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Abstract

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Entanglement swapping for generalized nonlocal correlations

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I. INTRODUCTION

One of the most surprising aspects of quantum theory is its ability to yield nonlocal correlations, which cannot be explained by any local hidden-variable model [1,2]. These correlations do not allow superluminal signalling, but are nevertheless useful in many information theoretic tasks [3,4]. An interesting question is whether quantum theory yields the maximal amount of nonlocal correlations consistent with causality. Perhaps surprisingly, it has been shown that this is not the case [5], and that it is possible to construct a theoretical system which does not allow superluminal signalling, yet which is more nonlocal than quantum theory. Such a system can achieve the maximal possible value of 4 for the Clauser-Horne-Shimony-Holt (CHSH) [2] expression, compared to 2√2 for quantum theory (the Cirel’son bound [6]), or 2 for any local hidden variable model.

Recently, the idea of nonlocal correlations stronger than those attainable in quantum theory has received considerable interest: van Dam [7] has shown that they allow any bipartite communication complexity problem to be solved with only one bit of communication. Wolf and Wullschleger [8], Buhrman et al. [9], and Short et al. [10] have considered whether they can be used for oblivious transfer and bit-commitment. Cerf et al. [11] have shown that they can be used to efficiently simulate measurements on a quantum singlet state, and Barrett et al. [12,13] and Jones and Masanes [14] have characterized and considered the inter convertibility of different nonlocal correlations.

To investigate the properties of general nonlocal no-signalling correlations, we consider an abstract correlation system composed of a number of black boxes (subsystems) held by different parties, each of which has an input (their measurement setting) and an output (their measurement result) [16]. We represent the combined state of all of the boxes by the conditional probability distribution for their outputs given their inputs.

The nonlocal correlations achievable using correlated box states are analogous to those achievable using entangled quantum states. It is therefore interesting to see what properties of entanglement have analogs for these more general nonlocal correlations. A property which does have such an analog can be viewed as a general property of nonlocal correlations (and therefore not specifically quantum), while a property without such an analog is specific to quantum theory, and therefore may reveal why quantum theory has the particular form that it does.

In this paper, we consider the analog of entanglement swapping [15] for correlated box states, in which nonlocal correlations between Alice and Bob, and between Bob and Charlie, are used to generate nonlocal correlations between Alice and Charlie.

In an attempt to achieve this, we consider the possibility of introducing a class of objects called “couplers,” which can perform an analog for boxes of measurements with entangled eigenstates in quantum theory. We first consider a natural potential coupler in detail, showing how it fails to provide a consistent solution, then proceed to develop a general framework with which to explore other possibilities. Surprisingly, we find that no couplers exist for two binary-input–binary-output boxes, and hence that there is no analog of entanglement swapping for such boxes.

The structure of the paper is as follows: In Sec. II we define general correlated box states, and in Sec. III we briefly review quantum entanglement swapping. In Sec. IV, we attempt to achieve an analog of entanglement swapping for nonlocally correlated boxes. However, we show this is impossible to achieve using a sequence of conditional measurements on individual boxes. In Sec. V we introduce couplers, and consider both general and specific cases, and in Sec. VI we present our conclusions.

II. CORRELATED NO-SIGNALLING BOXES

A. General case

Consider a general multipartite system composed of $N$ correlated subsystems, each of which can be moved about freely. We represent each subsystem by a black box, which has an input (corresponding to the choice of which measurement to perform on that subsystem), and an output (which is the result of the chosen measurement). We will assume that only one input can be made to each box, and that the corre-
FIG. 1. Two correlated boxes held by Alice and Bob, in the state $P(ab|xy)$. The dashed line between the two boxes represents that they are nonlocally correlated.

The probability distribution $P(ab|xy)$ for these extremal states is attained using quantum states such as the singlet state $|\Psi^+\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)_{bc}$ and the Bell state $|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)_{ac}$, where $a$, $b$, and $c$ are the inputs of Alice, Bob, and Charlie, respectively. These states are maximally entangled, and their correlations cannot be described by local hidden variable theories.

The complete class of no-signalling probability distributions for two binary-input–binary-output boxes has been investigated by Barrett et al. [13], and was found to form an eight-dimensional convex polytope with 24 vertices. Any such no-signalling probability distribution (including those attainable using quantum states) can be expressed as a convex combination (i.e., a mixture) of these 24 vertex states.

Sixteen of the 24 vertices represent the deterministic “local” states, for which Alice and Bob’s outputs are a function of their inputs alone, and hence their boxes are uncorrelated. These are the analog of quantum-mechanical product states. The probability distributions for these extremal local states are

$$P^L_{ab\gamma d}(ab|xy) = \begin{cases} 1 : a = ax \oplus \beta \\ b = \gamma y \oplus \delta \\ 0 : \text{otherwise} \end{cases} \quad (4)$$

where $\alpha, \beta, \gamma, \delta \in \{0,1\}$ parametrize the 16 states and $\oplus$ denotes addition modulo 2. A state will lie within the convex polytope $\mathcal{L}$ formed by these 16 vertices if and only if it can be simulated by local operations and shared randomness, and we will refer to any such state as a local state.

The remaining eight vertices represent nonlocal states (lying outside $\mathcal{L}$), with probability distributions given by

$$P^N_{ab\gamma d}(ab|xy) = \begin{cases} 1/2 : a \oplus b = xy \oplus ax \oplus \beta y \oplus \gamma \\ 0 : \text{otherwise} \end{cases} \quad (5)$$

where $\alpha, \beta, \gamma \in \{0,1\}$. Each nonlocal extremal state maximally violates a CHSH-type inequality (achieving a greater value than is attainable in quantum theory), and cannot be simulated with local operations and shared randomness.

These states are the analog of maximally entangled states in quantum theory. Following Ref. [13], we will refer to all of the nonlocal extremal states as PR states [5], although for simplicity we will usually consider the standard PR state $P^N_{\text{pr}(ab|xy)}$, for which $a \oplus b = xy$. All states which lie inside the no-signalling polytope (defined by all 24 vertices), yet outside the local polytope $\mathcal{L}$, represent nonlocal correlations. Some of these nonlocal states can be realized in quantum theory, while others (such as the PR states) cannot.

III. ENTANGLEMENT-SWAPPING IN QUANTUM THEORY

In the quantum case, the simplest example of entanglement swapping is as follows. Suppose that Alice shares a singlet state with Bob, and Bob shares a singlet state with Charlie, such that their combined state is

$$|\Psi\rangle = \frac{1}{2}(|0\rangle_A |1\rangle_B - |1\rangle_A |0\rangle_B)_{ac} = |0\rangle_B |1\rangle_C - |1\rangle_B |0\rangle_C. \quad (6)$$

In this state, Alice’s and Charlie’s qubits are completely uncorrelated. However, expanding the bipartite states held by Bob, and by Alice and Charlie, in the Bell basis of maximally entangled states,

$$|\psi^\pm\rangle = \frac{1}{\sqrt{2}}(|01\rangle \pm |10\rangle), \quad (7)$$

The no-signalling condition corresponds to the two requirements

$$\sum_a P(ab|xy) = P(b|y) \quad \forall \ x, y, b, \quad (2)$$

$$\sum_b P(ab|xy) = P(a|x) \quad \forall \ x, y, a, \quad (3)$$

for some pair of probability distributions $P(a|x)$ and $P(b|y)$. The meaning of Eq. (2) is that Alice cannot signal superluminally to Bob, while Eq. (3) means that Bob cannot signal superluminally to Alice.
shown in Fig. 2. Regardless of the values of inputs is then given by and Charlie. The probability distribution for their inputs and outputs is given by

\[ P'(ac|xz) = \begin{cases} 1 : a = \lambda x \oplus b_1 \\ c = (b_2 \oplus \nu)z \oplus b_2 \\ 0 : \text{otherwise} \end{cases} \]  

which is the probability distribution corresponding to the local state \( P'_{b_1b_2|a_1a_2}(ac|xz) \). By applying inputs to his boxes, Bob has therefore collapsed the state of the remaining two boxes to an extremal local state. This is analogous to the collapse of entangled quantum states after a measurement by one party.

By switching the ordering of his inputs, selecting his strategy choices \((\lambda, \mu, \nu)\) probabilistically, restricting the information he gives to Alice and Charlie, or sometimes announcing that the process has “failed,” Bob can cause the state shared by Alice and Charlie to be any probabilistic combination of the extremal local states (and hence any local state). However, there is no way that Bob can introduce nonlocal correlations between Alice and Charlie, since in each particular instance of the procedure Alice and Charlie share a local state.

The above result extends to the general case in which Bob shares any correlated box state with Alice, and any correlated box state with Charlie, but has no boxes which are correlated with both parties. In this case, the initial state of all of the boxes will be a product of two separate states:

\[ P(ab_1b_2c|xy_1y_2z) = P(ab_1|xy_1)P(b_2c|y_2z), \]

where \( x \) and \( a \), and \( z \) and \( c \), represent the inputs and outputs of Alice’s and Charlie’s boxes, respectively, and Bob’s boxes are partitioned into two sets (with inputs \( y_1 \) and \( y_2 \), and outputs \( b_1 \) and \( b_2 \), respectively) depending on whether they are correlated with Alice or with Charlie.

The most general strategy Bob can adopt is to choose which of his boxes to apply an input to, and what input to apply to that box, dependent on all earlier outputs. Using such an approach, he will generate a particular set of inputs and outputs with probability \( P(b_1b_2y_1y_2) \). As Bob applies all of his inputs first, \( P(b_1b_2y_1y_2) \) can be calculated without reference to Alice and Charlie, and will depend only on the reduced state of Bob’s boxes \( [P(b_1|y_1)P(b_2|y_2)] \), and his particular choice of strategy.

Unfortunately, once all of Bob’s inputs and outputs are known, the state \( P(ab_1|xy_1) \) collapses to \( P(a|xb_1y_1) \), which is a local probabilistic operation for Alice, and similarly \( P(b_2c|y_2z) \) collapses to \( P(c|zb_2y_2) \). For a given set of inputs and outputs for Bob, the final state of Alice’s and Charlie’s boxes will therefore take the form

\[ P'(ac|xz) = P(b_1y_1|a|x)P(b_2y_2|c|z) \]  

which is manifestly local between Alice and Charlie. Regardless of his strategy, it is therefore impossible for Bob to generate nonlocal correlations between Alice and Charlie, even with some small probability.
In this case, it is easy to see that each party’s output is random, and the outcomes of any two parties are uncorrelated [in particular $P'(ac | xz) = P(ac | xz) = 1/4$ as required]. Only by learning all three outputs is any information about the inputs obtained, and the coupler cannot therefore be used for signalling.

However, after Bob has applied the coupler and obtained an output $b'$ [with probability $P'(b') = 1/2$], Alice and Charlie will share the maximally nonlocal state $P_{001}(ac | xz)$. If Bob then announces his measurement result, and one party performs the local operation $a = a \oplus b'$ on their output (i.e., applies a NOT gate if $b' = 1$), Alice and Charlie will be left with the standard PR state $P_{001}(ac | xz)$ as desired. This procedure, in which Alice or Charlie must perform a local correction conditional on Bob’s measurement result in order to obtain a desired final state, is strongly analogous to the quantum case.

It therefore seems that, by enlarging the class of generalized nonlocal objects to include couplers in addition to boxes, we achieve generalized nonlocality swapping. This is in complete analogy with quantum mechanics where in addition to entangled states, we consider measurements with entangled eigenstates. However, as we will show below, the above coupler actually cannot exist.

\section*{B. Difficulties with the potential coupler}
Before allowing the coupler defined in the last section in our model, it is important to check that it gives consistent results when applied to all possible states. To investigate this, we consider the case in which Alice and Charlie apply inputs to their boxes before Bob applies the coupler to his boxes. If they initially share standard PR states as before, their inputs and outputs will obey the relations $a \oplus b_1 = x_1$ and $b_2 \oplus c = y_2$. After Alice and Charlie have applied inputs and obtained outputs from their boxes, the outputs of Bob’s two boxes will be given by $b_1 = c_1 \oplus a$ and $b_2 = c_2 \oplus a \oplus b_2$. The probability distribution for Bob’s two boxes has therefore collapsed to the extremal local state $P_{0110}(b_1 b_2 | y_1 y_2)$.

Suppose that Bob then applies a coupler to his two boxes. As is the case for the original box inputs, we assume that the final probability distribution given by Eq. (13) will be the same regardless of the timings of Alice and Charlie’s inputs, and of Bob’s application of the coupler. In order to satisfy Eq. (13), it is therefore necessary that the probability distribution $P'(b')$ for the coupler output when it is applied to two boxes in the state $P_{ab12}(b_1 b_2 | y_1 y_2)$ is given by

$$P'(b') = \begin{cases} 1 & : b' = a \alpha \oplus \beta \oplus \delta \\ 0 & : \text{otherwise} \end{cases}$$

Similarly, in the case in which Alice and Charlie measure first, all possible pairs of initial bipartite extremal-states shared by Alice-Bob and Bob-Charlie [e.g., $P_{ab1}(b_1 | x_1)$ and $P_{0110}(b_2 | y_2)$] will collapse to a local extremal state for Bob’s two boxes [e.g., $P_{X \oplus Y}(b_1 b_2 | y_1 y_2)$ for the example given]. If we assume that the coupler always acts in the same way when applied to the same state of Bob’s two boxes, then Eq. (14) can be used to deduce the action of the coupler on all initial states of this type.
The results obtained from the two decompositions of $P^\mathcal{N}(b')$ are therefore inconsistent, and cannot be reconciled by any physical probability distribution $P^\mathcal{N}(b')$. Only the nonphysical distribution

$$P^\mathcal{N}(b') = \frac{3}{2} - 2b',$$

which is not a valid probability distribution as it does not satisfy $0 \leq P^\mathcal{N}(b') \leq 1$, could recover the desired decomposition invariance.

Because of this inconsistency, the “naive” coupler defined by Eq. (13) is not an allowable object within the correlated-box model. However, this raises the question of whether another coupler exists which can achieve an analog of entanglement swapping.

### C. General couplers

To generalize the approach of the previous section, we will consider a coupler as any device which acts on $n$ boxes with a given range of inputs and outputs and produces a single output $b'$ (also with a given range), and which cannot be implemented by applying a sequence of individual inputs to the coupled boxes (as in Sec. IV). In this context, the coupler considered in the last section, were it to have proved consistent, would have been an $n=2$ coupler, with all inputs and outputs being binary.

We consider a general set of $N$ boxes divided between the person who is going to apply the coupler (who we call Bob) and the rest of the world (which we call Alice). In the most general case, Bob has the $n$ boxes to which the coupler is to be applied (with inputs $y$ and outputs $b$), and an additional set of $m$ boxes (with inputs $\bar{y}$ and outputs $\bar{b}$). Alice has the remaining $(N-n-m)$ boxes (with inputs $x$ and outputs $a$). In the last section, for example: $x=\{x,z\}, a=\{a,c\}, y=\{y_1,y_2\}, b=\{b_1,b_2\}, \bar{b}=\bar{y}=\emptyset$.

The coupler then performs the transformation

$$P(ab\bar{b}|xy\bar{y}) \rightarrow P'(ab\bar{b}'|\bar{x}\bar{y}).$$

Following the discussion in the previous section, we impose four natural constraints on the coupler’s action:

(i) **Universality**: The coupler must be applicable to any set of $n$ boxes with the appropriate range of inputs or outputs, that are part of any no-signalling correlated box state. Note that this is the condition for which the coupler proposed in the last section fails, as it cannot be applied to two boxes in a PR state.

(ii) **Completeness**: A correlated box state is completely specified by the conditional probability distribution for its outputs given its inputs. To respect this completeness, we
require that the probability distribution $P'(a^ib^j|x^i\bar{y}^j)$ obtained by applying the coupler to a set of boxes depends only on the probability distribution $P(a^ib^j|x^i\bar{y}^j)$ of those boxes to which it is applied.

The same probability distribution $P(a^ib^j|x^i\bar{y}^j)$ can be obtained from many different mixtures of extremal states [as in Eqs. (15) and (16)], or by collapsing a larger state by applying inputs to some of the boxes. This requirement ensures that the coupler gives the same outcome in all of these cases. It also ensures that the results do not depend on the time at which the coupler is applied, just as the timings of standard inputs do not affect the boxes’ outputs.

(iii) No signalling: The coupler must not allow signalling. Note that the most powerful situations for sending and receiving information are when all of the boxes that are not held by Bob (i.e., Alice’s boxes) are gathered in the same place, and hence all of their inputs and outputs are immediately accessible. We must rule out two possibilities:

(a) Signalling from Alice to Bob: We require that Bob cannot learn anything about Alice’s inputs from his coupler and box outputs. We therefore require that

$$\sum_a P'(a^ib^j|x^i\bar{y}^j) = P'(b^j|\bar{y}^j)$$

for some probability distribution $P'(b^j|\bar{y}^j)$ which is independent of $x$.

(b) Signalling from Bob to Alice: We require that Alice cannot learn anything about Bob’s box inputs, or about whether he has (or has not) applied his coupler. We therefore require that

$$\sum_{b,b'} P'(a^ib^j|x^i\bar{y}^j) = \sum_{b,b} P(a^ib^j|x^i\bar{y}^j) = P(a|x).$$

(iv) Nontriviality: The coupler must represent something that was not previously possible. As discussed in Sec. IV, the most general strategy that Bob can adopt without a coupler is to apply a sequence of individual inputs to his boxes, where later inputs may depend on earlier outputs. A coupler cannot be simulated using such a procedure (with $b'$ given by some function of the outputs).

We will now show that constraints (i)–(iii) allow only those couplers which act as linear maps on the reduced state of Bob’s boxes.

Let us consider the case in which Alice applies her inputs $x$, and Bob applies his inputs $\bar{y}$ before he applies the coupler. They will obtain outputs $a$ and $b$, respectively, with probability $P(a^ib^j|x^i\bar{y}^j)$, and the state of Bob’s boxes will collapse to a specific state $P(x^i\bar{y}^j|b^j)=P(b^j|x^i\bar{y}^j)$, which depends on the inputs and outputs obtained.

From the completeness constraint (ii), the coupler will then act on Bob’s $n$-box reduced state exactly as it would have if that $n$-box state were prepared directly (without collapsing a larger $N$ box system), giving the output probability distribution

$$P'(x^i\bar{y}^j|b^j) = C[P(x^i\bar{y}^j|b^j)],$$

where $C$ is some function characteristic of the coupler. The final probability distribution for Alice’s and Bob’s outputs must, therefore, given be

$$P'(a^ib^j|x^i\bar{y}^j) = P(a^ib^j|x^i\bar{y}^j)P'(x^i\bar{y}^j|b^j).$$

Note that as long as $P'(x^i\bar{y}^j|b^j)$ is a valid probability distribution [satisfying $\sum_b P'(x^i\bar{y}^j|b^j)=1$], Eq. (22) will always satisfy the no-signalling constraint (iii b).

To investigate the class of allowed coupler functions $C$, and to incorporate the no-signalling constraint (iii a), it is helpful to consider a particular class of $(n+1)$-box states, for which Alice has a single box with input $x$ and output $a$, and Bob has only those $n$ boxes to which he will apply the coupler. The states we will consider are given by

$$P(ab|x) = \sum_{a'} \lambda_{a'} P_{a'}(b|y) : x = 0,$$

$$\sum_{a'} \lambda_{a'} P_{a'}(b|y) : x = 1, a = 0,$$

$$0 : \text{otherwise},$$

where $P_{a'}(b|y)$ are a set of no-signalling $n$-box states labeled by $a$, and $\lambda_{a'}$ is the probability for Alice to obtain output $a$ when $x=0$ (satisfying $\sum_{a'} \lambda_{a'} = 1$). It is easy to check that $P(ab|x)$ is a valid no-signalling state. A state of this type corresponds to the case in which Bob is given an $n$-box state selected randomly from some set, and Alice can either discover which state Bob has been given (by making $x=0$ into her box) or not (by making $x=1$). In the latter case, the collapsed state of Bob’s boxes will be a probabilistic mixture of all of the boxes in the set.

Applying a coupler to Bob’s boxes [which must be possible due to the universality constraint (i)] yields the state

$$P'(ab'|x) = C \left[ \sum_{a'} \lambda_{a'} P_{a'}(b|y) \right] : x = 0,$$

$$\sum_{a'} \lambda_{a'} P_{a'}(b|y) : x = 1, \ a = 0,$$

$$0 : \text{otherwise},$$

which is no-signalling from Alice to Bob, as required by constraint (iii a), only when

$$\sum_{a} \lambda_{a} C[P_{a}(b|y)] = C \left[ \sum_{a'} \lambda_{a'} P_{a'}(b|y) \right] = P'(b').$$

In order for this relation to be satisfied for all choices of $\lambda_{a}$ and $P_{a}(b|y), C$ must be a linear function of Bob’s $n$-box probability distribution, of which the most general form is given by

$$P'(b') = \sum_{b} \chi(b',by) P(b|y) + \xi(b').$$

As $\sum_{b} P(b|y)=1$, we can always eliminate $\xi(b')$ by adding it to each of the coefficients $\chi(b',by_0)$ for a particular $y_0$. 

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hence Eq. (26) can be simplified to give the final coupler function

$$P'(b') = \sum_{b' y} \chi(b', b y) P(b | y). \quad (27)$$

Combining Eqs. (22) and (27), and using the fact that

$$P(a b b | x y y) = P(a b | x y) P(b y | a x), \quad (28)$$

we find that the effect of the coupler on a general box state is given by

$$P'(a b b | x y y) = \sum_{b'} \chi(b', b y) P(a b b | x y y). \quad (29)$$

Although we have so far only considered the non-signalling constraint (iii a) for a specific set of initial boxes given by Eq. (23), it is easy to see that Eq. (29) will obey (iii a) for any initial state. As the initial distribution $P(a b b | x y y)$ is non-signalling we have

$$\sum_{a} P'(a b b | x y y) = \sum_{b'} \chi(b', b y) P(b b | y y) \quad (30)$$

as required by Eq. (19). Hence any coupler obeying Eq. (27) cannot be used for signalling.

The only remaining constraints on $\chi(b', b y)$ are universality (i) and nontriviality (iv). The former requires that $P'(b')$ must be a valid probability distribution [satisfying $P'(b') > 0$ and $\sum_{b'} P'(b') = 1$] for all initial $n$-box states $P(b | y)$ to which the coupler can be applied, and the latter requires that the action of the coupler cannot be simulated using Bob’s standard box inputs.

D. Two-box binary-input–binary-output couplers

As a specific case of the general couplers introduced in the last section, we will investigate the class of couplers which act on two binary-input–binary-output boxes and generate a binary output $b'$. Following Eq. (27), if a coupler characterized by $\chi(b', b_1 b_2 y_1 y_2)$ is applied to the two-box state $P(b_1 b_2 | y_1 y_2)$, the final probability distribution will be

$$P'(b') = \sum_{b_1 b_2 y_1 y_2} \chi(b', b_1 b_2 y_1 y_2) P(b_1 b_2 | y_1 y_2). \quad (32)$$

The only constraints on $\chi(b', b_1 b_2 y_1 y_2)$ are that it satisfies the universality and nontriviality conditions introduced in the previous section. We first consider universality, which requires that $P'(b')$ is a valid probability distribution for all initial two-box binary-input–binary-output no-signalling states $P(b_1 b_2 | y_1 y_2)$, i.e.,

$$0 \leq P'(b') \leq 1, \quad (33)$$

$$\sum_{b'} P'(b') = 1. \quad (34)$$

It is easy to see that these properties are preserved under convex combination. It is therefore only necessary to ensure that Eqs. (33) and (34) hold for the 24 extremal states introduced in Sec. II. All other states can be represented as convex combinations of the extremal states, and will therefore satisfy Eqs. (33) and (34) automatically.

Note that once $P'(0)$ is determined for each extremal state, $P'(1)$ is fixed by Eq. (34) to be $P'(1) = 1 - P'(0)$. Given any $\chi(0, b_1 b_2 y_1 y_2)$ satisfying Eq. (33) it is always possible to find a $\chi(1, b_1 b_2 y_1 y_2)$ satisfying Eqs. (33) and (34) by defining $\chi(1, b_1 b_2 y_1 y_2) = 1/4 - \chi(0, b_1 b_2 y_1 y_2)$, as this gives

$$P'(1) = \sum_{b_1 b_2 y_1 y_2} \left( \frac{1}{4} - \chi(0, b_1 b_2 y_1 y_2) \right) P(b_1 b_2 | y_1 y_2)$$

$$= 1 - P'(0). \quad (35)$$

As all other choices of $\chi(1, b_1 b_2 y_1 y_2)$ must give the same values for $P'(1)$, they are all equivalent. Hence a coupler is completely specified by the 16 parameters $\chi(0, b_1 b_2 y_1 y_2)$.

The class $\mathcal{X}$ of $\chi(0, b_1 b_2 y_1 y_2)$ distributions satisfying the universality constraint are those which are consistent with the 48 linear inequalities that result from applying Eq. (33) to the 16 extremal local states $P_{a b | y}^{L}(b_1 b_2 | y_1 y_2)$ and 8 extremal nonlocal states $P_{a b | y}^{N}(b_1 b_2 | y_1 y_2)$. This approach yields a convex polytope for $\mathcal{X}$. It has 9 dimensions and 82 vertices, as well as 7 lineairies which have no effect on the final probability distributions (and therefore define an equivalence class of $\chi(0, b_1 b_2 y_1 y_2)$ which would correspond to the same coupler).

Each point in the polytope $\mathcal{X}$ corresponds to a different potential coupler, with the only remaining constraint on them being that of nontriviality. The 82 extremal points of $\mathcal{X}$ can all be expressed by $\chi(b_1 b_2 y_1 y_2) \in [0, 1]$, and can therefore be characterized by the values of $\chi(b_1 b_2 y_1 y_2)$ for which $\chi(b_1 b_2 y_1 y_2) = 1$ (with all other coefficients being zero).

Table I gives a representation of each extremal $\chi(b_1 b_2 y_1 y_2)$ distribution in this way. For simplicity, these

<table>
<thead>
<tr>
<th>Potential coupler classes</th>
<th>Number in class</th>
<th>Entries with $\chi(b_1 b_2 y_1 y_2) = 1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Deterministic ($\chi^D_{a b y}$)</td>
<td>2</td>
<td>$b_1 = y_1 = 0, b_2 = a$</td>
</tr>
<tr>
<td>One-sided ($\chi^O_{a b y}$)</td>
<td>8</td>
<td>$y_0 = y_1 = \alpha, b_2 = b_0 = \gamma$</td>
</tr>
<tr>
<td>XOR gated ($\chi^X_{a b y}$)</td>
<td>8</td>
<td>$y_1 = \alpha, y_2 = \beta, b_2 = b_0 + b_1 + \gamma$</td>
</tr>
<tr>
<td>AND gated ($\chi^A_{a b y}$)</td>
<td>32</td>
<td>$y_1 = \alpha, y_2 = \beta, b_1 = b_2 = b_0 + \gamma$</td>
</tr>
<tr>
<td>Sequential ($\chi^S_{a b y}$)</td>
<td>32</td>
<td>$y_0 = b_1, y_1 = b_2, b_2 = b_0 + \gamma, b_1 = b_0 + \gamma$</td>
</tr>
</tbody>
</table>
extremal points have been divided into five classes, each of which are parameterized by a subset of the binary coefficients \( \{ \alpha, \beta, \gamma, \delta, \epsilon \} \in \{0,1\} \).

Perhaps surprisingly, all 82 extremal points of \( X \) fail to satisfy the nontriviality constraint (iv). Each of these potential couplers can be generated by applying wiring and logic gates to Bob's two boxes (indeed, they represent every inequivalent wiring of this type). A circuit diagram for the standard potential coupler \( (\alpha=\beta=\gamma=\delta=\epsilon=0) \) in each class is shown in Fig. 5. The remaining potential couplers in each class can be obtained by adding NOT gates to some or all of the wires in the standard circuit, and/or by swapping the positions of Bob's two boxes. As every potential coupler in \( X \) can be realized by a probabilistic mixture of these extremal potential couplers (and hence a probabilistic wiring strategy for Bob), all of them fail to satisfy the nontriviality constraint. There are therefore no binary-output couplers for two binary-input–binary-output boxes.

Although we have so far only considered couplers with a binary output \( b' \), this result will hold for couplers with any range of outputs \( b \in \mathbb{N} \). If for some output \( b=b_0 \) a coupler characterized by \( \chi(b,b_1y_1y_2) \) existed, we could always construct a binary-output coupler characterized by \( \chi'(b',b_1y_1y_2) \) which output \( b'=0 \) if \( b=b_0 \) and \( b'=1 \) when \( b \neq b_0 \), by taking

\[
\chi'(b'=0,b_1y_1y_2) = \chi(b=b_0,b,b_2y_1y_2),
\]

and

\[
\chi'(b'=1,b_1y_1y_2) = \sum_{b \neq b_0} \chi(b,b,b_2y_1y_2).
\]

As we have shown that there are no binary-output couplers, there must also be no couplers with a larger output range. We can therefore conclude that, in the case of two binary-input–binary-output boxes, no couplers exist. As discussed in Sec. IV, this means that it is impossible to implement an analog of entanglement swapping for PR states.

VI. CONCLUSIONS

Considering correlation experiments in terms of abstract black boxes allows us to separate the information-theoretic content of nonlocality from the underlying physical detail.

For correlations between a set of time-independent measurements, no-signalling box states represent every possibility which is consistent with relativity. However, quantum theory also allows the possibility of entanglement swapping, in which nonlocal correlations can be introduced between two subsystems which are initially uncorrelated, by performing a joint measurement on two subsystems with which they are entangled and announcing the result.

We have shown that it is impossible to achieve an analog of entanglement swapping between two bipartite nonlocal box states using a sequence of individual (yet conditional) box inputs. This led us to introduce the concept of couplers, which are an analog of measurements with entangled eigenstates in quantum theory.

Under very general assumptions of universality (a coupler can be applied to any box with the appropriate inputs and outputs), completeness (a coupler acts identically on states with the same probability distribution), and no-signalling, we found that any allowed coupler must be linear. We then proceeded to investigate the allowed couplers which act on two binary-input–binary-output boxes. Perhaps surprisingly, we found that no couplers of this type exist. In particular, this means that when Alice and Bob share a PR state, and Bob and Charlie share a PR state, there is no way that Bob can generate any nonlocal correlations between Alice and Charlie.

As we have so far only explicitly considered couplers acting on binary-input–binary-output boxes, it would be interesting to see if couplers exist in more general cases. Of particular interest would be the class of couplers which act on two ternary-input–binary-output boxes and generate a two-bit output. As qubit states can be characterized by measurements of the three Pauli matrices, and Bell measurements have four outputs, this would enable a closer analogy between the quantum case and that of general correlated box states.

Couplers also correspond to a first step into the dynamics of correlated boxes, transforming \( n \) boxes in the initial state into one effective box (with output \( b' \) and no input) in the final state. It is straightforward to generalize the constraints introduced for couplers to apply to more general dynamical processes (taking \( n \) boxes in the initial state to \( m \) boxes in the final state). This opens the possibility for deeper studies of the dynamics of correlated boxes (as desired in Ref. [10]).

The results obtained so far for couplers suggest that by allowing stronger nonlocal correlations than are attainable in quantum theory, the dynamics of the model actually become weaker (to the extent that no couplers exist for two binary-input–binary-output boxes). This suggests that there is a tradeoff between the strength of correlations and of dynamics. Quantum theory not only contains nonlocal correlations but allows a wide range of ways to manipulate them, through entanglement swapping and related procedures. Perhaps, in some sense, quantum theory achieves the optimal tradeoff between correlations and dynamics.
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[16] Note that this setup differs from that of some other authors (e.g., Ref. [13]), who consider that all parties share a single extended black box. Treating the subsystems as separate and correlated boxes provides us with a closer analog to the quantum case, and allows us to utilize the concept of measurement-induced “collapse” in later sections. The mathematical formalism is the same in both cases.
[17] Note that when $P(O|I)$ satisfies the no-signalling condition, $P(O_R|I_R)$ will also satisfy it. $P(O_R|I_R)$ is therefore an allowed correlated box state.
[18] Note that we do not give the coupler an input, as we intend it to correspond to a specific measurement (analogous to the Bell measurement in quantum mechanics), whereas the standard box inputs correspond to a selection of possible measurements.
[19] We obtained the polytope $\mathcal{X}$ using the program LRS, written by D. Avis.