Testing the 'Inaction Corridor' in a three-regime threshold error correction model with an application to a Buffer-Stock model for US money demand

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Testing the ‘Inaction Corridor’ in a Three-Regime Threshold Error Correction Model with an Application to a Buffer-Stock Model for US Money Demand

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In this paper, we develop a test for the existence of a middle ‘band of inaction’ in a three-regime threshold vector error correction model (TVECM). A Wald statistic is proposed for this purpose, its limiting distribution is derived (which is non-standard) and critical values calculated through simulations. Our methodology is applied to estimate a buffer-stock model for the US money demand and to test for the existence of an inaction middle range within which the money balance is free to fluctuate without portfolio adjustments. Our data seem to corroborate this hypothesis.

(J.E.L.: C22, C51, Q41).

1. Introduction

While traditional cointegration theory assumes linearity in adjustment to long-run equilibrium, many economic situations are characterized by nonlinear adjustments that might be caused by transaction costs or other institutional market interventions. A simple and convenient way to capture such features is to use a threshold model for the equilibrium error processes. Many threshold models are available and have been used in the literature, see, for example, Balke and Fomby (1997), Enders and Siklos (2001) and Kapetanios et al. (2003) among others. Such models are particularly relevant for describing exchange rate dynamics, purchasing power parity and law of one price (Taylor, 2001; Lo and Zivot, 2001; Taylor et al.,

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For some economic issues, two regimes may prove too restrictive to describe the dynamic of the adjustment towards long-run equilibrium. For example, a money demand model may require three regimes if we recognize non-zero costs of adjustment of money balances, which imply that the agents allow short-run deviations of money balances from long-run equilibrium and adjust only for relatively (upper and lower) large deviations.

This note examines a three-regime threshold vector error correction model (TVECM) and investigates the existence of a middle ‘band of inaction’ in such a framework. We propose a Wald statistic, derive its limiting distribution and calculate the critical values through simulations. Finally, our methodology is illustrated by estimating a buffer-stock model for US money demand and testing for the existence of an inaction middle range within which the money balance is free to fluctuate without portfolio adjustments. We find that this null hypothesis is not rejected in our application.

Section 2 presents the three-regime model with a unit root in the middle regime and derives the vector error correction model associated with this threshold behaviour. Section 3 proposes a test for the presence of a unit root in one of the regimes, taking account of the fact that the threshold acts as a nuisance parameter under the null. The asymptotic distribution of the test statistic is derived in Section 4. Section 5 presents an empirical application of the model and the testing procedure using tabulated critical values. Finally, Section 6 concludes.

2. Threshold Error Correction Model with a ‘Band of Inaction’

Consider a $n$-dimensional $I(1)$ process $y_t = (y_{1t}, \ldots, y_{mt})'$ partitioned as $(y_{1t}, y_{mt}')$ for $t = 1, \ldots, T$, where $y_{mt}$ is a $m$-vector and $n = 1 + m$. Let us assume that the generating mechanism for $y_t$ is the following cointegrated system

\[
\begin{align*}
    y_{1t} &= \beta' y_{mt} + z_t, \\
    \Delta y_{mt} &= \eta_t,
\end{align*}
\]

where $u_t = (z_t, \eta_t')$ is an $I(0)$ process. The above system implies that while $y_{mt}$ is not cointegrated, the first equation is a single cointegrating regression with cointegrating vector $\alpha' = (1, -\beta')$. In what follows, we will assume that the cointegrating vector $\alpha$ is known. Taking first differences in (1) and rearranging, the system has the following VECM representation

\[
\begin{align*}
    \Delta y_t &= \gamma \alpha' y_{t-1} + v_t,
\end{align*}
\]

where $y_t' = (y_{1t}, y_{mt})$, $\gamma' = (-1, 0)$ and $v_t = \begin{pmatrix} 1 & \beta' \\ I_m \end{pmatrix} u_t$. 

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In what follows, we examine the case where the disequilibrium error follows a three-regime threshold process:

\[
\begin{align*}
    z_t &= \begin{cases} 
        \phi_1 z_{t-1} + \epsilon_t & \text{if } s_{t-d} \in (-\infty, \theta_1] = A_1, \\
        \phi_2 z_{t-1} + \epsilon_t & \text{if } s_{t-d} \in (\theta_1, \theta_2) = A_2, \text{ with } \theta_1 = -\theta_2, \\
        \phi_3 z_{t-1} + \epsilon_t & \text{if } s_{t-d} \in [\theta_2, +\infty) = A_3,
    \end{cases}
\end{align*}
\]

where \( \epsilon_t \) is a i.i.d. sequence with zero mean, constant variance and finite \( 2\delta \) moments for some \( \delta > 2 \), \( s_{t-d} \) is a univariate process called the transition variable and \( \theta_1 \) is a threshold parameter. The transition variable and the threshold parameter satisfy the following Assumption 1.

**Assumption 1.**
(i) \( s_{t-d} \) is a strictly stationary and ergodic sequence, whose distribution \( F \) is continuous everywhere. (ii) \( s_{t-d} \) admits a symmetric density function. (iii) The delay \( d \) is strictly positive in order to ensure the predetermination of the process. (iv) If \( d = 0 \) then the variable \( s_t \) has to be assumed exogenous. (v) \( \theta \in \Theta \), where \( \Theta \) is a closed and bounded subset of the sample space of the variable \( s_{t-d} \).

These assumptions are standard in the literature dealing with threshold models and they are important for the asymptotic theory. They have been widely discussed in Caner and Hansen (2001) and Gonzalo and Pitarakis (2006). However, because we deal with three regimes and not two as in the above mentioned references, we make an additional assumption on the distribution of \( s_{t-d} \) which is its symmetry. Without being really restrictive, this assumption facilitates the derivation and the tabulation of the test statistic that we propose in the next section. In particular, it allows us to restrict the number of nuisance parameters to one in the limiting distribution of the test statistic in writing \( F(\theta_1) = 1 - F(\theta_2) \). Equations (1) and (3) reflect the threshold cointegration idea developed in the early paper of Balke and Fomby (1997). The conditions for the stationarity of a TAR(1), as derived by Chan et al. (1985), are given by \( |\phi_1| < 1, |\phi_3| < 1 \) and \( \phi_1 \phi_3 < 1 \). We let \( \phi_2 = 1 \) so that the second regime is a ‘band of inaction’. In this middle band the process behaves as a random walk. Thus \( z_t \) can be viewed as a threshold unit root (TUR) process introduced by González and Gonzalo (1998). Caner and Hansen (2001) analyse inference in such a TAR process with unit root and call it a partial unit root process. Following this terminology, we can say that our variables are partially cointegrated when the disequilibrium of the long-run relationship follows such a TUR process. Let us write model (3) with \( \phi_2 = 1 \) as

\[
(\Phi_1(L)z_t)_{t \in A_1} + ((1 - L)z_t)_{t \in A_2} + (\Phi_2(L)z_t)_{t \in A_3} = \epsilon_t,
\]

where \( \Phi_j(L) = 1 - \phi_jL \) for \( j = 1, 3 \).
Proposition 1. The Threshold VECM associated with the system described by: (1), (4) and where the error \( \eta_t \) in (1) is a \( q \)-order autocorrelated process, is given by

\[
\Delta y_t = \pi_1 y_{t-1}^{(1)} + \pi_3 y_{t-1}^{(3)} + \Gamma(L) \Delta y_{t-1} + \varepsilon_t,
\]

where \( y_{t-1}^{(j)} = y_{t-1}I_{\{A_j\}} \), \( \pi_j = \gamma_j \alpha' \), \( \gamma_j = (\Phi_j(1) 0) \), for \( j = \{1, 3\} \),

\[
\Gamma(L) = \begin{pmatrix} 0 & \beta' \ g(L) \\ 0 & g(L) \end{pmatrix}, \quad \text{with} \quad g(L) = \sum_{i=1}^{q} g_i \ L_i^{-1}, \quad \text{and where} \ g_i \ \text{are} \ (m \times m) \ \text{coefficient matrices.}
\]

Proof. See Appendix A1.

Note that in the above TVECM only the loading factors switch with the regime. There is no error correction mechanism in the middle band due to the non-stationarity of the error process in the second regime, i.e. \( \pi(2) = 0 \). We therefore say that the TVECM only has two ‘active’ regimes instead of three.

3. Test Statistic for Inaction Corridor

A test for the ‘inaction band’ can be performed by estimating \( \pi_2 \) from the following general model

\[
\Delta y_t = \pi_1 y_{t-1}^{(1)} + \pi_2 y_{t-1}^{(2)} + \pi_3 y_{t-1}^{(3)} + \varepsilon_t,
\]

and testing \( \pi_2 = 0 \) or equivalently \( \gamma_2 = 0 \).

Stacking the observations, the above model can be written compactly as

\[
\Delta y = \begin{pmatrix} \gamma(2) \ \
\gamma(1,3) \end{pmatrix} + \varepsilon,
\]

with \( \mathbb{V}(\varepsilon) = \Omega \otimes I_T \), where \( \gamma(1,3) = (\gamma(1), \gamma(3)) \), \( \gamma(1,3) = (\gamma_1, \gamma_3)' \) and

\[
\begin{pmatrix} \gamma(2) \\
\gamma(1,3) \end{pmatrix}, \Delta y \ 	ext{and} \ \varepsilon \ \text{are} \ (T \times 1), \ (T \times n) \ \text{and} \ (T \times n) \ \text{which stack the}
\]

observations on \( \gamma_{t-1}^{(2)}, \Delta y_t' \) and \( \varepsilon_t' \), respectively, for \( t = 1, \ldots, T \). Applying partitioned regression results, we obtain

\[
(\hat{\gamma}_2 - \gamma_2) = (I_n \otimes (\varepsilon(2) M_2(2))^{-1} \varepsilon(2)) \ vec(\varepsilon),
\]

with

\[
\mathbb{V}(\hat{\gamma}_2) = (\Omega^{-1} \otimes (\varepsilon(2) M_2(2)))^{-1},
\]

where

\[
M = I_T - \gamma(1,3) (\gamma(1,3) \gamma(1,3))^{-1} \gamma(1,3)'.
\]

The Wald statistic is then given by
\( (7) \quad W_T(\theta_1) = tr\{\Omega^{-1}(z^{(2)'}\text{M}\varepsilon')(z^{(2)'}\text{M}z^{(2)})^{-1}(z^{(2)'}\text{M}\varepsilon)\}. \)

Even though the threshold parameter is identified under the null hypothesis, it still appears as a nuisance parameter whose support is continuous on the interval \([0, 1]\). Therefore the critical values have to be tabulated for a given value of \(\theta_1\). It is therefore impossible in our context to provide a tabulation of the test statistic’s distribution that is free of the nuisance parameter. When the threshold parameter is estimated, the uncertainty can be taken into account using a penalized statistic like the one usually used when the parameter is not identified under the null. One can thus use the supremum Wald statistic given by

\( (8) \quad \sup W = \sup_{\theta_1 \in \Theta} W_T(\theta_1). \)

4. Asymptotic Distribution

According to Assumption 1, it is always possible to trim the subset \(\Theta\) such that \(P(s_{t-d} \leq \theta) = \tau > 0\) and \(P(s_{t-d} \leq \bar{\theta}) = 1 - \tau\), with \(\theta < \theta_1 < \theta_2 < \bar{\theta}\). Let \(U_t = F(s_t)\) with \(U_t\) following a uniform distribution \(U_{[0,1]}\). As \(I_{(s_{t-d} \in A_j(\theta_j))} = I_{(F(s_{t-d}) \in A_j(F(\theta_j)))}\), we can use \(u_j = F(\theta_j) \in [\tau, 1 - \tau]\) for trimming (Caner and Hansen, 2001). Further we can write \(u = F(\theta_1) = 1 - F(\theta_2)\) since we assume that the threshold variable admits a symmetric density (Assumption 1, item(ii)) and \(\theta_1 = -\theta_2\). In what follows, integrals are taken over the unit interval and “\(\Rightarrow\)” indicates convergence in distribution.

**Theorem 1.** Let Assumption 1 and Assumption A1 (in Appendix A.2) hold. Then, under \(H_0 : \gamma_2 = 0\), we have

\[ W \Rightarrow Q(u), \]

\[ \sup W \Rightarrow \sup_{u \in [\tau, 1-\tau]} Q(u), \]

where

\[ Q(u) = tr \left\{ \left( \int B(r) \, dB(r, 1-2u) \right)' \left( \int B(r)B(r)' \right)^{-1} \times \left( \int B(r) \, dB(r, 1-2u) \right) \right\}, \]

and \(B(r)\) is a univariate standard Brownian motion and \(B(r, 1-2u)\) denotes a standard \(n\)-vector two-parameter Brownian motion.
Proof. See Appendix A.2.

In case the TVECM includes an intercept, the univariate Brownian motion $B(r)$ in the limiting distribution given in Theorem 1 should be replaced by the demeaned Brownian motion $\bar{B}(r) = B(r) - \int B(r).

The critical values of the limiting distribution of the Wald statistic can be calculated by Monte Carlo simulations for different values of $n$ and $\theta_1$. In the empirical application section, we compute the critical values for the standard Wald statistic with the value obtained for the (consistent) estimator of the threshold. The critical values of the supremum Wald statistic, which is free of the threshold parameter, are given in Appendix A.3.

Note that the previous statistics are given for a known cointegrated vector $\beta$. However, following Hansen and Seo (2002), it can be shown that when $\beta$ is estimated, the asymptotic distribution of the used statistic holds in a $T^{-1}$ neighbourhood of the true value of $\beta$ if $T(\hat{\beta} - \beta)$ is $O_p(1)$. Therefore, the limiting distributions of Theorem 1 hold while using the OLS estimator for $\beta$ from a two-step procedure.

5. Illustrative Application for US Money Demand

In this section, we turn to an illustration of an inaction band in a money demand function using data for the United States. The model is based on the buffer stock theory which states that the money balance absorbs unexpected shocks (i.e. unanticipated inflows or outflows). However, existence of adjustment costs of cash balances implies that the agents are willing to allow relatively weak short-run deviations of money balances from the long-run equilibrium. They only trigger the portfolio adjustment for relatively large deviations (variations), i.e. when an upper or lower threshold is reached. Within this range of ‘acceptable’ levels, the cash balance is free to fluctuate without portfolio adjustments from the agents. Let us denote the nominal money stock as $M$, the price level as $P$, the real income $Y$, and the long term interest rate $R$. The money demand function is usually defined as follows:

$$\frac{M}{P} = f(Y, R, X), \quad (9)$$

with $X$, a set of other possible variables. While the function $f(\cdot)$ is assumed to be linear, the buffer-stock model implies that the short-term disequilibrium can be formulated as a piecewise linear process described by (3) with $\phi_2 = 1$. Using a semi-log specification, our cointegrating equation is therefore given by\(^1\)

$$\left( m_t - p_t \right) = \alpha_1 y_t + \alpha_2 R_t + z_t, \quad (10)$$

\(^1\) ADF and KPSS tests lead to a presence of unit roots for $(m_t - p_t)$, $y_t$, and $R_t$. The results are not presented to save space.
where the lower case letters design the variables in logarithm, $z_t$ is defined by (3), $\alpha_1$ and $\alpha_2$ are the elasticity and semi-elasticity parameters respectively, both expected to satisfy: $\alpha_1 > 0$ and $\alpha_2 < 0$. According to the buffer stock theory, invariant thresholds over time are due to the slowness in response of price level, interest rates and income.

We use monthly series over the period January 1980 to December 2004, representing 300 observations. The series are: the CPI, M3 definition for nominal money, and the real GDP for the real income $Y$. These data are all taken from the FRED II database, except the monthly real GDP which is provided by Stock and Watson (2010). Our series are depicted in Figure 1. The choice of M3 aggregate is justified by Milbourne’s result which states that if the buffer stock model holds, it would be for a broad definition of money (Milbourne, 1987).

The three-regime TVECM corresponding to the above model is given by (6). In order to satisfy Assumption 1 (in particular that $s_t$ is required to be stationary), the disequilibrium dynamics (3) is specified as a Momentum-TAR (M-TAR) along the lines of Enders and Siklos (2001) and Caner and Hansen (2001), i.e. $s_{t-d} = \Delta z_{t-1}$.

Testing buffer-model’s inaction corridor consists in testing $\phi_2 = 1$, or equivalently $\gamma_2 = 0$ in (6). For this purpose, we estimate the three-regime TVECM by a two-stage procedure. We first estimate (10) by OLS to obtain
Table 1. Buffer-Stock Model Results: Estimates and Tests

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Estimate</th>
<th>Std. Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\hat{\alpha}_1$</td>
<td>0.409</td>
<td>0.002</td>
</tr>
<tr>
<td>$\hat{\alpha}_2$</td>
<td>-0.029</td>
<td>0.002</td>
</tr>
</tbody>
</table>

Symmetry Test

<table>
<thead>
<tr>
<th>Test Statistic</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>SkewT($\Delta \hat{z}_{t-1}$)</td>
<td>0.050</td>
</tr>
<tr>
<td>Pearson stat</td>
<td>0.356</td>
</tr>
<tr>
<td>P.Value</td>
<td>0.639</td>
</tr>
</tbody>
</table>

TVECM estimates

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Estimate</th>
<th>Std. Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\hat{\gamma}_{11}$</td>
<td>-0.018</td>
<td>0.010</td>
</tr>
<tr>
<td>$\hat{\gamma}_{12}$</td>
<td>-0.002</td>
<td>0.023</td>
</tr>
<tr>
<td>$\hat{\gamma}_{13}$</td>
<td>-1.585</td>
<td>1.117</td>
</tr>
<tr>
<td>$\hat{\gamma}_{21}$</td>
<td>0.0003</td>
<td>0.0019</td>
</tr>
<tr>
<td>$\hat{\gamma}_{22}$</td>
<td>-0.005</td>
<td>0.004</td>
</tr>
<tr>
<td>$\hat{\gamma}_{23}$</td>
<td>-0.213</td>
<td>0.210</td>
</tr>
<tr>
<td>$\hat{\gamma}_{31}$</td>
<td>-0.005</td>
<td>0.005</td>
</tr>
<tr>
<td>$\hat{\gamma}_{32}$</td>
<td>-0.027</td>
<td>0.012</td>
</tr>
<tr>
<td>$\hat{\gamma}_{33}$</td>
<td>-0.609</td>
<td>0.595</td>
</tr>
</tbody>
</table>

Threshold estimate

$\hat{\theta}_1 = -\hat{\theta}_2$ | 0.013 |

Inaction band existence test: $H_0 : \gamma_2 = 0$

<table>
<thead>
<tr>
<th>Statistic</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Wald</td>
<td>2.032</td>
</tr>
</tbody>
</table>

Notes: (a) $H_0$ cannot be rejected at 5% and 10%, i.e. the critical values are 7.834 and 6.163, respectively. Table 2 only reports the loading parameter estimates in order to save some space. Indeed, including the lags, the model has 36 parameters.

an estimate of the disequilibrium $\hat{z}_t$ (Table 1 shows that the sign of the elasticities are consistent with the theory). Since our transition variable is based on this estimate, another important item of Assumption 1 that one should pay attention to is the symmetry of its distribution. The second graph of Figure 2 shows that this assumption seems to hold for $\Delta \hat{z}_{t-1}$. We also perform a test of this hypothesis of symmetry in a parametric way using the Pearson test. The test statistic is given by $\frac{\text{Skew}_T}{(6/T)^{1/2}}$, where $\text{Skew}_T$ denotes the empirical skewness coefficient, and is distributed as $N(0, 1)$. The result of this test is reported in Table 1 and concludes that the null hypothesis of symmetry cannot be rejected for the empirical distribution of $\Delta \hat{z}_{t-1}$.

The second stage consists in estimating the three-regime TVECM using the disequilibrium estimate from the previous step. We run a multivariate least squares estimation over a grid of points on the threshold parameter in order to minimize the residual sum of squares. This grid set corresponds to 90 per cent of the estimated values of the transition variable which are taken as possible values of the threshold. The discarded values are the largest and smallest 5 per cent of the transition variable, i.e. $\tau = 0.05$. Considering our sample size, the choice of this percentage seems to be reasonable. The
order of TVECM is selected by minimizing the AIC criterion (a lag of 3 is chosen). Moreover, because the interest rates are strongly heteroskedastic, the standard deviations of parameter estimates are computed using the Newey–West covariance-matrix (Newey and West, 1987) in order to obtain a heteroskedasticity-consistent Wald statistic. Our results are summarized in Table 1. Note that we did not perform a constrained estimation of the TVECM according to the structure (5). However, looking at the standard deviations of the loading parameters, this structure seems to be consistent with the one derived in Proposition 1. To conduct our inaction middle band test, we tabulated, for \( \hat{\theta}_1 \) and for \( \tau = 0.05 \), the distribution of the Wald statistic given by Theorem 1 (Appendix A.3 for details of simulations). The critical values at 1 per cent, 5 per cent and 10 per cent, for \( n = 3 \), and for our threshold estimate \( \hat{\theta}_1 = 0.013 \), \( i.e. \) \( F_T(\hat{\theta}_1) = 0.070 \), where \( F_T \) denotes the empirical distribution of the transition variable, are 11.356, 7.834 and 6.163, respectively.\(^2\) The Wald statistic reported in Table 1 concludes that the null of existence of an inaction middle band cannot be rejected, and therefore argues in favour of the buffer effect for the US data.

\(^2\) We could also compute the sup-Wald statistic for an unknown threshold value (taking into account the uncertainty) and then use the critical values provided in Table A1 (\( n = 3 \), \( [\tau, 1 - \tau] = [0.05, 0.95] \) in our context) in order to test the existence of the inaction corridor.
6. Concluding Remarks

This note examines a three-regime threshold vector error correction model and investigates the existence of a middle ‘band of inaction’ in this framework. We propose a Wald statistic for testing for a unit root in this middle regime, derive its limiting distribution (which is not standard) and calculate the critical values through simulations. Finally, we illustrate our methodology by estimating a buffer-stock model for US money demand in which the money balance fluctuates freely without portfolio adjustments in the middle regime. The null of a middle inaction band is not rejected in our application.
REFERENCES


Appendix

A.1. Proof of Proposition 1.

Combining VECM (2) and equation (4), we have:

$$\Delta y_t = \gamma \alpha' y_{t-1} + \left( \begin{array}{c} 1 \\ 0 \\ I_m \end{array} \right) \left( \begin{array}{c} \phi_1 \alpha' y_{t-1}^{(1)} + \alpha' y_{t-1}^{(2)} + \phi_3 \alpha' y_{t-1}^{(3)} + \epsilon_t \\ \eta_t \end{array} \right),$$

$$\Delta y_t = \gamma \alpha' y_{t-1} + \left( \begin{array}{c} \phi_1 I_{\{A_1\}} + (1 - I_{\{A_1\}} - I_{\{A_3\}}) + \phi_2 I_{\{A_3\}} \\ 0 \end{array} \right) \alpha' y_{t-1}$$

$$+ \left( \begin{array}{c} \epsilon_t + \beta' \eta_t \\ \eta_t \end{array} \right),$$

Hence: $$\Delta y_t = \left( \begin{array}{c} (\phi_1 - 1) I_{\{A_1\}} + (\phi_3 - 1) I_{\{A_3\}} \\ 0 \end{array} \right) \alpha' y_{t-1} + \epsilon_t.$$

The expression $$\Delta y_t = \pi_1 y_{t-1}^{(1)} + \pi_3 y_{t-1}^{(3)} + \epsilon_t$$ is then immediate.

Let us assume now that $$\eta_t$$ of (1) has a finite $$q$$-autoregressive representation such that $$G(L)\eta_t = \zeta_t$$, where $$\zeta_t$$ is an iid process with 0 mean, $$G(L) = I_m - L g(L)$$, and $$g(L) = \sum_{i=1}^{q} g_i L^{i-1}$$ where $$g_i$$ are coefficient matrices of dimension $$(m \times m)$$.

Substituting in our VECM, we obtain: $$\Delta y_t = \pi_1 y_{t-1}^{(1)} + \pi_3 y_{t-1}^{(3)} + \left( \begin{array}{c} \beta' (g(L) \eta_{t-1}) \\ \eta_{t-1} \end{array} \right) + \left( \begin{array}{c} \beta' \zeta_t + \epsilon_t \\ \zeta_t \end{array} \right).$$ Denoting the last error term vector as $$\epsilon_t$$, we get the expression in (5).

A.2. Proof of Theorem 1.

For the derivation of the limiting distribution of the Wald and sup-Wald statistics (7)–(8), we need the following assumptions and theorem.

**Assumption A1.** Consider the sequence $$\{e_t, U_t\}$$, where $$U_t$$ is distributed as a Uniform $$U[0,1]$$, and let $$\mathcal{F}_{t-1}$$ be the natural filtration. We assume (i) $$e_t$$ and $$U_t$$ are strictly stationary, ergodic and strong mixing with mixing coefficients $$a_i$$ satisfying $$\sum_{i=1}^{\infty} a_i^{1/2-1/8} < \infty$$ for some $$\delta > 2$$, (ii) $$e_t$$ is independent of $$\mathcal{F}_{t-1}$$ and (iii) $$\mathbb{E}(e_t) = 0$$, $$\mathbb{E}|e_t|^4 < \infty$$. 

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Theorem A1. Consider the $n$-dimensional I(1)-vector process $y_t$, such that $\Delta y_t = e_t$, where $e_t$ satisfies Assumption A with covariance matrix $\Omega_e$. Then, as $T \to \infty$ and under Assumptions 1 and A1 (above), we have

(i) $T^{-1/2} \sum_{t=1}^{[Tr]} \mathbb{I}_{U_t \cdot \cdot \cdot \cdot \leq F(\theta_1)} e_t^\prime \to \mathcal{N}_n(0, \Omega_e^{1/2} B(r, u))$, on $(r, u) \in [0, 1]^2$, where $T^{-1/2} B(r, u) \sim N(0, ru)$ is a $n$-vector two-parameter Brownian motion.

(ii) $T^{-1} \sum_{t=1}^{T} y_t - \mathbb{I}_{U_t \cdot \cdot \cdot \cdot \leq F(\theta_1)} e_t^\prime \to \int B(r) d B(r, u) \Omega_e^{1/2}$,

(iii) $T^{-1} \sum_{t=1}^{T} y_t - \mathbb{I}_{U_t \cdot \cdot \cdot \cdot \in (F(\theta_1), F(\theta_2))} e_t^\prime \to \int B(r) d B(r, 1-2u) \Omega_e^{1/2}$,

(iv) $T^{-2} \sum_{t=1}^{T} \mathbb{I}_{U_t \cdot \cdot \cdot \cdot \in (F(\theta_1), F(\theta_2))} y_t - y_{t-1} y_{t-1}^\prime \to (1-2u) \Omega_e^{1/2} \{ \int B(r) B(r)^\prime \} \Omega_e^{1/2}$.

(v) $T \sum_{t=1}^{T} \mathbb{I}_{U_t \cdot \cdot \cdot \cdot \in (F(\theta_1))} y_t - y_{t-1} y_{t-1}^\prime \to u \Omega_e^{1/2} \{ \int B(r) B(r)^\prime \} \Omega_e^{1/2}$.

Proof of Theorem A1

These results are direct extensions of Theorems 1–3 in Caner and Hansen (2001, pp. 1560–1).

Let us now turn to the statistics (7)–(8) and derive their limiting distributions. For this purpose, we rewrite (7) as

$$\mathcal{W}_T(F(\theta_1)) = T \left\{ \frac{1}{T} (\zeta^{(2)} M \eta^*)^\prime T^2 (\zeta^{(2)} M \zeta^{(2)})^{-1} \frac{1}{T} (\zeta^{(2)} M \eta^*) \right\},$$

where $\eta^* = \varepsilon \Omega_e^{-1/2}$. Consider the general case where the TVECM includes an intercept. The matrix $M$ in the previous expression becomes $M = I_T - P_z$ with $P_z = Z^{(1,3)}(Z^{(1,3)}Z^{(1,3)})^{-1} Z^{(1,3)}$ with $Z^{(1,3)} = (t^{(1,3)} z^{(1,3)}), t^{(1,3)} = (t^{(1)}, t^{(3)})$, and where $t_T$ denotes a $T$-vector of ones.

Using the result that for conformable matrices $F$ and $G$ such that $E = (F, G), P_E = P_F + M_F G'M_F G^{-1} G'M_F$, we can write the different terms of the Wald statistic as follows. Starting with the first term, we have:

$$\frac{1}{T} (\zeta^{(2)} M \eta^*) = \frac{1}{T} (\zeta^{(2)} \eta^* - \zeta^{(2)} P_z \eta^* - \zeta^{(2)} M_z \zeta^{(1,3)}) \times (\zeta^{(1,3)} M_z \zeta^{(1,3)})^{-1} \zeta^{(1,3)} M_z \eta^*).$$

(A.1)

The middle term can be written as:

$$\frac{1}{T^2} (\zeta^{(2)} M \zeta^{(2)}) = \frac{1}{T^2} (\zeta^{(2)} \zeta^{(2)} - \zeta^{(2)} P_z \zeta^{(2)} - \zeta^{(2)} M_z \zeta^{(1,3)}) \times (\zeta^{(1,3)} M_z \zeta^{(1,3)})^{-1} \zeta^{(1,3)} M_z \zeta^{(2)}).$$

(A.2)

Let $\bar{z}^{(2)}$ be the demeaned $\zeta^{(2)}$, and $\bar{z}^{(1,3)}$ be the demeaned $\zeta^{(1,3)}$, we have for (A.1):

$$\frac{1}{T} (\zeta^{(2)} M \eta^*) = \frac{1}{T} \bar{z}^{(2)} \eta^* - \frac{1}{T} \sqrt{T} \bar{z}^{(2)} \bar{z}^{(1,3)} \bar{z}^{(1,3)} \zeta^{(1,3)} M_z \eta^*,$$

$$\frac{1}{T^2} (\zeta^{(2)} M \zeta^{(2)}) = \frac{1}{T^2} \bar{z}^{(2)} \bar{z}^{(2)} - \frac{1}{T} \sqrt{T} \bar{z}^{(2)} \bar{z}^{(1,3)} \bar{z}^{(1,3)} \zeta^{(1,3)} M_z \bar{z}^{(2)}.$$
and for (A.2):

\[ \frac{1}{T^2} (\varepsilon^{(2)'} M \varepsilon^{(2)}) = \frac{1}{T^2} \bar{\varepsilon}^{(2)'} \bar{\varepsilon}^{(2)} - \frac{1}{T \sqrt{T}} \bar{\varepsilon}^{(2)'} \bar{\varepsilon}^{(1,3)} (\bar{\varepsilon}^{(1,3)' \cdot \bar{\varepsilon}^{(1,3)}})^{-1} \frac{1}{T} \sqrt{T} \varepsilon^{(1,3)' \cdot \bar{\varepsilon}^{(2)}}. \]

Under \( H_0 \), \( \varepsilon^{(2)} \) is \( I(1) \) whereas \( \varepsilon^{(1,3)} \) is \( I(0) \). Thus:

\[ \frac{1}{T} (\varepsilon^{(2)'} M \varepsilon^{*}) = \frac{1}{T} \bar{\varepsilon}^{(2)'} \eta^* + o_p(1) \text{ and } \frac{1}{T^2} (\varepsilon^{(2)'} M \varepsilon^{(2)}) = \frac{1}{T^2} \bar{\varepsilon}^{(2)'} \bar{\varepsilon}^{(2)} + o_p(1). \]

Using Theorem A1, we have:

\[ \frac{1}{T} (\varepsilon^{(2)'} M \varepsilon^{*}) \implies \int B(u) dB(u, 1 - 2u)' \text{ and } \]

\[ \frac{1}{T^2} (\varepsilon^{(2)'} M \varepsilon^{(2)}) \implies (1 - 2u) \int B(u) \bar{B}(u)' , \]

where \( \bar{B}(u) = B(u) - \int B(u) du \).

### A.3. Critical Values For The sup \( W \)-Stat

In order to tabulate the limiting distribution of Theorem 1, the stochastic integrals have been evaluated at 5000 points over the argument \( r \) and 100 steps over the argument \( u \). The critical values have been computed as the corresponding empirical quantiles from 5,000 replications. Tables A1 and A2 report these values for various ranges \( [\tau, 1 - \tau] \).

Caner and Hansen (2001) discuss the inconsistency of tests for \( \tau = 0 \) in threshold models. Indeed, because the critical values of the statistic increase as \( \tau \) decreases, the rejection of the null requires a larger value of the statistic as \( \tau \) tends to 0. It follows that \( \tau \) should be set in the interior of \( (0, 1) \).

<table>
<thead>
<tr>
<th>( n )</th>
<th>( 90% )</th>
<th>( 95% )</th>
<th>( 99% )</th>
<th>( 90% )</th>
<th>( 95% )</th>
<th>( 99% )</th>
<th>( 90% )</th>
<th>( 95% )</th>
<th>( 99% )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( [\tau, 1 - \tau] )</td>
<td>( [0.15, 0.85] )</td>
<td>6.116</td>
<td>7.518</td>
<td>10.516</td>
<td>8.669</td>
<td>10.283</td>
<td>14.085</td>
<td>10.905</td>
<td>12.588</td>
</tr>
<tr>
<td>( [0.10, 0.90] )</td>
<td>6.221</td>
<td>7.626</td>
<td>10.639</td>
<td>8.810</td>
<td>10.398</td>
<td>14.194</td>
<td>10.971</td>
<td>12.815</td>
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<tr>
<td>( [0.05, 0.95] )</td>
<td>6.320</td>
<td>7.712</td>
<td>10.654</td>
<td>8.949</td>
<td>10.510</td>
<td>14.194</td>
<td>11.037</td>
<td>12.869</td>
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</tr>
<tr>
<td>( [0.05, 0.95] )</td>
<td>12.942</td>
<td>14.801</td>
<td>18.863</td>
<td>14.969</td>
<td>17.089</td>
<td>21.424</td>
<td>16.815</td>
<td>19.133</td>
<td>23.453</td>
</tr>
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</table>

**Note:** Calculated from 5000 simulations.

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### Table A2. Critical Values for the sup \( W \) Statistic (with Intercept)

<table>
<thead>
<tr>
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<th>99%</th>
<th>90%</th>
<th>95%</th>
<th>99%</th>
<th>90%</th>
<th>95%</th>
<th>99%</th>
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</thead>
<tbody>
<tr>
<td>([\tau, 1-\tau])</td>
<td></td>
<td></td>
<td></td>
<td></td>
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<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>[0.05, 0.95]</td>
<td>6.180</td>
<td>7.588</td>
<td>10.969</td>
<td>8.888</td>
<td>10.549</td>
<td>14.578</td>
<td>11.184</td>
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<table>
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<th>( n )</th>
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<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>([\tau, 1-\tau])</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>[0.10, 0.90]</td>
<td>13.135</td>
<td>14.865</td>
<td>18.893</td>
</tr>
<tr>
<td>[0.05, 0.95]</td>
<td>13.190</td>
<td>15.014</td>
<td>19.063</td>
</tr>
</tbody>
</table>

**Note:** Calculated from 5000 simulations.

### Non-technical Summary

While traditional cointegration theory assumes linearity in adjustment to long-run equilibrium, many economic situations are characterized by nonlinear adjustments that might be caused by transaction costs or other institutional market interventions. A popular route to capture such features is to use a threshold model for the equilibrium error processes (see Balke and Fomby (1997), Enders and Siklos (2001), Kapetanios et al. (2003), Taylor (2001), Lo and Zivot (2001), Taylor et al. (2001), Yoon (2010), among others). For some economic issues, two regimes may prove too restrictive to describe the dynamic of the adjustment towards long-run equilibrium. For example, a money demand model may require three regimes. In this note, we propose a Wald statistic to test the existence of a middle ‘band of inaction’, i.e. existence of a non-stationary middle regime, in a three-regime threshold vector error correction model (TVECM) framework. We apply this inference to a buffer-stock model for US money demand. According this model, existence of adjustment costs of cash balances implies that the agents are willing to allow relatively weak short-run deviations of money balances from the long-run equilibrium. They only trigger the portfolio adjustment for relatively large deviations (variations), i.e. when an upper or lower threshold is reached. Within this range of ‘acceptable’ levels, the cash balance is free to fluctuate without portfolio adjustments from the agents.

In order to assess such a model, we use monthly series over the period January 1980 to December 2004. Our empirical investigation concludes that the null of existence of an inaction middle band cannot be rejected, and therefore argues in favour of the buffer effect for the US data.