Dark matter halo assembly bias: environmental dependence in the non-Markovian excursion set theory

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Abstract

In the standard excursion-set model for the growth of structure, the statistical properties of halos are governed by the halo mass and are independent of the larger-scale environment in which the halos reside. Numerical simulations, however, have found the spatial distributions of halos to depend not only on their mass but also on the details of their assembly history and environment. Here we present a theoretical framework for incorporating this "assembly bias" into the excursion-set model. Our derivations are based on modifications of the path-integral approach of Maggiore & Riotto that models halo formation as a non-Markovian random-walk process. The perturbed density field is assumed to evolve stochastically with the smoothing scale and exhibits correlated walks in the presence of a density barrier. We write down conditional probabilities for multiple barrier crossings and derive from them analytic expressions for descendant and progenitor halo mass functions and halo merger rates as a function of both halo mass and the linear overdensity $\delta_e$ of the larger-scale environment of the halo. Our results predict a higher [...]
In hierarchical cosmological models such as ΛCDM (cold dark matter), dark-matter halos of lower mass form earlier on average than more massive halos. The virial mass of halos is a key parameter that governs many properties of galaxies and their host halos, e.g., galaxy morphology and color, baryonic feedback processes, formation redshift, and halo occupation number. Recent numerical simulations, however, have shown that a halo’s local environment—in addition to its mass—also affects the formation processes. At a fixed mass, older halos are found to cluster more strongly than more recently formed halos (Gottlöber et al. 2001; Sheth & Tormen 2004; Gao et al. 2005; Harker et al. 2006; Wechsler et al. 2006; Jing et al. 2007; Wang et al. 2007; Gao & White 2007; Maulbetsch et al. 2007; Angulo et al. 2008; Dalal et al. 2008; Li et al. 2008). Other halo properties such as concentration, spin, shape, velocity structure, substructure mass function, merger rates, and halo occupation distribution have also been shown to vary with halo environment (e.g., Avila-Reese et al. 2005; Wechsler et al. 2006; Jing et al. 2007; Gao & White 2007; Bett et al. 2007; Wetzel et al. 2007; Fakhouri & Ma 2009, 2010; Faltenbacher & White 2010; Zentner et al. 2013).

In comparison, the formation and properties of dark-matter halos depend only on the mass and not the environment in the extended Press–Schechter and excursion-set models (Press & Schechter 1974; Bond et al. 1991; Lacey & Cole 1993). These models are used widely for making theoretical predictions of halo and galaxy statistics and for Monte Carlo constructions of merger trees. The lack of environmental correlation arises from the Markovian nature of the random walks in the excursion-set model: the change of the matter overdensity as a function of the smoothing scale is treated as a Markovian process, which by definition decouples the density fluctuations on small (halo) and large (environment) scales. This limitation stems from the use of the Fourier-space top-hat window function as the mass filter. When a Gaussian window function is used, for instance, Zentner (2007) finds an environmental dependence in the halo-formation redshift, but the dependence is opposite to that seen in the numerical simulations cited above. Several other attempts at incorporating environmental effects into the excursion-set model were not able to reproduce the correlations seen in the simulations (e.g., Sandvik et al. 2007; Desjacques 2008).

In this paper, we aim to derive analytic expressions for halo statistics that depend on halo mass as well as large-scale environmental density. To achieve this goal, we begin with the non-Markovian extension of the excursion-set model by Maggiore & Riotto (2010a, MR10 hereafter). In this approach, a path-integral formalism is used to perform perturbative calculations for non-Markovian processes of Gaussian fields. A key quantity is the probability that the smoothed matter overdensity remains below a critical value down to a certain mass scale (Equation (40) of MR10). They show that this quantity can be written as a multivariable integral of a Gaussian distribution function, which can be worked out exactly in the Markovian case, and perturbatively for weakly non-Markovian processes (see Sections 3–5 of MR10 for details). This probability can be used to derive the first-crossing rate for the halo mass function as shown in Equation (42) of MR10.

To introduce environmental dependence, we modify Equation (40) of MR10 by first isolating (i.e., not integrating out) the dependence of the matter overdensity on the specified environmental scale in this equation. We then add to the path integral a portion that is between the descendant and progenitor halo mass scales with a slightly higher critical value for halo identification (corresponding to the halo formation criteria at a slightly higher redshift). The resulting new probability is a function of the environmental density and the descendant and progenitor halo masses. Its derivative with respect to the descendant and the progenitor masses yields the conditional density function of the environmental density as a function of the descendant and progenitor halo masses.

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mass function as a function of the overdensity of the large-scale environment, which will be the main result of this paper. In Section 2, we provide a summary of the excursion-set model and the path-integral approach to the non-Markovian extension. In Section 3, we introduce the formalism and perform the main calculation, including the simplification of the final result in the limit of the large-scale environment.

2. NON-MARKOVIAN EXTENSION TO THE EXCURSION-SET MODEL

2.1. Summary of the Excursion-set Model

At any given time \( t \) and position \( \mathbf{x} \), a virialized dark-matter halo is formed in the excursion-set model if the linear mass overdensity \( \delta(\mathbf{x}, R) \) smoothed on the scale of the halo size \( R \) exceeds a threshold \( \delta_i \) that is determined by the spherical-collapse model, and if no larger smoothing scales meet the criterion. The smoothed density field is given by

\[
\delta(\mathbf{x}, R) = \int d^3x' W(|\mathbf{x} - \mathbf{x}'|, R) \delta(x'),
\]

where \( \delta(\mathbf{x}) = \rho(\mathbf{x})/\bar{\rho} - 1 \) is the density contrast about the mean mass density \( \bar{\rho} \) of the universe, \( W(|\mathbf{x} - \mathbf{x}'|, R) \) is the smoothing filter function, and \( R \) is the smoothing scale. When \( W \) is a top-hat function in \( k \)-space, the overdensity traces out the smoothing scale as a Markovian random-walk process. Instead of \( R \) (or mass) scale, where \( \text{Var}(\delta_i) \rightarrow 0 \). In hierarchical models of structure formation, such as the \( \Lambda \)CDM model, \( S \) is a monotonically decreasing function of \( R \). The variables \( S \), \( R \), and the associated mass, \( M = (4/3)\pi R^3 \bar{\rho} \), can therefore be used interchangeably.

In the standard excursion-set model, the first-crossing distribution of random walks with a constant barrier \( \delta_i \) determines the halo mass function. Further refinement is achieved by the ellipsoidal collapse model with a scale-dependent \( \delta_i \) (Sheth et al. 2001; Sheth & Tormen 2002) or a diffusing barrier (Robertson et al. 2009; Maggiore & Riotto 2010b). The resulting halo mass functions are found to agree reasonably well with N-body simulation results (e.g., Tinker et al. 2008; Ma et al. 2011).

In addition to the halo mass function, the excursion-set model also predicts the halo assembly history. As the linear density field grows with time, halos are identified on increasingly larger mass scales, which signifies the gain of dark-matter mass through mergers or accretion. Statistics such as the halo merger rates, progenitor mass functions, and their relations with the large-scale environmental density can all be worked out in this framework.

The calculation of the halo statistics typically treats the change of the smoothed linear density field \( \delta \) with a decreasing smoothing scale \( S(R) \) as a Markovian process, in which each step of the random walk is uncorrelated with the previous one. The Markovian assumption therefore decouples the linear density fluctuations below and beyond the halo mass scale, causing the halo properties, such as its formation time and merger rate, to be independent of the density of the halo environment. This assumption greatly simplifies the calculations and has led to a number of useful analytic results. The Markovianity of the process, however, relies on the density smoothing filter being a top-hat function in \( k \) space, which does not correspond to a well-defined halo mass in real space. In addition, the decoupling between halo mass and halo environment is not seen in numerical simulations.

2.2. Introduce Non-Markovianity

A difficulty of the excursion-set model is that an unambiguous relation between the smoothing radius \( R \) and the mass \( M \) of the corresponding collapsed halo only exists when the filter is a top-hat function in real space: \( M(R) = (4/3)\pi R^3 \bar{\rho} \). For all other filter functions (e.g., top hat in \( k \) space, Gaussian), it is impossible to associate a well-defined mass \( M(R) \) (see, e.g., Bond et al. 1991; Zentner 2007).

To deal with this problem, Maggiore & Riotto (2010a) uses a path-integral approach to compute the probability associated with each trajectory \( \delta(S) \) and sum over all relevant trajectories. For convenience, the time variable is first discretized and the continuum limit is taken at the end. Specifically, we discretize the interval \([0, S]\) in steps \( \Delta S = \epsilon \), so \( S_k = k \epsilon \) with \( k = 1, \ldots, n \), and the end point is \( S_n = S \). A trajectory is defined by the collection of values \( \{\delta_1, \ldots, \delta_n\} \), such that \( \delta(S_k) = \delta_k \). All trajectories start at a value \( \delta_0 \) at “time” \( S = 0 \).

The basic quantity in this approach is the probability density in the space of trajectories, defined as

\[
W(\delta_0; \delta_1, \ldots, \delta_n; S_n) = \left\langle \delta_D[\delta(S_1) - \delta_1] \ldots \delta_D[\delta(S_n) - \delta_n] \right\rangle,
\]

where \( \delta_D \) is the Dirac delta function, and all trajectories start from \( \delta_0 \) at \( S = 0 \). For a Gaussian random density field, the only non-zero component in \( W \) is the connected two-point correlator \( \langle \delta_j \delta_k \rangle_c \), and \( W \) can be transformed into

\[
W(\delta_0; \delta_1, \ldots, \delta_n; S_n) = \int_{-\infty}^{\infty} \frac{d\lambda_1}{2\pi} \cdots \frac{d\lambda_n}{2\pi} \exp \left( i \sum_{j=1}^{n} \lambda_j \delta_j - \frac{1}{2} \sum_{j,k=1}^{n} \lambda_j \lambda_k \langle \delta_j \delta_k \rangle_c \right).
\]

If the density smoothing filter is a top-hat function in \( k \) space, the evolution of \( \delta(S) \) is Markovian, and the density correlation is

\[
\langle \delta_i \delta_j \rangle_c = \min(S_i, S_j). \]

In this case, the integrals in Equation (4) can be worked out directly to give

\[
W_{\text{GM}}(\delta_0; \delta_1, \ldots, \delta_n; S_n) = \frac{1}{(2\pi \epsilon)^{n/2}} \exp \left( -\frac{1}{2\epsilon} \sum_{i=0}^{n-1} (\delta_{i+1} - \delta_i)^2 \right),
\]

where the superscript “gm” refers to the “Gaussian and Markovian” case. When the density smoothing filter is not a top-hat function in \( k \) space, e.g., a top-hat function in real space or a Gaussian function, MR10 showed that an additional term appeared in the density correlation:

\[
\langle \delta_i \delta_j \rangle_c = \min(S_i, S_j) + \Delta(S_i, S_j),
\]
where $\Delta(S_i, S_j)$ is well approximated by

$$\Delta(S_i, S_j) \approx \kappa \frac{S_{\text{min}}(S_{\text{max}} - S_{\text{min}})}{S_{\text{max}}},$$

$$S_{\text{max}} = \max(S_i, S_j), \quad S_{\text{min}} = \min(S_i, S_j).$$ (8)

The parameter $\kappa$ characterizes the non-Markovian process, whose value depends on the shape of the smoothing filter, e.g., $\kappa \approx 0.44$ for a top-hat function in real space, and $\kappa \approx 0.35$ for a Gaussian function.

For convenience, we use $\Delta_j$ to denote $\Delta(S_i, S_j)$ in this paper. In the non-Markovian case, Equation (4) can be expanded perturbatively into

$$W(\delta_0; \delta_1, \ldots, \delta_n; S_n) \approx \left(1 + 2 \sum_{i,j=1}^n \Delta_j \frac{\partial^2}{\partial \delta_i \partial \delta_j} \right) W_{\text{EM}}(\delta_0; \delta_1, \ldots, \delta_n; S_n).$$ (9)

We will use Equation (9) for the rest of the paper, keeping in mind that this relation only includes the leading order non-Markovian corrections.

3. MAIN DERIVATION

3.1. Introduce the Environmental Variable

The new ingredient that we will introduce into the non-Markovian excursion-set model is the linear overdensity, $\delta_e$, which quantifies the larger-scale environment of a dark-matter halo. We denote the smoothing scale over which $\delta_e$ is evaluated as $S_e$, where $S$ is defined in Equation (2).

Throughout the paper, we use the subscripts "e," "d," and "p" to denote environment, descendant, and progenitor, respectively.

We consider a descendant halo of mass $M_d$, or $S_d = S(M_d)$, that forms at redshift $z_d$, when the barrier height is $\delta_{cd} = \delta_e / D(z_d)$, where $\delta_e = 1.68$ and $D(z)$ is the linear growth function.

We consider the probability for the descendant halo to have a progenitor halo of mass $M_p$, or $S_p = S(M_p)$, that forms at a higher redshift $z_p$ when the barrier height is higher: $\delta_{cp} = \delta_e / D(z_p)$. We adopt the convention that the critical overdensity, instead of the linear overdensity, is a function of redshift. The linear overdensity is always evaluated at redshift zero, including that on the environmental scale.

As an initial setup, we define three events $A$, $B$, and $C$ as follows.

A. At a location of interest, the overdensity smoothed over a scale $S_e$ (centered on the location) is $\delta_e$.

B. At the same location as in $A$, a halo of mass $S_d$ forms at redshift $z_d$, corresponding to barrier $\delta_{cd} = \delta_e / D(z_d)$, where $S_d > S_e$.

C. At the same location as in $A$, a progenitor halo of mass $S_p$ forms at redshift $z_p$, corresponding to barrier $\delta_{cp} = \delta_e / D(z_p)$, where $z_p > z_d$ and $S_p > S_d > S_e$.

We then define the following probabilities that relate the three events above.

1. $P(A) d\delta_e$ is the probability that the linear overdensity smoothed over scale $S_e$ is between $\delta_e$ and $\delta_e + d\delta_e$. For a Gaussian field, we have the simple relation

$$P(A) = \exp\left(-\delta_e^2 / 2S_e\right) / \sqrt{2\pi S_e}. $$ (10)

2. $P(A, B) d\delta_e dS_d$ is the probability that a halo of mass between $S_d$ and $S_d + dS_d$ forms at redshift $z_d$, and at the halo location, the linear overdensity smoothed over a larger scale $S_e$ is between $\delta_e$ and $\delta_e + d\delta_e$. More explicitly, we have

$$P(A, B) = P_{AB}(S_d, z_d, S_e, \delta_e).$$ (11)

3. $P(A, B, C) d\delta_e dS_d dS_p$ is the probability that a halo of mass between $S_d$ and $S_d + dS_d$ forms at redshift $z_d$, and the mass of this halo at an earlier redshift $z_p$ is in a progenitor of mass between $S_p$ and $S_p + dS_p$, and at the halo location, the linear overdensity on a scale of $S_e$ is between $\delta_e$ and $\delta_e + d\delta_e$. More explicitly, we have

$$P(A, B, C) = P_{ABC}(S_p, z_p, S_d, z_d, S_e, \delta_e).$$ (12)

Our goal is to derive expressions for the following conditional probabilities that depend on the halo environment parameterized by $\delta_e$ and $S_e$.

1. $P(B|A) dS_d$ is the probability that a halo of mass between $S_d$ and $S_d + dS_d$ forms at redshift $z_d$ in an environment of linear overdensity $\delta_e$ on a scale of $S_e$. More explicitly, we have

$$P(B|A) = P_{B|A}(S_d, z_d | S_e, \delta_e).$$ (13)

As we show in Section 3.2, this quantity is simply related to the environment-dependent halo mass function.

2. For a halo of mass $S_d$ forming at redshift $z_d$, located in the center of an environment of scale $S_e$ and linear overdensity $\delta_e$, $P(C|A, B) dS_p$ is the probability that the mass of this halo at an earlier redshift $z_p$ is in a progenitor of mass between $S_p$ and $S_p + dS_p$. More explicitly, we have

$$P(C|A, B) = P_{C|A,B}(S_p, z_p | S_d, z_d, S_e, \delta_e).$$ (14)

As we show in Section 3.2, this quantity is simply related to the environment-dependent progenitor mass function and the halo merger rate.

The probabilities are related by $P(B|A) = P(A, B) / P(A)$ and $P(C|A, B) = P(A, B, C) / P(A, B)$. When the smoothing filter is chosen to be a top-hat function in $k$ space, the random walk is a Markovian process. The environmental dependence drops out in this case, and we have $P(C|A, B) = P(C|B)$, which is related to the standard progenitor mass function.

3.2. Relate Probability Functions to Halo Mass Functions and Merger Rates

We define $n_{\mu}(M_d, z_d | S_e, \delta_e) dM_d$ as the mean number density of (descendant) halos of mass between $M_d$ and $M_d + dM_d$ at redshift $z_d$ residing in a region of linear overdensity $\delta_e$ and smoothed over scale $S_e$. This halo mass function is simply related to the conditional probability $P(B|A)$ (denoted as $P_{B|A}$ below) by

$$n_{\mu}(M_d, z_d | S_e, \delta_e) = \bar{\rho} \frac{dS_d}{dM_d} | P_{B|A}(S_d, z_d | S_e, \delta_e),$$ (15)

where $\bar{\rho}$ is the mean mass density.

Similarly, we define $N(M_p, z_p | M_d, z_d, S_e, \delta_e) dM_p$ as the mean number of progenitor halos of mass between $M_p$ and $M_p + dM_p$ at redshift $z_p$ for a descendant halo of mass $M_d$ and redshift $z_d$ residing in an environment of scale $S_e$ and linear overdensity $\delta_e$. This progenitor mass function is simply related
to the conditional probability $P(C|A, B)$ (denoted as $P_{(C|A,B)}$) below by

$$N(M_p, z_p| Md, z_d, S_r, \delta_e) = \frac{M_d}{M_p} \frac{dS_p}{dM_p} P_{(C|A,B)} (S_p, z_p| Md, z_d, S_r, \delta_e).$$

(16)

The halo merger rate can be written in terms of the progenitor mass function above. To this end, we adopt the binary-merger assumption as in Zhang et al. (2008) and define $R(M, \xi, z| S_r, \delta_e)$ (same as the B/M term in Equation (8) of Fakhouri & Ma 2009) to be the number of mergers per unit progenitor mass ratio $\xi$ (ratio of the small to the large progenitor mass) and unit redshift for each descendant halo of mass $M$ at redshift $z$, under the condition that the linear overdensity on the environmental scale $S_r$ is $\delta_e$. Due to the binary-merger assumption, the merger rate $R$ can be related to the progenitor mass function via

$$R(M_d, \xi, z_d| S_r, \delta_e) = \frac{M_d}{(1 + \xi)^2} \int d \delta_e N^{|z = \delta_e} \left( M_d \xi, z| M_d, z_d, S_r, \delta_e \right).$$

(17)

Equations (15)–(17) enable us to obtain the environment-dependent halo mass functions and halo merger rates from $P(B|A)$ and $P(C|A, B)$. Since $P(B|A) = P(A, B)/P(A)$ and $P(C|A, B) = P(A, B, C)/P(A, B)$, our next task is therefore to calculate $P(A, B)$ and $P(A, B, C)$.

3.3. Express $P(A, B)$ in Path-integral Form

According to the definition of $P(A, B)$ in Section 3.1, we have

$$\int_{S_l}^{\infty} dS_p^{d} P_{AB}(S_p^{d}, z_d, S_r, \delta_e) = \int_{-\infty}^{\delta_m} d\delta_1 \ldots d\delta_m \ldots d\delta_n$$

$$\times W(0; \delta_1, \ldots, \delta_m = \delta_e, \ldots, \delta_n; S_d),$$

(18)

where the positions of $S_r$ and $S_d$ are approximated as $me$ and $n\epsilon$, respectively, with $m$ and $n$ being integers. In other words, $S_m = S_r, S_n = S_d$, and $\delta_m = \delta_e$. The hat over $\delta_m$ means that $\delta_m$ is omitted from the list of integration variables.

By taking partial derivatives with respect to $\delta_m$ on both sides of Equation (18) and using Equation (9), we obtain

$$P(A, B) = P_{AB}(S_d, z_d, \delta_e, S_r)$$

$$= \frac{\partial}{\partial S_d} \int_{-\infty}^{\delta_m} d\delta_1 \ldots d\delta_m \ldots d\delta_n \left( 1 + \frac{1}{2} \sum_{i,j=1}^{n} \Delta_{ij} \delta_i \delta_j \right)$$

$$\times W^{\text{gm}}(0; \delta_1, \ldots, \delta_m = \delta_e, \ldots, \delta_n; S_d).$$

(19)

The terms proportional to $\Delta_{ij}$ are the non-Markovian corrections.

We rewrite the summation in the non-Markovian terms in Equation (19) as

$$\frac{1}{2} \sum_{i,j=1}^{n} \Delta_{ij} \delta_i \delta_j = \sum_{i=1}^{n-1} \Delta_{in} \delta_i \delta_n + \sum_{i \neq j}^{n} \Delta_{ij} \delta_i \delta_j,$$

(20)

where $\Delta_{ij} = 0$ for $i = 1, 2, \ldots, n$ based on Equation (8) and is therefore not included. It can be shown that the first term on the right-hand side of Equation (20) is zero. The second term can be broken into five pieces, representing all the possible locations of $i$ and $j$ with respect to $m$ and $n$:

$$\sum_{i < j < n} = \sum_{i=m+1}^{n-1} \sum_{j=m+1}^{n-1} \sum_{i < j < n} \cdot (i = m)$$

$$+ \sum_{j=m+1}^{n-1} \sum_{i=1}^{m-1} \sum_{j=i+1}^{m} \cdot (j = m) \cdot \sum_{i=1}^{m-1} \cdot \sum_{j=1}^{m-1} \cdot (m-1 \cdot j-1).$$

(21)

In total, $P(A, B)$ in Equation (19) is the sum of the Markovian term and the five terms in Equation (21). We write these six terms as

$$P(A, B) = P_{AB}^{M} + P_{NM1}^{NM} + \cdots + P_{NM5}^{NM}.$$

The superscripts $M$ and $NM$ refer to Markovian and non-Markovian, respectively, and the number following each NM refers to the order of the term on the right-hand side of Equation (21).

The algebra involved in deriving these six terms is straightforward but lengthy. We leave the details to Appendix A. The final expression for $P(A, B)$ is given by Equation (A7).

3.4. Express $P(A, B, C)$ in Path-integral Form

The derivation of $P(A, B, C)$ is similar to that of $P(A, B)$ above but is more complicated. According to the definition of $P(A, B, C)$ in Section 3.1, we have

$$\left( \int_{S_l}^{S_p} dS_d \int_{S_p}^{\infty} dS_p^{d} + \int_{-\infty}^{\infty} dS_p^{d} \int_{S_l}^{\infty} dS_p \right)$$

$$\times P_{ABC}(S_p^{d}, z_p, S_d, z_d, S_r, \delta_e)$$

$$= \int_{-\infty}^{\delta_m} d\delta_1 \ldots d\delta_m \ldots d\delta_n \int_{-\infty}^{\delta_m} d\delta_1 \ldots d\delta_m \ldots d\delta_n$$

$$\times W(0; \delta_1, \ldots, \delta_m = \delta_e, \ldots, \delta_n; S_p),$$

(23)

where the positions of $S_r, S_d, and S_p$ are approximated as $me, ne, and Ne$, respectively, with $m, n, and N$ being integers. In other words, $S_m = S_r, S_n = S_d, and \delta_m = \delta_e$. The hat over $\delta_m$ means that $\delta_m$ is omitted from the list of integration variables.

By taking partial derivatives with respect to both $S_p$ and $S_d$ on the two sides of Equation (23) and using Equation (9), we obtain

$$P(A, B, C) = P_{ABC}(S_p, z_p, S_d, z_d, S_r, \delta_e)$$

$$= \frac{\partial^2}{\partial S_d \partial S_p} \int_{-\infty}^{\delta_m} d\delta_1 \ldots d\delta_m \ldots d\delta_n \int_{-\infty}^{\delta_m} d\delta_1 \ldots d\delta_m \ldots d\delta_n$$

$$\times \left( 1 + \frac{1}{2} \sum_{i,j=1}^{n} \Delta_{ij} \delta_i \delta_j \right) W^{\text{gm}}(0; \delta_1, \ldots, \delta_n = \delta_e, \ldots, \delta_N; S_p).$$

(24)

The terms proportional to $\Delta_{ij}$ are the non-Markovian corrections.

Similar to Equation (20), we rewrite the summation in the non-Markovian terms above as

$$\frac{1}{2} \sum_{i,j=1}^{N} \Delta_{ij} \delta_i \delta_j = \sum_{i=1}^{N-1} \Delta_{iN} \delta_i \delta_N + \sum_{i < j < N} \Delta_{ij} \delta_i \delta_j.$$
As before, the first term here is always zero. We decompose the rest into 13 terms:

\[
\sum_{i<j<N} = \sum_{i,j \in \{1,\ldots,N\}} (i \neq j) + \sum_{i \in \{1,\ldots,N\}} (i = j)
\]

\[
= \sum_{i} \sum_{j < i} + \sum_{j} \sum_{i < j} = \sum_{i=1}^{N-1} \sum_{j=i+1}^{N-1} (i \neq j) + \sum_{i=1}^{N} \sum_{j=1}^{i-1} (i = j)
\]

\[
+ \sum_{i=1}^{N-1} \sum_{j=i+1}^{N-1} (i = n) + \sum_{i=1}^{N} \sum_{j=1}^{i-1} (j = n)
\]

Again, we denote the Markovian part of \(P(A, B, C)\) as \(P_{ABC}^M\), and the 13 non-Markovian terms on the right-hand side of Equation (26) as \(P_{ABC}^{NM13}\). The probability \(P(A, B, C)\) is then

\[
P(A, B, C) = P_{ABC}^M + P_{ABC}^{NM1} + \cdots + P_{ABC}^{NM13}.
\]

We leave the details of the derivation of these 14 terms to Appendix B. The final expression for \(P(A, B, C)\) is given by Equation (B5).

### 3.5. Asymptotic Forms in the Limit of Large Environmental Scale

As shown in Sections 3.3 and 3.4 and Appendices A and B, the general forms of \(P(A, B), P(A, B, C), \) and \(P(C|A, B)\) contain many terms. In practice, it is often unnecessary to consider the general case. Here, we derive the simplified forms of \(P(B|A)\) and \(P(C|A, B)\) in the limit of large environmental scale, which is usually the case considered in simulations and observations. We leave the details of the derivation to Appendix C and quote the final results here.

To linear order in \(\delta_e\), the probability of forming a descendant halo of mass \(S_d = S(M_d)\) at redshift \(z_d\) that resides in a larger environment of overdensity \(\delta_e\) smoothed over scale \(S_e\) is

\[
P(B|A) \approx \frac{\delta_{cd}}{\sqrt{2\pi S_d^{3/2}}} \exp \left( -\frac{\nu^2}{2} \right)
\]

\[
\times \left[ 1 - \kappa + \frac{\nu^2}{2} \right] \Gamma \left( 0, \frac{\nu^2}{2} \right)
\]

\[
+ \frac{\delta_e}{\delta_{cd}} \left[ \nu^2 - 1 + \kappa - \frac{\nu^2}{2} \right] \Gamma \left( 0, \frac{\nu^2}{2} \right)
\]

where \(\nu \equiv \delta_{cd}/\sqrt{S_e}\), \(\delta_{cd} = \delta_c/D(z_d)\) is the barrier height for forming a descendant halo at redshift \(z_d\), \(\Gamma(0, x)\) is the incomplete gamma function, and \(\kappa\) is the non-Markovian parameter defined in Equation (8). We note that this equation is identical to Equation (24) of Ma et al. (2011) for the conditional first-crossing rate, which was used to derive the halo bias parameter. In the limit of \(\delta_e \to 0\), we recover from Equation (28) the non-Markovian extension of the standard halo mass function (see, e.g., Table 1 of Ma et al. 2011):

\[
P(B) = \frac{\delta_{cd}}{\sqrt{2\pi S_d^{3/2}}} \exp \left( -\frac{\nu^2}{2} \right)
\]

\[
\times \left[ 1 - \kappa + \frac{\nu^2}{2} \right] \Gamma \left( 0, \frac{\nu^2}{2} \right)
\]

Similarly, the conditional probability (to linear order in \(\delta_e\)) that a descendant halo of mass \(S_d = S(M_d)\) at redshift \(z_d\), residing in a larger environment of overdensity \(\delta_e\) at scale \(S_e\), has a progenitor halo of mass \(S_p = S(M_p)\) at redshift \(z_p\) (assuming \(z_p \approx z_d\)) is

\[
P(C|A, B) \approx \frac{\delta_{cd} - \delta_{cp}}{\sqrt{2\pi (S_p - S_d)^{3/2}}} \left[ 1 + \kappa \beta \alpha \right] \left[ 1 - \sqrt{\frac{\nu^2}{2}} \right] \left[ 1 - \frac{\nu^2}{2} \right]
\]

\[
+ \pi \kappa \left( \nu^2 \delta_{cd} - 1 \right) (1 - \alpha)^{3/2} \exp \left( \frac{\nu^2}{2} \right) \text{erfc} \left( \sqrt{\frac{\nu^2}{2}} \right)
\]

where \(\alpha \equiv S_d/S_p\), and \(\beta\) is a simple algebraic function of \(\alpha\):

\[
\beta = -2 + \frac{(1 - \alpha)^{3/2}}{2\alpha} \ln \left( 1 + \sqrt{1 - \alpha} + \frac{1}{\sqrt{1 - 1 - \alpha}} \right) + \frac{1}{\alpha} + 2\alpha.
\]

The variables \(\delta_{cp} = \delta_c/D(z_p)\) and \(\delta_{cd} = \delta_c/D(z_d)\) specify the barrier heights for forming the progenitor and descendant halos at redshift \(z_p\) and \(z_d\), respectively. Equations (10) and (11) relate \(P(C|A, B)\) above to the mean progenitor mass function \(N(\delta_c, z_p, z_p)|M_p, S_e, \xi, \delta_c\) and the merger rate \(R(M_d, z_p, z_p)|M_d, S_e, \xi, \delta_c\). In the Markovian limit \((\kappa = 0)\), we note that Equation (30) reduces to the familiar conditional mass function of small look-back time \((\delta_{cp} - \delta_{cd})\) predicted by the excursion-set model, and the dependence on the environmental overdensity \(\delta_e\) drops out. This limit confirms that the introduction of the non-Markovian process to the excursion-set model is the key in introducing the environmental dependence of halo formation history.

Finally, the accuracy of the simple spherical-collapse model can be improved by considering a diffusing barrier instead of a constant one. The introduction of the diffusing barrier is motivated by both the elliptical-collapse model and \(N\)-body studies, for the reason that realistic halos are triaxial rather than spherical. For our purpose, we only need to replace \(\delta_e\) by \(\delta_e/\sqrt{1 + D_B}\) and \(\kappa\) by \(\kappa/(1 + D_B)\) in our formulae to take into account the diffusing barrier effect (Robertson et al. 2009; Maggiore & Riotto 2010b), with \(D_B = 0.25\).

### 3.6. Numerical Results

In Figure 1, we illustrate the numerical results from our analytic formulae for the halo mass function \(n(M_d, z_d|S_e, \delta_e)\) (upper panels) and the merger rate \(R(M_d, z_p, z_p)|M_d, S_e, \delta_c\) (lower panels) as a function of the halo environment \(\delta_e\). Three descendant halo masses at \(z_d = 0\) are shown for comparison: \(M_d = 10^{11}\) (blue), \(10^{12}\) (green), \(10^{13}\) \(M_{\odot}\) (red). The environmental mass scale \(S_e\) is chosen to be \(10^{17}\) \(M_{\odot}\). The full expressions (solid curves) are
Figure 1. Environmental dependence of the halo mass function $n(M_d, z_d|S_e, \delta_e)$ (upper panels) and the halo merger rate $R(M_d, z_d|S_e, \delta_e)$ for merger mass ratios above 0.01 (i.e., $\xi = 0.01$–1, lower panels). Two types of barriers in the excursion-set model are shown for comparison: constant expressions (valid to linear order in $\delta_e$ and diffusing $\delta_e/\xi$ ratios above 0.01 (i.e., computed from Equations (A7) and (B5), and the approximate expressions (valid to linear order in $\delta_e$) from Equations (28) and (30). The diffusing barrier effect is included in the right two panels with $D_B = 0.25$ and not included in the left two panels (i.e., $D_B = 0$). The cosmological model is a $\Lambda$CDM model with $\Omega_m = 0.25$, $\Omega_{\Lambda} = 0.045$, $\Omega_k = 0.75$, $h = 0.73$, and an initial power-law power spectrum of the density fluctuation with index $n = 1$ and normalization $\sigma_8 = 0.9$.

The lower panels of Figure 1 show a positive dependence of the merger rate on $\delta_e$. Since Equations (16) and (17) indicate that the progenitor mass function has the same dependence on $\delta_e$ as the halo merger rate, our results imply that progenitor mass functions are also higher in regions with higher $\delta_e$. This environmental trend is consistent with that seen for halo merger rates in the Millennium simulation (Fakhouri & Ma 2008, 2009, 2010), where the amplitudes of the merger rate and progenitor mass functions increase with the environmental overdensities. The black solid curves in the lower panels of Figure 1 show the environmental dependence of the second formula in Equation (11) of Fakhouri & Ma (2009). The larger-scale overdensity in this case is $\delta_e$ and is measured within a comoving radius of $R = 7\ h^{-1} \ Mpc$ centered at each halo in the simulation. As Figure 1 shows, the overall dependence of the merger rate on $\delta_e$ is similar, while the slope of the curves from our analytic model has a weak dependence on halo mass. As discussed in detail in Fakhouri & Ma (2009), there are various options for quantifying halo environment in simulations. For instance, the environmental overdensity can be computed by either including or excluding the virial mass of the central halo within the sphere of radius $R$ over which $\delta_e$ is computed. For simplicity, Equation (11) of Fakhouri & Ma (2009) provides two separate fits for $\delta_7$ and $\delta_{7-FOF}$, where the latter excludes the halo’s friends-of-friends algorithm (FOF) mass. They also noted that the difference between the two definitions, $\delta_7 - \delta_{7-FOF}$, is a function of halo mass, increasing from $\sim 0.01$ at $10^{12} M_\odot$ to $\sim 10$ at $10^{15} M_\odot$. Given this uncertainty and mass dependence, it is therefore not surprising that our analytic model predicts mass-dependent slopes in Figure 1. A closer comparison between our model prediction and simulation results would require a more elaborate mapping between the linear $\delta_e$ in the excursion-set model and the nonlinear $\delta_7$ and $\delta_{7-FOF}$ used in the simulation. We leave this step to future studies.

4. SUMMARY

We have presented a method to introduce “assembly bias” into the excursion-set model for the formation and growth of dark-matter halos. Our calculation is based on the barrier-crossing problem of non-Markovian processes, which we solve perturbatively using the path-integral formalism developed in MR10. The new variable that we introduced to parameterize a halo’s larger-scale environment is the linear overdensity field $\delta_e$ smoothed over a chosen scale of $S_e$, where $S_e$ is the variance of the linear density fluctuations and is a monotonically decreasing function of the smoothing radius $R$.

To introduce environmental dependence, we isolated $\delta_e$ from the path integral over the probability density of trajectories $W$ in Equation (18). We then derived the two main probability functions $P(A, B)$ and $P(A, B, C)$, defined in Section 3.1, for forming descendant and progenitor halos in an environment in which the linear overdensity smoothed over scale $S_e$ is given by $\delta_e$. The calculations are set up in Sections 3.3 and 3.4, and the details of how to manipulate the numerous integrals are given in Appendices A and B. The final analytic expressions for $P(A, B)$ and $P(A, B, C)$ are given by Equations (A7) and (B5), respectively.

The three key physical quantities that we investigated in this paper are the descendant halo mass function $n(M_d, z_d|S_e, \delta_e)$, the progenitor mass function $N(M_p, z_p|M_d, z_d, \delta_e)$, and the halo merger rate $R(M_{d}, \xi, z_d|S_e, d_e)$. These quantities are related to the conditional probabilities $P(B|A)$ and $P(C|A, B)$ by Equations (15)–(17), which in turn can be computed from our formulae for $P(A, B)$ and $P(A, B, C)$.

Since the full expressions for the mass functions and merger rates are complicated, we derived their asymptotic forms in the limit of large environmental scale (i.e., small $S_e$ and $\delta_e$) in Section 3.5 and Appendix C. This is a useful limit for many practical purposes. The approximate expressions for the descendant mass function and progenitor mass function are given by Equations (28) and (30), respectively. Figure 1 illustrates the environmental dependence predicted by our model. It is encouraging that both our analytic calculation and $N$-body results show that the halo merger rate and progenitor mass function correlate positively with the environmental density.

The recipe presented in this paper for incorporating environmental dependence into the excursion-set model is quite general. It should provide a useful theoretical framework for future investigations into how the spatial distributions and statistical properties of dark-matter halos depend on their mass as well...
as their assembly history and the larger-scale environment in which they reside.

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APPENDIX A
DEVIATION OF $P(A, B)$$P(A, B)$

In this appendix, we carry out the integral in Equation (19) explicitly and derive an expression for each of the six terms in the summation in Equation (22). We begin with the following relations from MR10:

$$W^{\text{mm}}(\delta_0; \delta_1, \ldots, \delta_n; S_n) = W^{\text{mm}}(\delta_0; \delta_1, \ldots, \delta_i; S_i)W^{\text{mm}}(\delta_i; \delta_{i+1}, \ldots, \delta_n; S_n - S_i),$$

$$\Pi^k(\delta_0; \delta_n; S_n) \equiv \int_{-\infty}^{\delta_n} d\delta_1 \ldots \int_{-\infty}^{\delta_{n-1}} d\delta_{n-1} W^{\text{mm}}(\delta_0; \delta_1, \ldots, \delta_n; S_n).$$  \hspace{1cm} (A1)

For the Markovian term in $P(A, B)$ of Equation (22), we find

$$P_{AB}^M = -\frac{\partial}{\partial S_n} \int_{-\infty}^{\delta_n} d\delta_n \Pi^k(0; \delta; S_m) \Pi^k(\delta_c; \delta_n; S_n - S_m).$$  \hspace{1cm} (A2)

For the five non-Markovian terms, we find

$$P_{AB}^{NM1} = -\sum_{j=m+1}^{n-1} \sum_{i=m+1}^{j-1} \Delta_{ij} \frac{d}{d\delta_n} \int_{-\infty}^{\delta_n} d\delta_n \Pi^k(0; \delta; S_m) \Pi^k(\delta_c; \delta_n; S_n - S_m)
\times \Pi^k(\delta_c; \delta; S_j - S_i) \Pi^k(\delta_c; \delta_n; S_n - S_j),$$

$$P_{AB}^{NM2} = -\sum_{j=m+1}^{n-1} \sum_{i=m+1}^{j-1} \Delta_{ij} \frac{d}{d\delta_n} \int_{-\infty}^{\delta_n} d\delta_n \Pi^k(0; \delta; S_m) \Pi^k(\delta_c; \delta_n; S_n - S_i)
\times \Pi^k(\delta_c; \delta; S_j - S_m) \Pi^k(\delta_c; \delta_n; S_n - S_j),$$

$$P_{AB}^{NM3} = -\sum_{i=1}^{m-1} \sum_{j=m+1}^{n-1} \Delta_{ij} \frac{d^2}{d\delta_n^2} \int_{-\infty}^{\delta_n} d\delta_n \Pi^k(0; \delta; S_m) \Pi^k(\delta_c; \delta_n; S_n - S_i)
\times \Pi^k(\delta_c; \delta; S_j - S_m) \Pi^k(\delta_c; \delta_n; S_n - S_j),$$

$$P_{AB}^{NM4} = -\sum_{i=1}^{m-1} \sum_{j=m+1}^{n-1} \Delta_{ij} \frac{d}{d\delta_n} \int_{-\infty}^{\delta_n} d\delta_n \Pi^k(0; \delta; S_m) \Pi^k(\delta_c; \delta; S_j - S_m)
\times \Pi^k(\delta_c; \delta_n; S_n - S_j),$$

$$P_{AB}^{NM5} = -\sum_{i=1}^{m-1} \sum_{j=m+1}^{n-1} \Delta_{ij} \frac{d}{d\delta_n} \int_{-\infty}^{\delta_n} d\delta_n \Pi^k(0; \delta; S_m) \Pi^k(\delta_c; \delta_n; S_n - S_i)
\times \Pi^k(\delta_c; \delta; S_j - S_m) \Pi^k(\delta_c; \delta_n; S_n - S_m).$$  \hspace{1cm} (A3)

To transform the summations into integrations and further simplify these expressions, we use the following relations from MR10:

$$\Pi^k_{\rightarrow 0}(\delta; \delta, S)(\delta_0 = \delta) = \frac{1}{\sqrt{2\pi S}}[e^{-(\delta - \delta_0)^2/(2S)} - e^{-(\delta_0, -\delta_0, -\delta)^2/(2S)}], \quad \Pi^k_{\rightarrow 0}(\delta_c; \delta, S) = \frac{e}{\sqrt{2\pi S^{3/2}}},$$

$$\Pi^k_{\rightarrow 0}(\delta_0; \delta_c; S)(\delta_0 \neq \delta_c) = \frac{\pi}{S^{3/2}} e^{\frac{\delta_c - \delta_0}{S^{3/2}} - e^{-(\delta_c, -\delta_c)^2/(2S)}}, \quad \Pi^k_{\rightarrow 0}(\delta_c; \delta; S)(\delta \neq \delta_c) = \frac{\sqrt{2\pi}}{\sqrt{S^{3/2}} e^{-(\delta_c, -\delta)^2/(2S)}}. $$  \hspace{1cm} (A4)

Substituting these expressions into Equations (A2) and (A3), we obtain

$$P_{AB}^M = -\Pi^k_{\rightarrow 0}(0; \delta_c; S_m) \frac{d}{d\delta_n} \int_{-\infty}^{\delta_n} d\delta_n \Pi^k(\delta_c; \delta_n; S_n - S_m).$$  \hspace{1cm} (A5)
Using Equation (7) for $\Delta(S_i, S_j)$, we can work out the integrals above. This step is straightforward but tedious, so we only present the final results here. We also replace $S_m$ and $S_n$ with the more physical notation for the environment and descendant: $S_m = S_e$ and $S_n = S_d$. Our final expression for $P(A, B)$ is given by

$$P(A, B) = P_{AB}^M + P_{AB}^{NM1} + \cdots + P_{AB}^{NM5}$$

$$= \frac{1}{\sqrt{2\pi}} \frac{\delta_{cd} - \delta_e}{(S_d - S_e)^{3/2}} \left[ 1 + \frac{\kappa S_e}{S_d} \left( 1 - \frac{(\delta_{cd} - \delta_e)^2}{S_d - S_e} \right) + \frac{\kappa}{\pi} \frac{\delta_e (\delta_{cd} - \delta_e)}{S_d} \right] E_1 \Pi$$

$$+ \frac{k}{2\sqrt{2\pi}} \frac{\delta_{cd} - \delta_e}{S_d} S_e^{-3/2} E_2 F_1 \Pi + \frac{k}{\pi} \frac{\delta_{cd} (\delta_{cd} - \delta_e)^2}{S_d - S_e} S_e^{-3/2} E_2 F_1 F_2$$

$$+ \frac{k}{2\sqrt{2\pi}} \frac{\delta_{cd} (\delta_{cd} - \delta_e)^3 S_e^{-1}}{S_d - S_e} (S_d - S_e)^{-3/2} E_2 E_1 F_2 + \frac{k}{\pi} \frac{\delta_{cd} (\delta_{cd} - \delta_e)^2 S_e^{-3/2}}{S_d - S_e} (S_d - S_e)^{-3/2} E_2 E_1 F_2,$$  (A7)

where

$$E_1 = \exp \left[ \frac{(\delta_{cd} - \delta_e)^2}{2(S_d - S_e)} \right], \quad E_2 = \exp \left[ \frac{(\delta_{cd} - \delta_e)^2}{2S_e} \right], \quad E_3 = \exp \left[ -\frac{(\delta_{cd} - \delta_e)^2}{2S_e} \right],$$

$$E_{R1} = \text{erfc} \left[ \frac{2\delta_{cd} - \delta_e}{\sqrt{2S_e}} \right], \quad E_{R2} = \text{erfc} \left[ \sqrt{\frac{S_d}{2(S_d - S_e)}} \frac{(\delta_{cd} - \delta_e)}{S_d} \right], \quad E_{R3} = \text{erfc} \left[ \frac{\delta_{cd} - \delta_e}{\sqrt{2(S_d - S_e)}} \right],$$

$$\Pi = \Pi_0^{\frac{e-}{a}}(0; \delta_e; S_e), \quad F[a(>0), b] = \int_{a}^{+\infty} \frac{dx}{x} e^{-(ax+b)^2},$$

$$F_1 = F \left[ \frac{\sqrt{S_d / S_e}}{S_d - S_e - 1} \frac{(\delta_{cd} - \delta_e)}{\sqrt{2S_e}}, \frac{(\delta_{cd} - \delta_e)}{\sqrt{2S_e}} \right] - F \left[ \frac{\sqrt{S_d / S_e} + 1}{\sqrt{2S_e}} \frac{(\delta_{cd} - \delta_e)}{\sqrt{2S_e}}, \frac{(\delta_{cd} - \delta_e)}{\sqrt{2S_e}} \right],$$

$$F_2 = F \left[ \frac{\delta_{cd} - \delta_e}{\sqrt{2S_e}}, \frac{\delta_{cd} - \delta_e}{\sqrt{2S_e}} \right].$$  (A8)
APPENDIX B

DERIVATION OF $P(A, B, C)$

In this appendix, we carry out the integral in Equation (24) explicitly and derive an expression for each of the 14 terms in the summation in Equation (27). For $P(A, B, C)$ in Equation (27), we find the Markovian term to be

$$P_{ABC}^M = \frac{\partial^2}{\partial S_a \partial S_N} \int_{-\infty}^{\delta_{b,v}} d\delta_n \int_{-\infty}^{\delta_{h,v}} d\delta_N \Pi_{e}^{\delta_{v}}(0; \delta_e; S_m) \Pi_{e}^{\delta_{u}}(\delta_u; \delta_n; S_n - S_m) \Pi_{e}^{\delta_{v}}(\delta_v; \delta_N; S_N - S_n),$$

(B1)

and the 13 non-Markovian terms to be

$$P_{ABC}^{NM1} = \sum_{j=1}^{m} \sum_{i=1}^{m-1} \partial^2 \Pi_{e}^{\delta_{u}}(0; \delta_u; S_m) \Pi_{e}^{\delta_{v}}(\delta_v; \delta_n; S_N - S_m) \times \Pi_{e}^{\delta_{v}}(\delta_v; \delta_N; S_N - S_j),$$
\[ P_{ABC}^{NM12} = \sum_{i=1}^{m-1} A_{im} \frac{\alpha^3}{\partial \delta_c \partial \delta_N \partial S_{N}} \int_{-\infty}^{\delta_N} d\delta_n \int_{-\infty}^{\delta_N} d\delta_N \Pi^{\delta_N}(0; \delta; S_{N} \mid S_{m} - S_{N}) \times \Pi^{\delta_N}(\delta; S_{N} - S_{m}) \Pi^{\delta_N}(\delta; S_{N} - S_{m}). \]

\[ P_{ABC}^{NM13} = \sum_{j=1}^{m-1} \sum_{i=1}^{n-1} A_{ij} \frac{\alpha^2}{\partial \delta_c \partial \delta_N \partial S_{N}} \int_{-\infty}^{\delta_N} d\delta_n \int_{-\infty}^{\delta_N} d\delta_N \Pi^{\delta_N}(0; \delta; S_{N} \mid S_{m} - S_{N}) \times \Pi^{\delta_N}(\delta; S_{N} - S_{m}) \Pi^{\delta_N}(\delta; S_{N} - S_{m}). \]

The 14 expressions above can again be written out as

\[ P_{ABC}^{M} = \Pi^{\delta_N}(0; \delta; S_{m}) \delta_{cd} \delta_{cp} \int_{-\infty}^{\delta_N} d\delta_n \int_{-\infty}^{\delta_N} d\delta_N \Pi^{\delta_N}(0; \delta; S_{N} \mid S_{m} - S_{N}) \Pi^{\delta_N}(\delta; S_{N} - S_{m}) \Pi^{\delta_N}(\delta; S_{N} - S_{m}). \]

\[ P_{ABC}^{NM1} = \frac{1}{\pi} \frac{\alpha^2}{\partial \delta_c \partial \delta_N \partial S_{N}} \int_{-\infty}^{\delta_N} d\delta_n \int_{-\infty}^{\delta_N} d\delta_N \Pi^{\delta_N}(0; \delta; S_{N} \mid S_{m} - S_{N}) \Pi^{\delta_N}(\delta; S_{N} - S_{m}) \Pi^{\delta_N}(\delta; S_{N} - S_{m}) \exp \left[ -\frac{\delta_{cp}^2}{2S_{N}} - \frac{\delta_{cp}^2 - \delta_{cp^2}}{2(S_{N} - S_{j})} \right]. \]

\[ P_{ABC}^{NM2} = \frac{1}{\pi} \frac{\alpha^3}{\partial \delta_c \partial \delta_N \partial S_{N}} \int_{-\infty}^{\delta_N} d\delta_n \int_{-\infty}^{\delta_N} d\delta_N \Pi^{\delta_N}(0; \delta; S_{N} \mid S_{m} - S_{N}) \Pi^{\delta_N}(\delta; S_{N} - S_{m}) \Pi^{\delta_N}(\delta; S_{N} - S_{m}) \exp \left[ -\frac{\delta_{cp}^2}{2S_{N}} - \frac{\delta_{cp}^2 - \delta_{cp^2}}{2(S_{N} - S_{j})} \right]. \]

\[ P_{ABC}^{NM3} = \frac{1}{\pi} \frac{\alpha^3}{\partial \delta_c \partial \delta_N \partial S_{N}} \int_{-\infty}^{\delta_N} d\delta_n \int_{-\infty}^{\delta_N} d\delta_N \Pi^{\delta_N}(0; \delta; S_{N} \mid S_{m} - S_{N}) \Pi^{\delta_N}(\delta; S_{N} - S_{m}) \Pi^{\delta_N}(\delta; S_{N} - S_{m}) \exp \left[ -\frac{\delta_{cp}^2}{2S_{N}} - \frac{\delta_{cp}^2 - \delta_{cp^2}}{2(S_{N} - S_{j})} \right]. \]

\[ P_{ABC}^{NM4} = 0. \]

\[ P_{ABC}^{NM5} = \frac{1}{\pi \sqrt{2\pi}} \Pi^{\delta_N}(0; \delta; S_{m}) \delta_{cd} \delta_{cp} \int_{-\infty}^{\delta_N} d\delta_n \int_{-\infty}^{\delta_N} d\delta_N \Pi^{\delta_N}(0; \delta; S_{N} \mid S_{m} - S_{N}) \Pi^{\delta_N}(\delta; S_{N} - S_{m}) \exp \left[ -\frac{\delta_{cp}^2}{2S_{N}} - \frac{\delta_{cp}^2 - \delta_{cp^2}}{2(S_{N} - S_{j})} \right]. \]

\[ P_{ABC}^{NM6} = P_{ABC}^{NM7} = P_{ABC}^{NM8} = 0. \]

\[ P_{ABC}^{NM9} = \frac{1}{\pi} \frac{\alpha^2}{\partial \delta_c \partial \delta_N \partial S_{N}} \int_{-\infty}^{\delta_N} d\delta_n \int_{-\infty}^{\delta_N} d\delta_N \Pi^{\delta_N}(0; \delta; S_{N} \mid S_{m} - S_{N}) \Pi^{\delta_N}(\delta; S_{N} - S_{m}) \exp \left[ -\frac{\delta_{cp}^2}{2S_{N}} - \frac{\delta_{cp}^2 - \delta_{cp^2}}{2(S_{N} - S_{j})} \right]. \]

\[ P_{ABC}^{NM10} = \frac{1}{\pi} \frac{\alpha^2}{\partial \delta_c \partial \delta_N \partial S_{N}} \int_{-\infty}^{\delta_N} d\delta_n \int_{-\infty}^{\delta_N} d\delta_N \Pi^{\delta_N}(0; \delta; S_{N} \mid S_{m} - S_{N}) \Pi^{\delta_N}(\delta; S_{N} - S_{m}) \exp \left[ -\frac{\delta_{cp}^2}{2S_{N}} - \frac{\delta_{cp}^2 - \delta_{cp^2}}{2(S_{N} - S_{j})} \right]. \]

\[ P_{ABC}^{NM11} = \frac{1}{\pi \sqrt{2\pi}} \Pi^{\delta_N}(0; \delta; S_{m}) \delta_{cd} \delta_{cp} \int_{-\infty}^{\delta_N} d\delta_n \int_{-\infty}^{\delta_N} d\delta_N \Pi^{\delta_N}(0; \delta; S_{N} \mid S_{m} - S_{N}) \Pi^{\delta_N}(\delta; S_{N} - S_{m}) \exp \left[ -\frac{\delta_{cp}^2}{2S_{N}} - \frac{\delta_{cp}^2 - \delta_{cp^2}}{2(S_{N} - S_{j})} \right]. \]

\[ P_{ABC}^{NM12} = \frac{1}{\pi} \frac{\alpha^3}{\partial \delta_c \partial \delta_N \partial S_{N}} \int_{-\infty}^{\delta_N} d\delta_n \int_{-\infty}^{\delta_N} d\delta_N \Pi^{\delta_N}(0; \delta; S_{N} \mid S_{m} - S_{N}) \Pi^{\delta_N}(\delta; S_{N} - S_{m}) \Pi^{\delta_N}(\delta; S_{N} - S_{m}) \exp \left[ -\frac{\delta_{cp}^2}{2S_{N}} - \frac{\delta_{cp}^2 - \delta_{cp^2}}{2(S_{N} - S_{j})} \right]. \]
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For simplicity, the calculation of $P_{ABC}$ is done in the limit of $\delta_{cp} - \delta_{cd} \ll 1$. The lowest order term of the final result is proportional to $\delta_{cp} - \delta_{cd}$. This is because when $\delta_{cp} = \delta_{cd}$, the integrals in Equation (24) are independent of the descendent halo mass $S_d$.

We now replace $S_m$, $S_n$, and $S_N$ with the more physical notation for the environment, descendant, and progenitor: $S_m = S_e$, $S_n = S_d$, and $S_N = S_p$. Our final expression for $P(A, B, C)$ is

$$P(A, B, C) = P_{ABC}^M + P_{ABC}^{NM1} + \cdots + P_{ABC}^{NM13}$$

$$= \frac{1}{2\pi} \delta_{cp} - \delta_{cd} \left( (S_d - S_e)^{-3/2} (S_p - S_d)^{-3/2} E_1 \Pi \right)$$

$$+ \kappa(\delta_{cp} - \delta_{cd}) \left\{ -\frac{1}{2\sqrt{2\pi}} \delta_{cd} \delta_{e} S_d^{-3/2} S_p^{-3/2} (S_d - S_e)^{-1/2} \left[ 1 - \frac{\delta_{cd} - \delta_{e}}{S_e} + \frac{(\delta_{cd} - \delta_{e})^2}{S_d - S_e} \left( 1 - \frac{2S_d^2}{S_e^2} \right) \right] E_1 \Pi \right\}$$

$$+ \frac{1}{2\sqrt{2\pi}} \delta_{cd} \delta_{e} S_d^{-3/2} (S_p - S_e)^{-1/2} \left[ \frac{S_d}{S_p} G_1 + \frac{\delta_{cd} - \delta_{e}}{S_d} - \frac{(\delta_{cd} - \delta_{e})^2}{S_d - S_e} \right] E_1 \Pi$$

$$+ \frac{1}{2\sqrt{2\pi}} \delta_{cd} (\delta_{cd} - \delta_{e})^3 S_e^{-1} (S_d - S_e)^{-1/2} (S_p - S_d)^{-1/2} \left( S_d - \frac{1}{2} S_p \right) \Pi$$

$$\times \exp \left[ -\frac{(\delta_{cd} - \delta_{e})^2}{2S_e} \left( \frac{S_d}{S_d - S_e} + 2 \sqrt{\frac{S_d}{S_d - S_e}} \right) \right]$$

$$+ \frac{1}{\pi \sqrt{2\pi}} \delta_{cd} \delta_{e} S_d^{-3/2} (S_d - S_e)^{-1/2} (S_p - S_d)^{-1/2} E_1 (G_3 + F_2) \right\} ,$$

where

$$G_1 = -1 + \frac{1}{2} S_d^{-1} S_p^{-1/2} (S_d - S_p)^{3/2} \ln \frac{\sqrt{S_d} + \sqrt{S_p} - S_d}{\sqrt{S_d} - \sqrt{S_p} - S_d} + S_p S_d^{-2} (S_d + S_e)$$

$$+ S_d^{-2} S_p^{-1} \left( S_d - \frac{1}{2} S_p \right) \left[ 2S_d^2 + (S_d + S_e)(S_p - S_d) \right] - S_d^{-3/2} S_p^{-1} (S_d - S_e)^{1/2} (S_p - S_d) \left( S_d - \frac{1}{2} S_p \right)$$

(B6)

and

$$G_2 = 6S_d - 2S_e - 4S_p - (S_d - S_e)(S_p - S_d)^{-1} (S_p - S_d) + S_d^{-1}(2S_d - S_e)(S_p - S_d)^2(S_p - S_d)^{-1}.$$  (B7)

APPENDIX C

$P(A, B)$ AND $P(A, B, C)$ IN THE LIMIT OF LARGE ENVIRONMENTAL SCALE

In the limit of large environmental scale, i.e., small $S_e$, the overdensity smoothed over this scale, $\delta_e$, also becomes a small parameter because $\delta_e^2 \sim S_e$. We will therefore assume $S_e$ is of the same order as $\delta_e^2$, while keeping in mind that $\delta_e^2 / S_e$ is not necessarily small. The conditional probability $P(C|A, B)$ is equal to the ratio of $P(A, B, C)$ in Equation (B5) and $P(A, B)$ in Equation (A7), each of which contains special functions defined in Equation (A8). The key step in simplifying $P(C|A, B)$ is to find the behavior of these
special functions in the limit of small $S_e$. After some algebra, we obtain

\[
\begin{align*}
E_1 & \approx \exp \left[ -\frac{(\delta_{cd} - \delta_e)^2}{2S_d} \right], \quad \Pi \approx \frac{1}{\sqrt{2\pi}S_e} \exp \left[ -\frac{\delta_e^2}{2S_e} \right], \quad E_{\Pi} \approx \frac{\sqrt{2S_e}}{\sqrt{\pi}(\delta_{cd} - \delta_e)} \exp \left[ -\frac{(\delta_{cd} - \delta_e)^2}{2S_e} \right], \quad E_{\Pi_1} \approx -\frac{\sqrt{2S_e}}{\sqrt{\pi}(\delta_{cd} - \delta_e)} \exp \left[ -\frac{(\delta_{cd} - \delta_e)^2}{2S_e} \right], \\
E_{r_2} & \approx \frac{\sqrt{2S_e}}{\sqrt{\pi}(\delta_{cd} - \delta_e)} \exp \left[ -\frac{(\delta_{cd} - \delta_e)^2}{2S_e} \right], \quad E_{r_3} \approx \text{erfc} \left[ \frac{\delta_{cd} - \delta_e}{\sqrt{2S_d}} \right], \quad F_1 \approx \exp \left[ -\frac{(\delta_{cd} - \delta_e)^2}{2S_e} \right] \Gamma \left[ 0, \frac{(\delta_{cd} - \delta_e)^2}{2S_d} \right],
\end{align*}
\]

(C1)

Note that $E_2$ and $E_3$ in Equation (A8) are not included here, because their forms cannot and need not be further simplified. The new forms of $E_{r_1}$, $E_{r_2}$, and $E_{r_3}$ are based on the formula

\[
\lim_{a \to +\infty} \text{erfc}[a] \to \frac{1}{a \sqrt{\pi}} \exp[-a^2],
\]

(C2)

which can be derived from

\[
\begin{align*}
\lim_{a \to +\infty} \text{erfc}[a] & = \lim_{a \to +\infty} \frac{2}{\sqrt{\pi}} \int_a^\infty \exp(-x^2)dx \\
& = \lim_{a \to +\infty} \frac{2}{\sqrt{\pi}} \exp(-a^2) \int_a^\infty \exp(a^2 - x^2)dx \\
& = \lim_{a \to +\infty} \frac{1}{\sqrt{\pi}} \exp(-a^2) \int_0^\infty \frac{\exp(-t)dt}{\sqrt{t + a^2}} \quad \text{ (Let } t = x^2 - a^2) \\
& = \lim_{a \to +\infty} \frac{1}{a \sqrt{\pi}} \exp(-a^2) \int_0^\infty \exp(-t) \left[ 1 + O \left( \frac{t}{a^2} \right) \right] dt \\
& = \lim_{a \to +\infty} \frac{1}{a \sqrt{\pi}} \exp(-a^2)[1 + O(a^{-2})] \\
& \to \frac{1}{a \sqrt{\pi}} \exp[-a^2].
\end{align*}
\]

(C3)

The simplifications of $F_1$ and $F_2$ are similar. We need to use the relations

\[
\lim_{a, b \to +\infty} F(a, b) \to \frac{1}{2a(a + b)} \exp[-(a + b)^2], \quad \lim_{b \to +\infty} [F(a, b) - F(a + 2b, -b)] \to \exp[-b^2] \Gamma(0, 2b).
\]

(C4)

Equation (C4) can be worked out as follows:

\[
\begin{align*}
\lim_{a, b \to +\infty} F[a, b] & = \lim_{a, b \to +\infty} \int_a^\infty \frac{\exp(-x + b)^2}{x} dx \\
& = \lim_{a, b \to +\infty} \exp[-(a + b)^2] \int_a^\infty \frac{\exp[(a + b)^2 - (x + b)^2]}{x} dx \\
& = \lim_{a, b \to +\infty} \frac{1}{2} \exp[-(a + b)^2] \int_0^\infty \frac{\exp(-t)dt}{\sqrt{t + (a + b)^2}[\sqrt{t + (a + b)^2} - b]} \quad \text{ (Let } t = (x + b)^2 - (a + b)^2) \\
& = \lim_{a, b \to +\infty} \frac{1}{2a(a + b)} \exp[-(a + b)^2] \int_0^\infty \left[ 1 + O \left( \frac{t}{a(a + b)} \right) + O \left( \frac{t}{a(a + b)} \right) \right] \exp(-t)dt \\
& \to \frac{1}{2a(a + b)} \exp[-(a + b)^2].
\end{align*}
\]

(C5)
Finally, using the results of Equation (C7) to Equations (A7) and (B5) for \( P(A, B) \) and \( P(A, B, C) \), respectively. Keeping terms up to first order in \( \delta_e \) and \( \kappa \) as well as terms proportional to \( \delta_e \kappa \), we obtain

\[
P(A, B) \approx \frac{\delta_{cd}}{2\pi S_d \sqrt{S_p S_c}} \exp \left( -\frac{\delta_e^2}{2S_c} - \frac{\nu^2}{2} \right) \left\{ 1 - \kappa + \frac{\kappa}{2} \exp \left( \frac{\nu^2}{2} \right) \Gamma \left( 0, \frac{\nu^2}{2} \right) \right\} + \frac{\delta_e}{\delta_{cd}} \left[ \frac{\nu^2 - 1}{2} + \kappa - \frac{\kappa}{2} \exp \left( \frac{\nu^2}{2} \right) \Gamma \left( 0, \frac{\nu^2}{2} \right) \right],
\]

(C7)

\[
P(A, B, C) \approx \frac{(\delta_{cp} - \delta_{cd})b_{cd}}{[2\pi S_d (S_p - S_d)]^{1/2} \sqrt{S_c}} \exp \left( -\frac{\delta_e^2}{2S_c} - \frac{\nu^2}{2} \right) \times \left\{ 1 - \kappa + \beta \alpha \kappa - (1 - \alpha)^{3/2} \kappa \left[ \sqrt{2\pi \nu + \pi} \exp \left( \frac{\nu^2}{2} \right) \text{erfc} \left( \frac{\nu}{\sqrt{2}} \right) \right] + \frac{\kappa}{2} \exp \left( \frac{\nu^2}{2} \right) \Gamma \left( 0, \frac{\nu^2}{2} \right) + \frac{\delta_e}{\delta_{cd}} \left( \kappa + (1 + \beta \kappa \alpha)(\nu^2 - 1) + \sqrt{2\pi \nu \alpha} \nu (1 - \alpha)^{3/2} (1 - \nu^2) \right) + \pi \kappa (1 - \alpha)^{3/2} \exp \left( \frac{\nu^2}{2} \right) \text{erfc} \left( \frac{\nu}{\sqrt{2}} - \kappa \right) - \frac{\kappa}{2} \exp \left( \frac{\nu^2}{2} \right) \Gamma \left( 0, \frac{\nu^2}{2} \right) \right\},
\]

(C8)
in which

\[
\nu \equiv \frac{\delta_{cd}}{\sqrt{S_d}}, \quad \alpha \equiv \frac{S_d}{S_p}, \quad \beta \equiv -2 + \frac{(1 - \alpha)^{3/2}}{2\alpha} \ln \left( \frac{1 + \sqrt{1 - \alpha}}{1 - \sqrt{1 - \alpha}} \right) + \frac{1}{\alpha} + 2\alpha.
\]

(C9)

Finally, using the results of Equations (C7) and (C8), we reach the simplified expressions for \( P(B|A) \) and \( P(C|A, B) \) in Equations (28) and (30) of Section 3.5.

REFERENCES

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