs goes along the same lines. The expected excess return is given under the original probability measure by

\[ \mu_t^o - r_t = R \lambda \sigma_t + \frac{\sigma_t}{\lambda_t} \left[ \gamma_t - \left( \frac{c_1}{c_1} m_{1t} + \frac{c_2}{c_1} m_{2t} \right) \right]. \]  \hspace{1cm} (37)

Once the steady state is reached, that is, when the conditional variances of the two agents are identical, the behavior of \(\mu_t^o - r_t\) is similar to the homogeneous case. On average, high past dividend growth implies low future returns. The heterogeneity in beliefs makes the relationship less stable, and before the steady state is reached various patterns can be observed. Using the equilibrium stock price, we can write the dividend yield, which is defined as the ratio of the dividend to the stock price as

\[ Dividend = \frac{D_t}{E_t} \int_t^T \xi_{t,s} D_s \, ds, \]  \hspace{1cm} (38)

which simplifies to

\[ Dividend = \frac{1}{E_t} \int_t^T \xi_{t,s} \exp \left[ \int_s^T (m_{1s} - \frac{1}{2} \lambda^2) \, ds + \int_s^T \lambda dX_{1s} \right] \, ds. \]  \hspace{1cm} (39)

According to Equation (39), the dividend yield is a decreasing function of the current belief \(m_{1t}\). High dividend yield should be observed when the conditional expectation of \(\gamma_t\) is low. Recall that in this case the excess return in the objective probability measure is high, since it is an increasing function of \(\gamma_t - m_{1t}\).
To numerically assess the implication of incomplete information for the predictability of excess returns, we perform the following simulations. Using the homogeneous information setting,\(^8\) we simulate the evolution of the stock price based on the present value formula. From that simulation, we construct the following variables (observed at discrete time intervals):

\[
\text{DivYield}_t = \frac{D_t}{S_t},
\]

\[
\text{DivGrowth}_t = \frac{D_t - D_{t-h}}{D_{t-h}},
\]

\[
\text{Excess}_t = \frac{S_t - S_{t-h} + D_t h}{S_{t-h}}.
\]

We then estimate the following linear regression:

\[
\text{Excess}_t = \alpha + \beta_1 \text{DivYield}_{t-h} + \beta_2 \text{DivGrowth}_{t-h} + \varepsilon_t. \tag{40}
\]

Based on the previous theoretical analysis, we expect \(\beta_2\) to be negative and \(\beta_1\) to be positive. A summary of the estimation results is presented in Table 3. The theoretical result is obtained strongly for the dividend yield with approximately two-thirds of the estimations yielding a significant positive coefficient. The coefficient for the growth rate of dividends is in general not significant. Notice, again, that predictability is apparent only to the outside observer, since the deviations from the full information benchmark are a function of the ex post error made by the agents. Our simulation results confirm the results of Timmermann (1996) and Lewellen and Shanken (2002), which both found, in a discrete-time setting, that predictability could be induced by estimation risk. We show here that the results still hold in a continuous-time pure exchange economy setting, when agents display

\begin{table}[h]
\centering
\caption{Estimation results of the linear regressions.}
\begin{tabular}{lllll}
\hline
& 5% & 10% & 15% & Positive & Negative \\
\hline
\(\beta_1\) & 20\% & 40\% & 60\% & 96\% & 4\% \\
\(\beta_2\) & 12\% & 24\% & 24\% & 56\% & 44\% \\
\hline
\end{tabular}
\end{table}

The dependent variable is the excess return (cum dividend), the independent variables are one-period lagged dividend yield and dividend growth.

\(^8\) We use the homogeneous information setting to simplify the procedure and limit the number of simulations required. For the question of predictability, the homogeneous and heterogeneous settings are similar; the effect should only be stronger for the homogeneous economy. The heterogeneity in beliefs remains a crucial assumption when considering deviations from the CCAPM, as explained in the previous sections.
risk aversion and have intermediate consumption. Recently Menzly, Santos, and Veronesi (2004) have shown that return predictability based on dividend yield in a model with habit formation is compatible with equilibrium and efficient markets. An equilibrium with heterogeneous beliefs is isomorphic to an equilibrium with homogeneous beliefs and state-dependent preferences, as shown in Riedel (2001). Therefore we might conjecture that the simulation results obtained in this section for homogeneous beliefs would be strengthened with the introduction of heterogeneity.

4 CONCLUSION

We have shown in this article how incomplete information and heterogeneity in beliefs affect the statistical properties of asset prices, making an important distinction between objective and subjective probability measures. We have seen that it is possible, under particular beliefs, preference configurations, and information structures, to observe large deviations from the full information market price of risk and riskless rate. Empirical values were matched with a low level of aggregate risk aversion (4.5) and a low level of aggregate consumption volatility (3.6%). This was obtained with an average divergence less than 2%, in a two-agent economy, where the pessimistic agent was endowed with lower initial conditional variance and risk aversion. We have also shown how the volatility of stock prices is affected by incomplete information and heterogeneous beliefs and have demonstrated that a high level of volatility can be obtained with a low level of dividend volatility. Finally, we discussed how the predictability of returns based on dividend yield could be motivated by incomplete information.

APPENDIX A: PROOF OF PROPOSITION 2

Consider the probability space $(\Omega, P^1, \mathcal{F}_t^D)$ on which we define the Brownian motion $\chi_t := \int_0^t \lambda_s^{-1} \left( \frac{dD}{D} - m_I ds \right)$. From the equilibrium stock price and first-order condition, we write the stock price as

\[
S_t = E^1 \left[ \int_t^T \frac{\xi_v}{\xi_t} D_v dv \mid \mathcal{F}_t^D \right].
\]  

(41)

Let $M_t$ be the following $(P^1, \mathcal{F}_t^D)$-martingale:

\[
M_t := \xi_t S_t = E^1 \left[ \int_t^T \xi_v D_v dv \mid \mathcal{F}_t^D \right].
\]  

(42)

it has a representation in terms of $\chi_1$, namely,

\[
M_t = M_0 + \int_0^t \phi_s d\chi_1 s,
\]  

(43)

for some unique square integrable process $\phi$. Applying Itô's lemma to the product $\xi_t S_t$, we identify its diffusion term and obtain
\[ \xi_{1t} S_t [-\theta_{1t} + \sigma_t] = \phi_t \]

and therefore the volatility is given by

\[ \sigma_t = \xi_{1t}^{-1} S_t^{-1} \phi_t + \theta_{1t}. \]

To identify the process \( \phi_t \), we need the following lemma.

**Lemma 3** [from Proposition 1.3.5 (Clark-Ocone) and Nualart (1995)]. Let \( S \) be the class of smooth functionals of the Brownian motion \( \chi_1 \). Define the operator \( D \) as the Malliavin derivative with respect to the one-dimensional Brownian motion \( \chi_1 \). Let \( F \in D^{1,2} \), where \( D^{1,2} \) is the closure of \( S \) with respect to the norm

\[ \|F\|_{1,2} = \left[ E\left(|F|^2\right) + E\left(\|DF\|_{L^2(T)}^2\right)\right]^{1/2}, \]

then

\[ F = E(F) + \int_0^T E(D_s F \mid \mathcal{F}_s) d\chi_{1s}, \]

and taking conditional expectations

\[ E(F \mid \mathcal{F}_t) = E(F) + \int_0^t E(D_s F \mid \mathcal{F}_s) d\chi_{1s}. \]


Note that the dividend process \( D_t \) and agent 1’s state price density process \( \xi_{1t} \) belong to \( S \). Indeed, we have for \( D_t \),

\[ D_t = D_0 \exp \left[ \int_0^t \left( m_{1s} - \frac{1}{2} \lambda^2 \right) ds + \int_0^t \lambda_s d\chi_{1s} \right], \]

and \( m_t \) is a functional of \( \chi_1 \) by construction. The state price density is a root of the polynomial

\[ \left( \frac{y_1 \xi_{1t}}{\rho_t} \right)^{-\frac{1}{\pi_1}} + \left( \frac{y_2 \xi_{1t} \eta_{2t}}{\rho_t} \right)^{-\frac{1}{\pi_2}} - D_t = 0 \]

and the solution can only depend on the processes \( D_t \) and \( \eta_{2t} \). The process \( D_t \) satisfies the condition as we have just seen, and by construction \( \eta_{2t} \) is a functional of \( \chi_1 \),

\[ \eta_{2t} = \eta_{20} \exp \left[ \frac{1}{2} \int_0^t \left( \frac{\Delta_{2s}}{\lambda} \right)^2 dt + \int_0^t \frac{\Delta_{2s}}{\lambda} d\chi_{1t} \right]. \]

\( \Delta_{2s} \) is a function of \( m_{1t} \) and \( m_{2t} \), which are both adapted to the natural filtration of \( D_t \) and can therefore also be written as a function of \( \chi_1 \).

The process \( \phi_t \) can be identified from Lemma 3:
\( \phi_t = E_t^1 \int_t^T D_t(\xi_{1v}D_v)dv, \tag{49} \)

which we rewrite using the chain rule of Malliavin calculus as

\[
\phi_s = E_t^1 \int_t^T D_v D_t(\xi_{1v})dv + E_t^1 \int_t^T \xi_{1v} D_t D_v dv.	ag{50}
\]

Using the market clearing condition, \( c_{1t} + c_{2t} = D_t \), and the optimal consumptions \( c_{1t} = \left( \frac{y_t \xi_{it}}{\mu} \right)^{-\frac{1}{\gamma_t}} \) and \( c_{2t} = \left( \frac{y_t \eta_{2v} \xi_{1v}}{\mu} \right)^{-\frac{1}{\gamma_t}} \) to identify the state price density process, we obtain

\[
\begin{align*}
\phi_t &= E_t^1 \int_t^T D_v \left[ \xi_{1v}D_v \left[ \lambda + \int_t^s D_t m_{1s} ds \right] \right] dv \\
+ E_t^1 \int_t^T \frac{\partial \xi_{1v}}{\partial \eta_{2v}} \left( \eta_{2v} \left[ \int_t^s \lambda^{-1} \Delta_{2u} \lambda^{-1} \Delta_{2v} ds \\
+ \int_t^s \lambda^{-1} D_t \Delta_{2u} ds \lambda_{1s} + \lambda^{-1} \Delta_{2u} \right] \right) dv \\
+ E_t^1 \int_t^T \xi_{1v} D_v \left[ \lambda + \int_t^s D_t m_{1s} dv \right] dv.
\end{align*}
\tag{51}
\]

Finally, we use the equilibrium market price of risk \( \theta_1 = R_{at} \lambda_t + \frac{\xi^2}{\lambda_{t} R_{at}} \lambda_t^{-1} \Delta_{2t} \) to obtain the result in Proposition 2:

\[
\sigma_t = \lambda + \Omega_t + \Pi_t + \Xi_t,
\tag{52}
\]

where

\[
\begin{align*}
\Omega_t &= \left( \frac{c_{2t} R_{at}}{C_t R_2} + k_{1t}^{-1} S_t^{-1} E_t^1 \int_t^T \frac{\partial \xi_{1v}}{\partial \eta_{2v}} \left( \eta_{2v} \right) \right) \lambda_t^{-1} \Delta_{2t} \\
\Pi_t &= R_{at} \lambda + \xi_{1t}^{-1} S_t^{-1} E_t^1 \int_t^T D_v \left[ \xi_{1v} D_v \left[ \lambda + \int_t^s D_t m_{1s} ds \right] \right] dv \\
+ \xi_{1t}^{-1} S_t^{-1} E_t^1 \int_t^T \xi_{1v} D_v \int_t^s D_t m_{1s} ds dv \\
\Xi_t &= \xi_{1t}^{-1} S_t^{-1} E_t^1 \int_t^T \frac{\partial \xi_{1v}}{\partial \eta_{2v}} \left( \eta_{2v} \right) \left[ \int_t^s \lambda^{-1} \Delta_{2u} \lambda^{-1} \Delta_{2v} ds \\
+ \int_t^s \lambda^{-1} D_t \Delta_{2u} ds \lambda_{1s} \right] dv.
\end{align*}
\]

When preferences are homogeneous, the volatility simplifies to

\[
\begin{align*}
\sigma_t &= \lambda \\
+ \left( \frac{c_{2t}}{C_t} + \xi_{1t}^{-1} S_t^{-1} E_t^1 \int_t^T \frac{\partial \xi_{1v}}{\partial \eta_{2v}} \left( \eta_{2v} \right) \right) \lambda_t^{-1} \Delta_{2t} \\
\left( 1 - R \right) \xi_{1t}^{-1} S_t^{-1} E_t^1 \int_t^T \xi_{1v} D_v \int_t^s D_t m_{1s} ds dv \\
+ \xi_{1t}^{-1} S_t^{-1} E_t^1 \int_t^T \frac{\partial \xi_{1v}}{\partial \eta_{2v}} \left( \eta_{2v} \right) \left[ \int_t^s \lambda^{-1} \Delta_{2u} \lambda^{-1} \Delta_{2v} ds \\
+ \int_t^s \lambda^{-1} D_t \Delta_{2u} ds \lambda_{1s} \right] dv.
\end{align*}
\tag{53}
\]
The Malliavin derivative of $m_1$ and $\Delta_2$ are given by the following expressions when the long-term mean is known:

$$D_t(m_{1s}) = \frac{\delta_{2\lambda} + \varepsilon_{1t}}{\lambda} \exp[\alpha(t - s)]$$ \hspace{1cm} (54)

$$D_t(\Delta_{2r}) = \frac{\varepsilon_{1t} - \varepsilon_{2t}}{\lambda} \exp \left[ - \int_t^\tau \left( \frac{\delta_{2\lambda} + \varepsilon_{2s}}{\lambda^2} + \alpha \right) ds \right].$$ \hspace{1cm} (55)

Using Equations (54) and (55) with Equation (53) gives Equation (32) in the text.

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