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Reference

BRUNNER, Nicolas, GISIN, Nicolas, SCARANI, Valerio. Entanglement and non-locality are different resources. New Journal of Physics, 2005, vol. 7, p. 88-88

DOI : 10.1088/1367-2630/7/1/088
Entanglement and non-locality are different resources

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Abstract. Bell’s theorem states that, to simulate the correlations created by measurement on pure entangled quantum states, shared randomness is not enough: some ‘non-local’ resources are required. It has been demonstrated recently that all projective measurements on the maximally entangled state of two qubits can be simulated with a single use of a ‘non-local machine’. We prove that a strictly larger amount of this non-local resource is required for the simulation of pure non-maximally entangled states of two qubits $|\psi(\alpha)\rangle = \cos \alpha |00\rangle + \sin \alpha |11\rangle$ with $0 < \alpha \lesssim \frac{\pi}{7.8}$. 
1. Introduction

There exists in Nature a channel that allows one to distribute correlations between distant observers, such that (i) the correlations are not already established at the source, and (ii) the correlated random variables can be created in a configuration of space-like separation, i.e. no normal signal can be the cause of the correlations [1]. This intriguing phenomenon, often called quantum non-locality, has been repeatedly observed, and it is natural to look for a description of it. A convenient description is already known: quantum mechanics (QM) describes the channel as a pair of entangled particles. In recent years, there has been a growing interest in providing other descriptions of this channel, mainly assuming a form of communication. Usually, the interest in these descriptions does not come from a rejection of QM and the desire to replace it with something else; rather the opposite: the goal is to quantify how powerful QM is by comparing its achievements to those of other resources.

For instance, one may naturally ask how much information should be sent from one party (Alice) to the other (Bob) in order to reproduce the correlations that are obtained by performing projective measurements on entangled pairs (to ‘simulate entanglement’). The amount of communication is something that we are able to quantify, thus the answer to this question provides a measure of the non-locality of the channel. Bell’s theorem implies that some communication is required, but does not quantify this amount. Several works [2, 3] underwent the task of estimating the amount of communication required to simulate the maximally entangled state of two qubits (singlet). These partial results were superseded in 2003, when Toner and Bacon [4] proved that the singlet can be simulated exactly using local variables plus just one bit of communication per pair. This amount of communication is tight, in the absence of block-coding—which is indeed the way Nature does it: in an experiment, each pair of entangled particles is ‘processed’ independently of those that preceded and those which will follow it.
More recently, something other than communication has been proposed as a tool to study non-locality: the non-local machine (NLM) described by Popescu and Rohrlich [5], sometimes called the PR box—actually, the first appearance of this ‘machine’ is equation (1.11) of [6]. This hypothetical machine was constructed to violate the Clauser–Horne–Shimony–Holt (CHSH) inequality \( B \leq 2 \) up to its algebraic bound of \( B = 4 \) (while it is known that QM reaches up only to \( B = 2\sqrt{2} \)) without violating the no-signalling constraint; and it would also provide a very powerful primitive for information-theoretical tasks [8, 9]. Cerf et al [10] have shown that the singlet can be simulated by local variables plus just a single use of the NLM per pair: this is the analogue of the Toner–Bacon result for communication. While the NLM is by far a less familiar object than bits of communication, in the context of simulation of entanglement it has a very pleasant feature: it automatically ensures that the no-signalling condition is respected. In contrast, bits of communication imply signalling: to reproduce quantum correlations, as in the Toner–Bacon model, one must cleverly mix different communication strategies in order to hide the existence of communication. The idea itself of hidden communication between quantum particles has several drawbacks [11] and is hard to reconcile with the persistency of correlations in experiments with moving devices [12].

Thus, to date, the simulation of quantum non-locality has been studied for two resources (communication and the NLM) and the results are similar: the basic unit of the resource (one bit, or a single use of the NLM) is sufficient for the simulation of the singlet. Very little is known beyond the case of the singlet. Even staying with just two qubits, the only known result is that two bits of communication are enough to simulate all states [4], but this is not claimed to be tight. In this paper, we study the analogue problem using the NLM and demonstrate that, in order to simulate the correlations of some pure non-maximally entangled state of two qubits, a single use of the NLM is not sufficient: an amount of non-local resources strictly larger than amount for the simulation of the maximally entangled state is needed. Curious as it may seem, this is not the first example in the literature where entanglement and non-locality do not behave monotonically with one another: Eberhard proved that non-maximally entangled states require lower detection efficiencies than maximally entangled ones, in order to close the detection loophole [13]; Bell inequalities have been found whose largest violation is given by a non-maximally entangled state [14] and this has some consequences on the communication cost as well [15]. It is also known that some mixed entangled states admit a local variable model, even for the most general measurements [16].

The present paper is structured as follows. As a necessary introduction, we start by recalling the meaningful mathematical tools for this investigation (section 2). In section 3, we demonstrate the main claim, by showing that there exists a unique Bell-type inequality using three settings for both Alice and Bob which is not violated by any strategy using the NLM at most once, and which is violated by all the states of the form \( |\psi(\alpha)\rangle = \cos \alpha |00\rangle + \sin \alpha |11\rangle \) for \( 0 \leq \alpha \leq \frac{\pi}{19} \) (the sign of approximate inequality means that these are numerical, not analytical results). Thus, these states cannot be simulated by a single use of the NLM. In the same section, we show how this new inequality can be violated by two uses of the NLM or by one bit of communication, and comment on these features. In section 4, we consider extension to more settings on Alice’s and/or on Bob’s side. The case of four settings for Alice and three settings for Bob allows us to extend the result to the range \( 0 < \alpha \lesssim \frac{\pi}{78} \) (in particular, we prove that at least two uses of the NLM are required to simulate pure states arbitrary close to the product state \( \alpha = 0 \)). The (admittedly incomplete) survey of other cases did not provide further improvements. The concluding remarks are given in section 5.
Figure 1. Representation of the regions in probability space. The thick line represents the polytope of shared randomness; its vertices are deterministic strategies (D). Above it lies the new polytope obtained with a single use of the NLM. The curved line encloses the points achievable with measurements on quantum states. The black dot represents the measurement on the singlet that gives the maximal violation of the inequality corresponding to the facet (CHSH or I3322). The existence of the grey regions above the I3322-facet is the result of section 3.

2. Tools: polytopes and the no-signalling condition

Instead of tackling the issue of simulating all possible measurements done on an entangled state, we consider a restricted protocol, as typical in Bell’s inequalities. Obviously, if this restricted protocol cannot be simulated, a fortiori it will be impossible to simulate all the correlations. We allow then each of the two physicists, called Alice and Bob, to choose between a finite set of possible measurements \( \{A_i\}_{i=1}^{m_A} \), \( \{B_j\}_{j=1}^{m_B} \). As a result of each measurement on a pair, they get an outcome noted \( r_A, r_B \). We focus here on dichotomic observables (like von Neumann measurements on qubits), with the convention \( r_A, r_B \in \{0, 1\} \). An ‘experiment’ is fully characterized by the family of probabilities \( P(r_A, r_B|A_i, B_j) \equiv P_{ij}(r_A, r_B) \). There are \( d = 4m_A m_B \) such probabilities, so each experiment can be seen as a point in a region of a \( d \)-dimensional space, bounded by the conditions that probabilities must be positive and sum up to one. By imposing restrictions on the possible distributions, the region of possible experiment shrinks, thus adding non-trivial boundaries [6, 17, 18]. For instance, one may require that the probability distribution must be built without communication, only with shared randomness. In this case, the bounded region actually forms a polytope, that is a convex set bounded by hyperplanes (‘facets’) which are Bell’s inequalities in the usual sense. The vertices of this polytope are deterministic strategies, that is, probability distributions obtained by setting \( r_A, r_B \) always at 0 or always at 1 for each setting. The vertices are thus easily listed, but to find the facets given the vertices is a computationally hard task. The probability distributions obtained with a single use of the NLM also form a polytope, obviously larger than the one of shared randomness—actually, the vertices associated to deterministic strategies remain vertices of this new polytope, but more vertices are added. Finally, the probability distributions that can be obtained from measurements on quantum states form a convex set which is not a polytope. ¹ A sketch of this structure is given in figure 1, which will be commented in more detail in the following.

¹ This statement is valid for the set of all possible measurements on all possible quantum states, see [17], as well as paragraph V.C. of [19]. If one restricts to the measurements on a given state, or even to the von Neumann measurements on a Hilbert space with given dimension, convexity is not proved (although no counterexample is known, to our knowledge).
Since our goal is to simulate QM, we impose from the beginning the constraints of no-signalling; that is, we focus only on those probability distributions which fulfil

\[ \sum_{r_A} P_{ij}(r_A, r_B) = P_j(r_B) \quad \text{for all } i \]  

(1)

and a similar condition for the marginal of A. Under no-signalling, the full probability distribution is entirely characterized by \(d_{ns} = m_A m_B + m_A + m_B\) probabilities, which we choose conventionally to be the \(P_i(r_A = 0)\), \(P_j(r_B = 0)\) and \(P_{ij}(r_A = r_B = 0)\) as in [20].

3. Main result

3.1. Basic notations

Let us focus more specifically on the first case of interest for this paper, \(m_A = m_B = 3\). The no-signalling probability space is 15-dimensional. All the facets of the deterministic polytope are known [20]: up to relabelling of the settings and/or of the outcomes, they are equivalent either to the usual two-settings CHSH inequality, or to the truly three-settings inequality \(I_{3322}\) that reads

\[
I_{3322} = \begin{bmatrix}
-2 & 1 & 0 & 0 \\
 1 & 1 & 1 & 1 \\
 1 & 1 & 1 & -1 \\
 0 & 1 & -1 & 0
\end{bmatrix} \leq 0. \tag{2}
\]

Here, the notation represents the coefficients that are put in front of the probabilities, according to

\[
\begin{array}{c|cc}
 & P_i(r_A = 0) & P_j(r_B = 0) \\
\hline
P_j(r_B = 0) & P_j(r_B = 0) & P_{ij}(r_A = r_B = 0). \\
\end{array} \tag{3}
\]

The maximal violation allowed by QM is obtained for the singlet and is \(\langle I_{3322} \rangle = \frac{1}{2}\). To become familiar with the notations, the deterministic strategy in which Alice outcomes \(r_A = 0\) for \(A_0, A_1\) and \(r_A = 1\) for \(A_2\), and Bob outcomes always \(r_B = 0\), corresponds to the probability point

\[
[0_d 0_d 1_d; 0_d 0_d 0_d] = \begin{bmatrix}
1 & 1 & 0 \\
1 & 1 & 0 \\
1 & 1 & 0 \\
1 & 1 & 0
\end{bmatrix}. \tag{4}
\]

To see the result of \(I_{3322}\) on this strategy, one has simply to multiply the arrays term-by-term: here we find \(I_{3322} = 0\). There are obviously \(2^6 = 64\) deterministic strategies; among these, 20 saturate the inequality \(I_{3322} = 0\) (i.e., they lie on the facet) while the others give \(I_{3322} < 0\). To verify that this is indeed a facet, it is enough to show that the rank of the matrix containing the 20 points that saturate the inequality is \(d_{ns} - 1 = 14\), so that the condition \(I_{3322} = 0\) really defines a hyperplane [21].
3.2. The polytope of a single use of the NLM

The NLM is defined as a two-input and two-output channel. Alice inputs $x$ and gets the outcome $a$, Bob inputs $y$ and gets the outcome $b$; all these numbers take the values 0 or 1, the marginal distribution on each side is completely random, $P(a = 0) = P(b = 0) = \frac{1}{2}$, but the outcomes are correlated as

$$a + b = xy. \quad (5)$$

Explicitly, if either $x = 0$ or $y = 0$, then $(a, b) = (0, 0) \text{ or } (1, 1)$ with equal probability; if $x = y = 1$, then $(a, b) = (0, 1) \text{ or } (1, 0)$ with equal probability.

The most general strategy allowing a single use of the NLM is sketched in figure 2. Alice and Bob share some random variable $\lambda$. Alice inputs in the machine the bit $x = x(A_i, \lambda)$; the machine gives the output $a$, and Alice outputs $r_A = r_A(A_i, \lambda, a)$. The extremal strategies are such that a given $x$ is associated to each $A_i$, and the outcome is either $r_A = a$ or $r_A = 1 - a$. Similarly for Bob. Note that it is also possible that for a given pair $(A_i, B_j)$ Alice uses the machine while Bob outputs a deterministic bit; in this case, we can suppose that Bob inputs $y = 0$ in the machine but does not use the output $b$. We shall come back below to the listing of extremal strategies. Let us now see how the polytope of possible probability distributions is enlarged by allowing a single use of the NLM.

By construction, with a single use of the machine, one can violate the CHSH inequality more than is possible in QM. In the polytope picture, one new point appears above any face corresponding to CHSH: the facets of the enlarged polytope should now pass through this points. They must pass through the deterministic points as well, because these points are still extremal. All this is sketched in figure 1. We will not study this example further, however, since there is no hope of finding something interesting above a CHSH-like facet: all the probability distributions involving two settings which are no-signalling (in particular all distributions arising from measurements on a quantum state) can be simulated by a single use of the NLM [18].

So let us consider the facet defined by $I_{3322} = 0$. Again, a single use of the NLM allows to violate $I_{3322}$, which is expected since the NLM can in particular simulate the singlet. For instance, consider the strategy in which Alice inputs $x = 0$ in the NLM for $A_0$ and $A_1$, and $x = 1$ for $A_2$, while Bob inputs $y = 0$ for $A_0$ and $B_2$, and $y = 1$ for $B_1$; in each case, the outputs are $r_A = a$,
This strategy gives the probability point

\[
\[0_m 0_m 1_m; 0_m 1_m 0_m\] = \frac{1}{2} \times \begin{pmatrix}
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 0 \\
1 & 1 & 1
\end{pmatrix} \tag{6}
\]

yielding \(I_{3322} = \frac{1}{2}\): the machine can violate \(I_{3322}\) more than QM. However, here we are going to show what is graphically represented in figure 1: for non-maximally entangled states, some points achievable with quantum states lie outside the enlarged polytope. In other words, the facets of this polytope define generalized Bell’s inequalities that can still be violated by QM.

To find the facets of the polytope allowing a single use of the machine, one must first list all the vertices (extremal strategies). This can be done systematically on a computer, once having noticed that for each setting, Alice and Bob have six choices: deterministically output 0 or 1 (noted \(0_d, 1_d\)), input 0 or 1 in the machine and keep the output of the machine (noted \(0_m, 1_m\)), input 0 or 1 in the machine and flip the output of the machine (noted \(0_f, 1_f\)).\(^2\) This listing gives \(a \text{ priori} 6^6 = 46656\) strategies, although many of them are equal\(^3\) and only 3088 different strategies are left after inspection. Note that some of these are not even extremal points of the polytope: for instance, the strategy \([0_f, 1_f, 0_f; 0_f, 1_f, 1_f]\) in which both Alice and Bob input 0 in the NLM for all settings, yields the same probability point as the equiprobable mixture of the deterministic strategies \([0_d, 0_d, 0_d; 0_d, 0_d, 0_d]\) and \([1_d, 1_d, 1_d; 1_d, 1_d, 1_d]\). Certainly not equivalent to a mixture of deterministic strategies, however, are those strategies like (6) which violate \(I_{3322}\): upon counting, there are 28 of these, all giving the same violation \(I_{3322} = \frac{1}{2}\). To find the facets of the new polytope, we use available computer programs.\(^4\) We find some trivial facets, plus a single non-trivial one\(^5\) which reads

\[
M_{3322} = \begin{pmatrix}
-2 & 0 & 0 \\
1 & 1 & 1 \\
-1 & 1 & -1 \\
0 & 1 & -1 \\
0 & -1 & 0
\end{pmatrix} \leq 0. \tag{7}
\]

This new inequality is extremely similar to \(I_{3322}\), equation (2): only the coefficient of \(P_0(r_A = 0)\) is now \(-2\) instead of \(-1\). The origin of this difference can be appreciated, at least to some extent: one of the biggest difficulties found in adapting the Toner–Bacon model \([4]\) to non-maximally entangled states lies in the need of simulating not only the correlations, but also the non-trivial marginal distributions. It is thus a good idea, for our purpose, to add penalties on the marginal distributions.

Summarizing, we have found a tight inequality \(M_{3322} \leq 0\) which is satisfied by all the 3088 extremal points of the polytope of probabilities achievable by shared randomness plus a single use of the NLM. Now, we move on to show that QM violates this inequality.

\(^2\) This takes automatically into account the possibility that Alice and Bob use the alternative version of the NLM defined by \(a + b = xy + 1\).

\(^3\) For instance, \([0_f, 1_f, 0_f; 0_f, 0_f, 1_f]\) and \([0_f, 1_f, 1_f; 1_f, 0_f, 1_f]\) define the same probability point.

\(^4\) We used the function \texttt{convhulln} of \texttt{MATLAB} on the 28 points which yield \(I_{3322} = \frac{1}{2}\), or on these points plus some of the 20 deterministic points which yield \(I_{3322} = 0\). The result is the same.

\(^5\) Actually, the program outputs two facets: inequality (7), and a similar one where the first line \((2, 0, 0)\) is replaced by \((1, 1, 0)\). If one writes down the inequality explicitly, it becomes evident that one is transformed into the other by exchanging \(A_0\) and \(A_1\) and by flipping the bit \(r_{B_2}\).
3.3. Violation of the inequality with pure non-maximally entangled states

We consider states of the form

$$|\psi(\alpha)\rangle = \cos \alpha |00\rangle + \sin \alpha |11\rangle$$  \hspace{1cm} (8)

with $\cos \alpha \geq \sin \alpha \geq 0$. Up to local operations, this is the most general pure state of two qubits (Schmidt decomposition). We form the Bell operator $M$ as usual, by replacing the probabilities in (7) by the corresponding one-dimensional projectors, like

$$P_0(r_A = 0) \rightarrow \frac{1}{2}(1 + \vec{a}_0 \cdot \vec{\sigma}).$$  \hspace{1cm} (9)

For each $\alpha$, we have to find the settings which maximize $M(\alpha) = \langle \psi(\alpha)|M|\psi(\alpha)\rangle$. We have not found a closed analytical formula, but it is easy for a computer to optimize over twelve real parameters. The result is shown in figure 3: one finds $M(\alpha) > 0$ for $0 < \alpha < 0.219 \pi$, with a maximal violation $M(\bar{\alpha}) \approx 0.0061$ at $\bar{\alpha} \approx 0.0712 \pi$. It appears that all the optimal settings are of the form

$$\hat{a}_i = \cos \theta^i_A \hat{z} + \sin \theta^i_A \hat{x}, \quad \hat{b}_j = \cos \theta^j_B \hat{z} + \sin \theta^j_B \hat{x}.$$  

The curve of figure 3 is the exact version of the pictorial argument of figure 1. Note in particular the following features: (i) as expected, there is no violation for the singlet ($\alpha = \pi/4$), because this state can be simulated with the NLM; even more, $M(\pi/4) = -\frac{1}{4}$ which is the difference between $\langle I_{3322} \rangle = \frac{1}{4}$ on the singlet and the maximal value $I_{3322} \approx \frac{1}{2}$ achievable with the NLM, the picture of figure 1 yields in this case even a quantitative prediction; and (ii) as mentioned [19], it is not obvious that the set of probabilities obtained from quantum measurements on
$|\psi(\alpha)\rangle$ is convex; so at this point it is not proved that states arbitrarily close to the product state $|00\rangle$ cannot be simulated by a single use of the NLM—the proof will be provided in section 4.

3.4. Violation of the inequality with two NLMs

As we mentioned in section 1, two bits of communications are a sufficient resource to simulate any state of two qubits. The analogue simulation by twice using the NLM is still missing, and may even not exist. While waiting for more clarification, we have found a way of violating inequality (7) by using the NLM twice. The settings are coded as $A \rightarrow (x', x'')$, $B \rightarrow (y', y'')$ according to: $A_0 \rightarrow (0, 0)$, $A_1 \rightarrow (0, 1)$, $A_2 \rightarrow (1, 0)$; $B_0 \rightarrow (0, 0)$, $B_1 \rightarrow (1, 0)$ and $B_2 \rightarrow (0, 1)$. Then, $x'$ and $y'$ are used as inputs in the first use of the machine, whose outcomes are denoted $a'$ and $b'$; $x''$ and $y''$ are used as inputs in the second use of the machine, whose outcomes are denoted $a''$ and $b''$. Finally, Alice outputs $r_A = a' + a''$, Bob outputs $r_B = b' + b''$ (sum modulo 2). This strategy gives the probability point

$$[\text{Two uses of the NLM}, \ldots] = \frac{1}{2} \times \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

yielding $I_{3322} = 1$ and $M_{3322} = \frac{1}{2}$.

We can go a step further. Consider a mixture of two strategies: with probability $p$, the strategy just described which uses two NLMs; with probability $1 - p$, the deterministic strategy in which Alice and Bob output always $r_A = r_B = 1$ for all settings (all entries of the table are zeros), and which obviously does not require any use of the NLM. Such a mixed strategy yields a violation $M_{3322} = \frac{p}{2}$ of the inequality. Now, if $p < \frac{1}{2}$, this strategy uses less than one NLM on average. This is not a contradiction with our main result: at least two NLMs must be available to simulate non-maximally entangled states, albeit possibly this resource is not used for all items.

3.5. Violation of the inequality with one bit of communication

It is natural to ask whether inequality (7) provides also a Bell inequality for one bit of communication. Bacon and Toner [22] had studied such inequalities for three settings, but they restricted themselves to correlation inequalities, whereas inequality (7) is a probability inequality. The problem is complex because, as we mentioned, pure strategies with one bit of communication imply signalling; we must find a mixture of such strategies which is no-signalling and which violates our inequality. It turns out that such mixtures do exist, so that our inequality is not an inequality for one bit of communication. In other words, the polytope of the probability distributions obtained with one bit of communication plus the no-signalling constraint is larger than the one associated to a single use of the NLM.

As an explicit example [23], it can be verified that the no-signalling strategy

$$[\text{One bit, no-signalling}] = \frac{1}{5} \times \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

This was pointed out to us by [23].
yields the violation $M_{3322} = \frac{1}{3}$ and can be obtained as the equiprobable mixture of the following five one-bit strategies:

\[
\begin{align*}
[r_A]; [r_B]|c_0, c_1, c_2] &= \begin{cases} [1, 1, 0; c, 1, c|1, 1, 0] \\
[1, 0, 1; c, c, 1|1, 0, 1] \\
[0, 1, 1; c, c, c|0, 1, 1] \\
[1, 1, 1; 1, 1, c|0, 1, 0] \\
[1, 1, 1; c, 1, 1|1, 1, 0] \end{cases}.
\end{align*}
\]

(12)

In these notations, $c_i$ is the value of the bit that Alice sends to Bob when she has used the setting $A_i$; $r_{Bj} = c$ means that, upon choosing the setting $B_j$, Bob outputs the value of the bit received from Alice.

The fact that the inequality (7) can be violated by one bit of communication shows that a single use of the NLM does not correspond to a single bit of communication plus no-signalling: the NLM is a resource strictly weaker than communication, as argued in [10], and grasps finer details of the structure of quantum non-locality. The question whether one bit of communication is sufficient to simulate non-maximally entangled states is obviously still open.

4. Extensions to more settings

In the previous section, we have provided a complete study of the case $m_A = m_B = 3$: there cannot be any inequality other than (7) which has the desired properties. In this section, we explore other cases, starting from the next easiest, namely $m_A = 4$ and $m_B = 3$.

4.1. The case $m_A = 4$ and $m_B = 3$

In the case $m_A = 4$ and $m_B = 3$, the no-signalling probability space is 19-dimensional. All the facets of the deterministic polytope have been listed in appendix A of [20]: one finds of course CHSH, $I_{3322}$, plus three new inequalities. The one which turns out to be of interest is (A.2) of that; using the properties of no-signalling distributions, and providing Alice instead of Bob with the four settings, this inequality can be rewritten in the form

\[
I_{4322}^{(2)} = \begin{array}{c|cccc}
-2 & -1 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 \\
-1 & 0 & -1 & 0 \\
0 & 1 & -1 & 0 & -1 \\
\end{array} \leq 0.
\]

(13)

The polytope of a single use of the NLM can be found as in section 3. After listing, one finds that the $6^7$ possible extremal strategies produce 17272 different points, 63 of which violate (13)

7 Actually, we did not perform a systematic search because the polytope becomes rather big. Based on the result of section 3, we have taken inequalities (A.1)-(A.3) of [20], verified that all of them can be violated by a single use of the NLM, then modified the marginals. It turns out that, even by modifying the marginals by 1, the new inequalities derived from (A.1) and (A.3) are not violated by QM.
by $I_{4322}^{(2)} = \frac{1}{2}$. By numerical inspection,\footnote{Finding a polytope given 63 vertices is too hard a task for the function convhulln of MATLAB. Therefore, we have run this function more than one million times, each time randomly selecting 30 points out of the 63. When the selected points defined a non-trivial facet, it was invariably (14). Thus we are confident that this is the only interesting inequality for one use of the NLM above the $I_{4322}^{(2)}$ deterministic facet.} we found that these points define a single non-trivial new facet

$$M_{4322} = \begin{bmatrix}
-2 & 1 & 1 & 1 & 0 \\
-1 & 1 & 0 & -1 & 0 \\
0 & 1 & -1 & 0 & -1
\end{bmatrix} \preceq 0. \quad (14)$$

Quite similarly to the case $m_A = m_B = 3$, the difference between the original inequality and the new one is just a larger penalty on one marginal. Also similar is the fact that (14) can be violated by strategies which use twice the NLM or one bit of communication, as can be easily verified. In fact, simply by taking the corresponding strategies of section 3 and adding the condition $A_3 = A_1$, we can produce the probability point

$$[\text{two NLMs, one bit}] = \lambda \times \begin{bmatrix} 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 0 \\
1 & 1 & 0 & 1
\end{bmatrix} \quad (15)$$

which gives $M_{4322} = \lambda$, with $\lambda = \frac{1}{2}$ in the case of two NLMs and $\lambda = \frac{1}{5}$ in the case of one bit of communication.

The interest of $M_{4322}$ comes from the quantum violation, which (i) is larger than the violation of $M_{3322}$, thus allowing one to extend the range of $\alpha$ for which one NLM is not enough, and (ii) is obtained for a family of settings which can be easily parametrized (see appendix A). Specifically, one finds $M(\alpha) > 0$ for $0 < \alpha \lesssim \frac{\pi}{7}$, with a maximal violation $M(\tilde{\alpha}) \approx 0.0102$ at $\tilde{\alpha} \approx \frac{\pi}{12}$ (figure 3). For small values of $\alpha$, moreover, one can prove

$$M(\alpha) \geq \frac{1}{4} \alpha^2 + O(\alpha^4). \quad (16)$$

Thus $M(\alpha) > 0$ as soon as $\alpha > 0$: the simulation of pure states with arbitrarily weak entanglement requires more than one NLM—again, as we noticed at the end of section 3.4, it may be the case that correlations can be reproduced by using this resource only on a subset of the particles.

### 4.2. Other inequalities

Beyond the 3322 and 4322 cases, the facets of the deterministic polytope have not been listed exhaustively, but several examples of facets are available [20, 24]. On these, we searched for possible extensions of our results by increasing the penalties in some marginals.

Starting from the inequality $I_{4422} \leq 0$ given in [20], the corresponding inequality $M_{4422} \leq 0$ is obtained exactly as above, just replacing $-1$ with $-2$ as the coefficient of $P_0(r_A = 0)$. The result is similar: $M_{4422} \leq 0$ indeed holds for all strategies allowing a single use of the non-local machine, and QM violates it. If all four settings are used, the range of values of $\alpha$ in which
we found a violation is however smaller than for $M_{3322}$, only up to $\sim \frac{\pi}{17}$. Note that $I_{4422}$ is less violated than $I_{3322}$ by the singlet [20], while the violation achievable by the NLM is $\frac{\pi}{2}$ for both; by looking at figure 1, it becomes intuitive that the range of violation should decrease. Interestingly, one can recover the (better) result of figure 3 by setting $a_3$ and $b_0$ to the value 1, thus reducing $M_{4422}$ to $M_{3322}$. This assignment reads $P_3(r_A = 0), P_0(r_B = 0) \rightarrow 0$ and is thus not of the form (9): it describes a degenerate measurement.

The other 4422 deterministic facets, as well as some 5522 and 6622 ones, did not appear to be worth a closer study after our survey. We have considered neither inequalities with larger number of outcomes, nor multi-partite scenarios.

5. Conclusion and perspectives

In conclusion, we have shown that the simulation of non-maximally entangled states of qubits requires a strictly larger amount of resources (use of the NLM) than the simulation of the singlet. We have completely solved the problem in the case of three settings and two outcomes for both Alice and Bob, and found an extension in the case where Alice chooses among four settings. At present, therefore, we know that the singlet can be simulated by a single use of the NLM, while the simulation of states with $0 < \alpha \lesssim \frac{\pi}{7.8}$ requires that more than one NLM be available (even though possibly this resource is seldom used). It will be of great interest to fill the gap, and to see whether a similar result holds when the resource used to simulate correlation are bits of communications instead of the NLM.

From a very fundamental point of view, we have discovered a new surprising feature of the quantum world, which shows once more how far this world lies from our intuition. But a precise understanding of the incommensurability between entanglement and non-locality would be of interest for applications as well. For instance, it would allow us to study whether in a given quantum information protocol (cryptography, teleportation, an algorithm . . .) it is better to look for the largest amount of entanglement or for the largest amount of non-locality.

Acknowledgments

We acknowledge financial support from the EU Project RESQ and from the Swiss NCCR ‘Quantum photonics’.

Appendix A. Optimal settings for $M_{4322}$

To compute the maximal violation of $M_{4322}$, equation (14), on qubit states, one must perform an optimization over 14 parameters. This we first performed numerically; by looking at the result, however, an analytical form for the settings has been guessed. We give the settings by indicating the azimuthal and polar angle of the vector in the Bloch sphere $\vec{n} \equiv (\theta, \phi)$.

For $0 \leq \alpha \lesssim \frac{\pi}{10.6}$, the optimal settings lie in the $(x, z)$ plane and only two parameters depend on $\alpha$; specifically

$A_0 = (\pi, 0) = -\hat{z}$, $A_1 = (\theta_A, \pi)$, $A_2 = \left(\frac{\pi}{2}, \pi\right) = -\hat{x}$, $A_3 = (\theta_A, 0)$,

$B_0 = (\theta_B, \pi)$, $B_1 = (\theta_B, 0)$, $B_2 = (\pi, 0) = -\hat{z}$. 

This gives
\[ M(\alpha) = \frac{1}{4} [ -3 + \cos 2\alpha + \cos \theta_A - \cos \theta_B (1 + \cos 2\alpha) + \cos \theta_A \cos \theta_B + \sin \theta_B (1 + \sin \theta_A) \sin 2\alpha ] . \]

We have not been able to find a closed formula for \( \theta_{A,B} \) as a function of \( \alpha \).

For \( \frac{\pi}{10} \leq \alpha \leq \frac{\pi}{4} \), the optimal settings do not lie in the \((x, z)\) plane any longer, and only one parameter depends on \( \alpha \); specifically
\[
\begin{align*}
A_0 &= (\pi, 0) = -\hat{z}, & A_1 &= \left( \frac{\pi}{2}, \frac{5\pi}{6} \right), & A_2 &= \left( \frac{\pi}{2}, \frac{\pi}{2} \right) = +\hat{y}, & A_3 &= \left( \frac{\pi}{2}, \frac{7\pi}{6} \right), \\
B_0 &= \left( \theta_B, \frac{4\pi}{3} \right), & B_1 &= \left( \theta_B, \frac{2\pi}{3} \right), & B_2 &= (\theta_B, 0).
\end{align*}
\]

This gives
\[ M(\alpha) = \frac{1}{4} [ -7 + \cos 2\alpha - 3 \cos \theta_B (1 + \cos 2\alpha) + 3\sqrt{3} \sin \theta_B \sin 2\alpha ] \]
which can easily be maximized to find
\[ \theta_B(\alpha) = \pi - \arctan \left( \frac{\sqrt{3} \sin 2\alpha}{1 + \cos 2\alpha} \right). \]

Even though these are not the best settings for small values of \( \alpha \), we can study the limit \( \alpha \to 0 \) and we find \( M(\alpha) = \frac{1}{4} \alpha^2 + \mathcal{O}(\alpha^4) \) which means a violation of the inequality for arbitrary small values of \( \alpha \). Thus, we prove analytically that at least this value can be reached, as written in the main text, equation (16).

References

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