Dimension Witnesses and Quantum State Discrimination

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Abstract

Dimension witnesses allow one to test the dimension of an unknown physical system in a device-independent manner, that is, without placing assumptions about the functioning of the devices used in the experiment. Here we present simple and general dimension witnesses for quantum systems of arbitrary Hilbert space dimension. Our approach is deeply connected to the problem of quantum state discrimination, hence establishing a strong link between these two research topics. Finally, our dimension witnesses can distinguish between classical and quantum systems of the same dimension, making them potentially useful for quantum information processing.

Reference


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Dimension Witnesses and Quantum State Discrimination
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Dimension witnesses allow one to test the dimension of an unknown physical system in a device-independent manner, that is, without placing assumptions about the functioning of the devices used in the experiment. Here we present simple and general dimension witnesses for quantum systems of arbitrary Hilbert space dimension. Our approach is deeply connected to the problem of quantum state discrimination, hence establishing a strong link between these two research topics. Finally, our dimension witnesses can distinguish between classical and quantum systems of the same dimension, making them potentially useful for quantum information processing.

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Recently, the problem of testing the dimension of an unknown physical system has attracted quite some attention. Here dimension represents, loosely speaking, the number of degrees of freedom of the system. For quantum systems this corresponds to the Hilbert space dimension. The main point of this line of research is to assess the dimension of an unknown physical system in a device-independent manner, that is, from measurement data alone, without any a priori assumption about the devices used in the experiment.

This is in contrast with the more usual approach in physics, in which, when constructing a theoretical model aiming at explaining some experimental data, the dimension of the system is a parameter that is defined a priori. For instance, when describing quantum systems, one generally starts by fixing the Hilbert space dimension, given reasonable assumptions about the nature of the system and its dynamics. Then the model may or may not reproduce the experimental data. If the model fits the data, one can make a statement about the system’s dimension. If the model does however not work, nothing can be said since, in principle, there could be a different model using the same dimension that could explain the data. Obviously, testing all possible models with a fixed dimension is impossible; hence, a better approach is required in order to determine the minimal dimension of the system compatible with some data.

The concept of a dimension witness was recently introduced to address this problem. First discussed in the context of Bell inequalities [1–5], dimension witnesses were also derived in the case of quantum random access codes [6] and the time evolution of quantum observables [7].

More recently a framework was developed in order to derive dimension witnesses for systems of arbitrary Hilbert space dimension is challenging, and no general solution has been provided yet. The approach of Ref. [8] can be used to derive quantum dimension witnesses, but the validity of these can generally be tested only in one out of $N$ possible states $\rho_x$. A second device then performs one out of $m$ possible measurements, giving an outcome $b = 1, \ldots, k$. The experiment is described by a set of conditional probabilities $P(b|x, y)$, which represent the probability of observing outcome $b$ when state $x$ was prepared and measurement $y$ was performed. In each round of the experiment, which state $x$ is prepared and which measurement $y$ is performed is chosen by the observer. However, the important point is that what the states and measurements actually are is unknown to the observer. The observer’s task will then be to estimate the minimal dimension $d$ of the system that is compatible with the experiment. More precisely what is the minimal dimension necessary to reproduce a given set of conditional probabilities $P(b|x, y)$?

In the case of classical systems, the above problem can be tackled using the framework of Ref. [8], which allows one to derive dimension witnesses for classical systems, based on geometrical ideas. For quantum systems however, finding dimension witnesses for systems of arbitrary Hilbert space dimension is challenging, and no general solution has been provided yet. The approach of Ref. [8] can be used to derive quantum dimension witnesses, but the validity of these can generally be tested only

FIG. 1. Testing the dimension of an unknown system in a prepare-and-measure scenario.
numerically. Hence this approach can be used only for systems of relatively low dimension. The approach of Ref. [6] does not rely on numerics, but gives witnesses for which strong and/or tight bounds can be derived only in particular cases. Finally, the approach of Ref. [7] does not apply to the static case.

The problem of testing the dimension of quantum systems in a prepare-and-measure scenario is well motivated, and not only from a purely conceptual point of view. Indeed, in quantum information, the dimensionality of quantum systems represents a resource for information processing. In general, systems of larger dimension offer more power for computation and communication. In particular, they are known to simplify quantum logic [9], enable for the optimal implementation of certain quantum protocols [10], and allow for lower detection efficiencies in Bell tests [11,12]. Moreover, the dimension of quantum systems plays a crucial role in the security of standard quantum cryptography [13]. Finally, dimension witnesses are relevant in practice, as demonstrated by two recent results, it is instructive to understand why the distinguishability of states is important here. Consider first the case

$$d \geq N.$$ Here, the dimension $d$ of the system, the mediating particle, is in fact large enough to encode perfectly the choice of preparation $x$. Hence, it is then possible that the state preparator simply sends $x$ to the measuring device. The latter then has all information about both $x$ and $y$, and can thus simulate any statistics $P(b|x,y)$. Thus, if by measuring the mediating particle it is possible to perfectly identify which preparation $x$ was chosen, no relevant device-independent statement can be made about the dimension of the system.

On the other hand, when $d < N$, the choice of preparation $x$ cannot be encoded perfectly in the system anymore, since the latter cannot be prepared in sufficiently many perfectly distinguishable states. Therefore, not all statistics $P(b|x,y)$ can be reproduced when $d < N$, and relevant device-independent statements can be made.

Thus, a central aspect of the problem is how well one can distinguish between a set of $N$ quantum states. Intuitively, a good strategy consists in choosing the states to be as far from each other as possible in the Hilbert space, in order to make them as distinguishable as possible. However, the dimension of the Hilbert space in which the set of states is embedded will clearly put restrictions on their distinguishability. It is exactly this aspect that we will exploit to derive our dimension witnesses.

The distinguishability of quantum states is captured by the notion of trace distance. Given two quantum states $\rho_x$ and $\rho_y$, how well they can be distinguished from each other, allowing for the most general measurement, is quantified by the trace distance

$$D(\rho_x, \rho_y) = \frac{1}{2} \| \rho_x - \rho_y \|_1.$$ (3)

Operationally, the trace distance represents the maximal distance in probabilities that a positive operator valued measure (POVM) element $M$ may occur depending if the state is $\rho_x$ or $\rho_y$, that is

$$D(\rho_x, \rho_y) = \max_M \text{tr}( (\rho_x - \rho_y) M ).$$ (4)

Also, the trace distance can be related to another notion of distinguishability, the fidelity $F$ between $\rho_x$ and $\rho_y$,

$$1 - F(\rho_x, \rho_y) \leq D(\rho_x, \rho_y) \leq \sqrt{1 - F^2(\rho_x, \rho_y)}.$$ (5)

Indeed if both states are pure, $F(\rho_x, \rho_y) = |\langle \Psi_x | \Psi_y \rangle|$, and the second inequality in Eq. (5) is saturated.

Consider the scenario of Fig. 1. Take the simple case in which there are $N$ possible preparations, and a single measurement (i.e., $m = 1$) with $N$ outcomes. In this case, we can build a dimension witness based on the average guessing probability, i.e., the function
\[ U_N = \frac{1}{N} \sum_{x=1}^{N} P(b = x| x) . \]  

(6)

In order to show that \( U_N \) works as a quantum dimension witness for any \( d < N \), we must find an upper bound on \( U_N \) depending on \( d \). This can be done as follows:

\[ U_N = \frac{1}{N} \sum_{x=1}^{N} \mathrm{tr}(\rho_x M_x) \leq \frac{1}{N} \sum_{x} \mathrm{tr}(M_x) = \frac{d}{N} = Q_d. \]  

(7)

hence leading to a dimension witness for any \( d < N \). Note however, that for this witness we have that \( C_d = Q_d \) (for any \( d \leq N \)); hence, the witness cannot distinguish between classical and quantum states of the same dimension.

After this warm up, let us now see how to construct dimension witnesses for systems of arbitrary dimension \( d \) such that \( C_d < Q_d \). Consider again \( N \) possible preparations, but now \( m = N(N-1)/2 \) dichotomic measurements (with outcomes ±1), labeled as \( y = (x, x') \), with \( x, x' \in \{1, \ldots, N\}, x \neq x' \). Consider the following expression:

\[ W_N = \sum_{x \neq x'} |P(x, (x')) - P(x', (x'))|^2, \]  

(8)

where we used the simplified notation \( P(x, (x')) = P(b = 1|x, (x')) \).

We will now see how to upper bound \( W_N \) depending on \( d \). For each measurement \( y = (x, x') \), call \( M_{(x,x')i} \) the POVM element corresponding to outcome \( b = 1 \). From the structure of \( W_N \), it is clear that each \( M_{(x,x')i} \) can be taken to be the projector onto the subspace generated by the positive eigenvectors of \( \rho_x - \rho_{x'} \). Also, since \( W_N \) is a convex functional, it follows that, in order to compute its maximum value, we can assume each of the states \( \{\rho_x\} \) to be pure, i.e., \( \rho_x = |\Psi_x\rangle \langle \Psi_x| \). We thus have that

\[ W_N = \sum_{x \neq x'} |\mathrm{tr}(\rho_x - \rho_{x'}) M_{(x,x')}|^2 \leq \sum_{x \neq x'} |D(\rho_x, \rho_{x'})|^2 \leq \sum_{x \neq x'} (1 - |\langle \Psi_x| \Psi_{x'}\rangle|^2), \]  

(9)

where we have used Eqs. (4) and (5). Next we use the fact that

\[ \sum_{x \neq x'} |\langle \Psi_x| \Psi_{x'}\rangle|^2 = \frac{1}{2} \left[ \sum_{x \neq x'} |\langle \Psi_x| \Psi_{x'}\rangle|^2 - N \right] \]  

(10)

\[ = \frac{N^2}{2} \mathrm{tr}(\Omega^2) - \frac{N}{2}, \]  

(11)

with \( \Omega = \frac{1}{N} \sum_{x} |\Psi_x\rangle \langle \Psi_x| \) being a normalized quantum state. Since the purity of any \( d \)-dimensional normalized state \( \Omega \) is lower bounded by

\[ \mathrm{tr}(\Omega^2) \geq \frac{1}{d}, \]  

(12)

we obtain that

\[ W_N = \sum_{x \neq x'} |P(x, (x')) - P(x', (x'))|^2 \leq Q_d \]  

(13)

Thus \( W_N \leq Q_d \) is a quadratic quantum dimension witness for any \( d < N \).

An interesting feature of the above witness is its tightness. That is, for any dimension \( d \), there exists an ensemble of states \( \{\rho_x\}_{x=1}^{N} \subset B(C^d) \) and measurement operators \( \{M_{(x,x')i}\} \subset B(C^d) \) which saturate the inequality of Eq. (13).

Suppose that, for any \( d \leq N \), there exists a set of pure states \( \{|\Psi_x\rangle\}_{x=1}^{N} \subset B(C^d) \) such that \( \Omega = \frac{1}{N} \sum_{x=1}^{N} |\Psi_x\rangle \times \langle \Psi_x| = \frac{1}{d} \mathbb{I}_d \). Then \( \mathrm{tr}(\Omega^2) = \frac{1}{d} \). Thus, by choosing \( M_{(x,x')i} \) to be the measurement that optimally discriminates between \(|\Psi_x\rangle\) and \(|\Psi_{x'}\rangle\), all inequalities incurred into the derivation of the bound will turn into equalities, therefore proving the attainability of the bound. Hence it suffices to prove that such vectors exist. It can be verified that the normalized vectors

\[ |\Psi_x\rangle = \frac{1}{\sqrt{d}} \sum_{k=0}^{d-1} e^{2\pi ikx/N} |k\rangle \]  

(14)

do the trick.

Next, we can derive a tight upper bound for classical systems of arbitrary dimension for our witness Eq. (13). By convexity, it follows that the quantity \( W_N \) is maximized using a deterministic strategy [8]. It then easily follows that

\[ W_N \leq C_d = \frac{N(N-1)}{2} - \left[ \frac{N}{d} \left( N - d \left( \frac{N}{d} \right) + 1 \right) \right], \]  

(15)

where \( [x] \) is the integer part of \( x \). Unless \( N \) is a multiple of \( d \), we have that \( C_d < Q_d \); hence, our witness can distinguish between classical and quantum systems of the same dimension. In particular, when \( N \) is prime we have that \( C_d < Q_d \) for any \( d < N \). This is illustrated in Table I, for the case \( N = 7 \). Notably, dimension witnesses featuring such a quantum advantage are potentially useful, for instance for the use of dimension witnesses in information-theoretic tasks [16,17], and might be relevant to discuss problems in the foundations of quantum mechanics [22].

It turns out that our quadratic dimension witness can also be linearized, which might prove to be useful in certain situations. Here our main ingredient is the Cauchy-Schwartz (CS) inequality: for any real vector \( \vec{v} \), we have that

\[
\begin{array}{|c|c|c|c|c|c|c|c|}
\hline
\text{Table I. Tight bounds for } d \text{-dimensional classical } (C_d) \text{ and quantum } (Q_d) \text{ systems for the dimension witness } W_N. \text{ Notably, the witness can distinguish between classical and quantum systems of the same dimension for any } d < N, \text{ since } C_d < Q_d. \\
\hline
\text{ } & d & 2 & 3 & 4 & 5 & 6 & 7 \\
\hline
C_d & 12 & 16 & 18 & 19 & 20 & 21 & \\
\hline
\end{array}
\]
Applying the CS inequality to our problem, i.e., to numerically the tightness of the witness (see Table II), using the techniques of Ref. [23]. The case as well. For small values of \( x \) and all \( d \), we investigated numerically the tightness of the witness (see Table II), using the techniques of Ref. [23]. The case \( N = d + 1 \). Consider quantum states as given by Eq. (14). Then, one can check that |\( \langle \Psi_x | \Psi_x' \rangle \)| = \( \frac{1}{2} \), for all \( x \neq x' \). This means that, for optimal measurements, \( P(x, (x, x')) - P(x', (x, x')) = \sqrt{1 - \frac{1}{4d^2}} \) for all \( x, x' \), and so the CS inequality Eq. (16) is saturated. Thus we finally get the tight linear dimension witness,

\[
V_N \equiv \sum_{x' > x} P(x, (x, x')) - P(x', (x, x')) \leq \frac{N(N - 1)}{2} \sqrt{1 - \frac{1}{\min(d, N)}} = Q_d.
\]

Note that the above inequality cannot always be saturated; hence, the witness is not tight in general. However, one notable exception is the case \( N = d + 1 \). Consider quantum states as given by Eq. (14). Then, one can check that |\( \langle \Psi_x | \Psi_x' \rangle \)| = \( \frac{1}{2} \), for all \( x \neq x' \). This means that, for optimal measurements, \( P(x, (x, x')) - P(x', (x, x')) = \sqrt{1 - \frac{1}{4d^2}} \) for all \( x, x' \), and so the CS inequality Eq. (16) is saturated. Thus we finally get the tight linear dimension witness,

\[
V_N \equiv Q_d = \frac{(d + 1)\sqrt{d^2} - 1}{2}.
\]