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Counting Statistics and Detector Properties of Quantum Point Contacts

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The dynamical process of quantum measurement is a crucial element of quantum-mechanical evolution [1,2]. This fact should make it surprising that considerations related to quantum measurements do not typically play a role in the theory of mesoscopic conductors despite the quantum nature of electron transport in them. The main reason for this is that the dynamics of measurement does not affect single-time averages of individual operators that are typically studied in mesoscopic transport, and manifests itself explicitly only in the higher-order correlators, when the state obtained as a result of a measurement evolves and undergoes subsequent measurements. One example of this in mesoscopic transport is electron “counting statistics” which determines full statistical properties of electron transfer through a conductor and requires an explicit discussion of the measurement procedure [3]. From the point of view of quantum measurement, the common assumption underlying different ways [2,3] of obtaining this statistics is the statement that electron states in different electrodes of the structure after scattering represent classical, noninterfering events. The purpose of this work is to use this understanding to analyze one of the most basic mesoscopic detectors, the quantum point contact (QPC) beyond the linear-response regime and show that its properties are determined by the counting statistics.

The QPC as the charge detector was suggested first in [4] and studied experimentally in [5]. Currently, the QPCs are the main detectors used for measurements of the quantum-dot qubits [6–9]. Theoretical description of the QPC detector has been worked out so far only in situations when the counting statistics does not play an essential role: in the detector has been worked out so far only in situations when dot qubits [6–9]. Theoretical description of the QPC detector has been worked out so far only in situations when dot qubits [6–9].

The operating principle of the QPC detector is that the state of the measured system is reflected in the electron current \(I\) between the QPC reservoirs [Fig. 1(a)]. For the QPC to act as a detector, the spectral components of this current at relevant frequencies should be effectively classical; i.e., both the “attempt frequency” \(eV/h\), where \(V\) is the QPC bias voltage, and the inverse scattering time \(\tau_0^{-1}\) should be much larger than the typical frequencies of the measured system. In this case, we can neglect the system dynamics and assume that its Hamiltonian is vanishing. It is convenient then to discuss the measurement in the basis of the eigenstates \(|j\rangle\) of the system operator which couples it to the QPC, with each state producing its own scattering matrix \(S_j\) for the QPC electrons.

Measurement-induced dephasing.—The basic physics of the measurement process is a trade-off between acquisition of information about the state of the measured system and backaction dephasing of this system (see, e.g., [11,15–17]). For the quantum-limited detectors, the rates of the two processes coincide; for less efficient detectors, the dephasing is more rapid than information gain. We begin by calculating the backaction dephasing rate for the QPC detector. Evolution in the scattering process of the total wave function \(|\psi_0\rangle\) composed of the wave function \(|\psi_j\rangle\) of an individual electron incident from one, say left, electrode, and the wave function of the measured system, \(\sum_j c_j |j\rangle\), can be represented as

\[ |\psi(t)\rangle = \int d\epsilon \, \sum_j c_j |\epsilon, j\rangle e^{-i \epsilon t} \]

\[ = \int d\epsilon \, |\psi_\epsilon\rangle \langle \psi_\epsilon | \sum_j c_j |j\rangle \]

\[ = \sum_j c_j |\psi_j\rangle \]

FIG. 1. (a) Real- and (b) energy-space diagrams of the quantum point contact (QPC) detector. The scattering potential \(U_j(x)\) for electrons in the QPC, and the current \(I\) driven by the voltage \(V\) are controlled by the state \(|j\rangle\) of the measured system.
\(|\text{in}\rangle \otimes \sum_j c_j |j\rangle \rightarrow \sum_j c_j (r_j |L\rangle + t_j |R\rangle) \otimes |j\rangle, \)  \hspace{1cm} (1)

where \(|L\rangle\) and \(|R\rangle\) are the outgoing electron states in the left and right electrodes, respectively. The transmission and reflection amplitudes, \(t_j\) and \(r_j\), are the elements of the scattering matrix \(S_j\):

\[
S_j = \left( \begin{array}{cc} r_j & t_j \\ t_j^* & r_j^* \end{array} \right) \hspace{1cm} (2)
\]

that depends on the state \(|j\rangle\) of the measured system. The resulting evolution of the density matrix \(\rho\) of the system, \(\rho = \text{Tr}_{\text{el}} \rho_0(\psi_0(\psi_0)\), is obtained by tracing over the electron states in Eq. (1):

\[
\rho_{jk} = c_j c_k^* \rightarrow c_j c_k^* (t_j t_k^* + r_j r_k^*), \hspace{1cm} (3)
\]

Equation (3) shows that the absolute value of the off-diagonal elements \(\rho_{jk}\), \(j \neq k\), of \(\rho\) are suppressed: \(|t_j t_k^* + r_j r_k^*| \leq 1\), as a result of averaging over the two possible outcomes of scattering.

We note that the process (3) requires that the state with energy \(e\) in the left electrode is occupied with probability \(f_1 = f_K(e)\) (where \(f_K\) is the Fermi function), while the corresponding state in the right electrode is empty with the probability \(1 - f_2\), where \(f_2 = f_P(e + eV)\) [see Fig. 1(b)]. If both states are empty, there is no effect on the system density matrix. If both are occupied, they renormalize the energies of the measured system and therefore shift the phase of \(\rho_{jk}\). Multiplying all types of evolution for a number \(n\) of different scattering events at energy intervals \(de\) during the time interval \(t\), we find that the density matrix evolves as \(|\rho_{jk}(t)\rangle = |\rho_{jk}(0)\rangle e^{-\Gamma_{jk}t}\), where \(\Gamma_{jk} = -\int_{2\pi\hbar} \ln[f_1 f_2 e^{i\alpha} + f_1 (1 - f_2)(t_j t_k^* + r_j r_k^*) + f_2 (1 - f_1)(1 - f_2)] \)  \hspace{1cm} (4)

is the dephasing rate.

The above heuristic derivation is presented to illustrate the measurement by QPC as the process of repeated interaction of the system with individual electrons and collection of statistics over the large number of scattering events, the concept that will be utilized below. Formally, Eq. (4) can be obtained by starting with the expression for the time evolution of the density matrix \(\rho\) due to interaction with the QPC: \(\rho_{jk}(t) = \rho_{jk}(0)(e^{H_i t} e^{-\hat{H}_i t})\), where \(H_i\) is the Hamiltonian of the QPC systems which includes the scattering potential \(U_j\) controlled by the measured system, and the average \((\cdots)\) is taken over the stationary state of the QPC. Following the same steps as in the calculations of the counting statistics [3] one gets in the limit \(t \rightarrow \hbar/eV\)

\[
\frac{\rho_{jk}(t)}{\rho_{jk}(0)} = \exp \left[ \int \frac{d\epsilon}{2\pi\hbar} \text{Tr} \ln[1 - f + S_k^d S_j f] \right]. \hspace{1cm} (5)
\]

where \(f\) is the matrix of occupation factors of the incident electron states: \(f = \text{diag} (f_1, f_2)\). Evaluation of the trace in Eq. (5) leads to Eq. (4) with \(e^{\hat{H}_i t} = \text{det}(S_k^d S_j f)\).

For a two-state system, there is only one rate \(\Gamma_{12} = \Gamma\), and Eq. (4) generalizes to arbitrary transmission properties of the QPC previous results for the “back-action dephasing” rate of the QPC detector obtained for energy-independent scattering amplitudes \(t_j\), \(r_j\). Indeed, in this case, and for vanishing temperature \(T = 0\), Eq. (4) gives

\[
\Gamma = -(eV/2\pi\hbar) \ln[1 + r_j t_k^*], \hspace{1cm} (6)
\]

the expression that reduces to the dephasing rate [10] \(\Gamma = (eV/4\pi\hbar) (\sqrt{D_1} - \sqrt{D_2})^2\) in the tunnel limit, \(D_j \equiv |t_j|^2 \ll 1\), or to the rate \(\Gamma = (eV/2\pi\hbar)(\Delta D)^2/[8D(1 - D)]\) in case of weak coupling, \(\Delta D \equiv D_1 - D_2 \ll (D_1 + D_2)/2 = D\) [12,13], if the coupling modulates only the absolute value of the transmission amplitude.

Detector counting statistics.—The backaction dephasing rates (4) characterize only one part of the dynamics of measurement, the other side being the information acquisition by the QPC detector. The information about the state \(|j\rangle\) of the measured system is contained in the distribution of probabilities \(P_j(n)\) for \(n\) electrons to be transmitted through the QPC during the time interval \(t\) if the system is in the state \(|j\rangle\). With increasing \(t\), the distributions \(P_j(n)\) (and, accordingly, the states \(|j\rangle\)) can be distinguished with more and more certainty since the differences in their average positions and the widths grow, respectively, as \(t^1/2\).

Quantitatively, the rates of information acquisition depend on the information measure used to characterize distinguishability of the distributions \(P_j(n)\). In the context of quantum measurement, it is more appropriate [19,20] to use for this purpose the measure given by the statistical overlap \(M_{jk}(t) = \sum_n [P_j(n) P_k(n)]^{1/2}\), which is related to one of the Rényi entropies [21] and not to the better-known Shannon entropy. At large time, \(eVt/\hbar \gg 1\), when many electrons interact with the system, transport becomes a Markovian stochastic process. The generating functions of the cumulants of \(n\) are then proportional to time: \(\ln[\sum_n P_j(n) \exp(\lambda n)] = i\hat{H}_j(\lambda)\). As a result, the overlap \(M_{jk}\) exponentially decays in time,

\[
M_{jk}(t) \equiv \sum_n [P_j(n) P_k(n)]^{1/2} = \exp[-W_{jk}t], \hspace{1cm} (7)
\]

and one can introduce the notion of the measurement rate \(W_{jk}\). A simple evaluation that involves the stationary point approximation expresses then the measurement rate in terms of the generating functions \(\hat{H}_j(\lambda)\):

\[
W_{jk} = -(1/2) \min_\lambda [\hat{H}_j(\lambda) + \hat{H}_k(-\lambda)], \hspace{1cm} (8)
\]

which is so far a general result.

For a QPC, the counting statistics results [3] can be summarized as follows. With probabilities \(p_1 = \ldots \).
$f_1(1 - f_2)D_j(e)$ and $p_{-1} = f_2(1 - f_1)D_j(e)$ electrons are transmitted, respectively, forward and backward through the QPC changing $n$ by ±1. With the probability $p_0 = 1 - p_1 - p_{-1}$ the charge $n$ is not changed (when the states in both electrodes are simultaneously empty or occupied, or the incident electrons are reflected back into the electrodes). Using the expansion formula for the probabilities of a polynomial process, $(p_0 + p_1 + p_{-1})^N = \sum_{m_+} P(m_+, m_-) = 1$, and inverting it to evaluate the moment generator $\sum_{m} P(m_+, m_-) \exp[\lambda(m_+ - m_-)] = \{p_0 + p_1 e^{\lambda} + p_{-1} e^{-\lambda}\}^N$ for the number of transmitted electrons $n = m_+ - m_-$ per energy interval $de = 2\pi h/\lambda$.) we obtain the generating functions

$$\mathcal{H}_j(\lambda) = \int \frac{de}{2\pi h} \ln[1 + f_1(1 - f_2)D_j(e^{\lambda} - 1) + f_2(1 - f_1)D_j(e^{-\lambda} - 1)]. \quad (9)$$

As shown below, Eq. (9) means that $W_{jk} = \Gamma_{jk}$.

Quantum-limited detection.—Equations (4) and (7)–(9) fully determine the detector properties of the QPC. In particular, they set the conditions for its “ideal” or quantum-limited operation characterized by the fact that the quantum coherence between the states $|j\rangle$ of the measured system is suppressed only by the acquisition of information about them, so that the measurement and dephasing rates coincide (see, e.g., Ref. [22]):

$$W_{jk} = \Gamma_{jk}. \quad (10)$$

A first, obvious, requirement for this is sufficiently low temperature $T \ll eV$, when electrons pass through the QPC in only one direction, say from the left to the right electrode. The next requirement follows from Eq. (4), which shows that the dephasing rate $\Gamma$ can increase without any changes in the probabilities $P_j(n)$, only if the phases of the scattering amplitudes deviate from the relation

$$\phi_j = \phi_k, \quad \phi_j = \arg(t_j/r_j), \quad (11)$$

under which they do not contain any information about the measured system. Similar to the case of weak coupling [15–17], an important example when Eq. (11) is true is given by the symmetric potentials: $U_j(-x) = U_j(x)$. The unitarity of $S_j$ implies then that $\phi_j = \pi/2$.

Finally, the last condition of the quantum-limited operation restricts the energy dependence of the QPC transparency $D_j(e)$. This condition is irrelevant when the bias is small, $eV \ll \tau_0^{-1}$, and $D_j$’s are effectively constant. In this case, the distributions $P_j(n)$ have the usual binomial form,

$$P(n) = C_N^n D^n e^{R(n-n)}, \quad N = eVt/2\pi h \quad \text{and} \quad R = 1 - D.$$ 

Taking the sum over $n$ explicitly in Eq. (7) one sees then that if the other two ideality conditions are also satisfied, the QPC is the quantum-limited detector with

$$W_{jk} = \Gamma_{jk} = \frac{eV}{2\pi h} \ln((D_j D_k)^{1/2} + (R_j R_k)^{1/2}). \quad (12)$$

In what follows we relax the last two requirements and show how the information contained in the scattering phases can be extracted using tunable detectors, and how the energy dependence of the transmission can be optimized to reach the quantum-limited detection at large $V$.

Tunable quantum detector.—If the condition (11) is not satisfied, the relative phases $\phi_j$ contain information about the quantum state that is lost in the measurement process. However, the fact that before the final state projection in the course of measurement the quantum coherence can be preserved and utilized leads to the following idea of a tunable detector: The information read via the interaction with the measured system, $|in\rangle \rightarrow S_j|\psi_j\rangle$, can be manipulated coherently, e.g., transported to a different location, before detection: $|\psi_j\rangle \rightarrow |\varphi_j\rangle = S_m|\psi_j\rangle$. These operations conserve the dephasing rate, $\langle \varphi_j|\varphi_j\rangle = \langle \psi_j|\psi_j\rangle$, but change the measurement rate, and can be used to tune the detector to the optimal point (10).

Here we consider the QPC detector included in the electronic Mach-Zehnder (MZ) interferometer [23] (Fig. 2) as an example of the setup that provides such a new mode of operation. In contrast to the case of a single QPC, where the transmitted and reflected components of the electron wave function represent different classical outcomes, in the MZ interferometer, they can be mixed coherently at the next QPC after accumulating additional phase difference $\chi$ [24]. The phase information is converted in the second QPC into the transmission probabilities, and by adjusting $\chi$ and transmission $D_m$ of the second QPC, one should be able to achieve the quantum-limited detection even when Eq. (11) is not true.

Quantitatively, the transformation of the amplitudes $t_j$, $r_j$ in the output amplitudes $t'_j$, $r'_j$ (Fig. 2) is unitary:

$$S_m = \left( \begin{array}{cc} i(R_m)^{1/2} & (D_m)^{1/2} e^{-i\chi} \\ (D_m)^{1/2} e^{i\chi} & i(R_m)^{1/2} \end{array} \right). \quad (13)$$

For illustration purposes we assume the simplest case of the measured system having only two states, and of sufficiently small bias, $eV \tau_0 \ll 1$, when transmission $D_m$ is energy independent. If the measured system changes only the phases $\phi_j$ (11) of the scattering amplitudes, i.e., $D_1 = D_2 = D$, the single-QPC measurement rate vanishes.

FIG. 2. The QPC detector included in the Mach-Zehnder interferometer: two chiral electronic modes mixing at the two QPCs (indicated by dashed lines) and accumulating the relative phase $\chi$ between them. The scattering amplitudes $t_j$, $r_j$ of one incident mode in the first QPC are controlled by the measured system. The interferometer enables one to use the phase information in these amplitudes for detection.
contrast, the MZ interferometer converts the phase into the transmission probabilities and the measurement rate can be tuned to the maximum value (10) even in this case: $W = \Gamma = -(eV/4\pi\hbar)\ln[1 - 4D\sin^2(\phi/2)]$, where $\phi = \phi_1 - \phi_2$. This happens at the optimal point
\[ \chi = (\phi_1 + \phi_2)/2, \quad D_m = R_m = 1/2. \quad (14) \]
where $\chi$ is fixed by Eq. (14) only if $D \neq 1/2$ and only up to the shift $\chi \rightarrow \chi + \pi$. For $D = 1/2$, $\chi$ is arbitrary.

In general, if $D_1 \neq D_2$, it is also possible to find the optimal point, which turns the interferometer in the optimal detector. For instance, in the linear-response regime, when $\Delta D = (D_1 - D_2)/2$ and $\phi$ are small, one can see from Eqs. (13) and (12) that this happens when the phase $\chi$ is given by Eq. (14) and
\[ D_m = 1/2 - (\Delta D/2)[(\Delta D)^2 + \phi^2(DR)^2]^{-1/2}. \quad (15) \]
This equation describes the transition from a maximally mixing second QPC for $\phi \gg \Delta D$, as in Eq. (14), when the measurement information is contained mostly in the phase, to the situation with no mixing for $\phi \ll \Delta D$, from which the very beginning the information is already in the transmission probabilities.

**Optimization in energy space.**—If the bias voltage is not small, $eV\tau_0 \approx 1$, the condition on $D_j(\epsilon)$ for the quantum-limited operation is found as follows. Equations (8) and (9) imply that the measurement rate $W$ is obtained by minimizing the function $\int d\epsilon \ln[(R_j + D_j e^{\epsilon})(R_k + D_k e^{-\epsilon})]$ with respect to the variable $\lambda$. The dephasing rate (4) can also be written as a minimum of the same function but taken locally for every energy $\epsilon$. Therefore $W_{jk} \geq \Gamma_{jk}$ with equality reached when the ratio $D_j(\epsilon)/R_j(\epsilon)$ has the same energy dependence in all states $|j\rangle$, differing only by the overall scale factor $C_j$:
\[ D_j(\epsilon)/R_j(\epsilon) = C_j D(\epsilon)/R(\epsilon). \quad (16) \]
This equation determines when the measurement is the quantum-limited detector (10) even for large bias $V$. It generalizes to arbitrary coupling strength the results of the linear-response theory [16,17] obtained when the measured system simply shifts the energy of the QPC electrons. Indeed, in this case Eq. (16) reduces to $D(\epsilon) \propto D(\epsilon)R(\epsilon)$ in agreement with Refs. [16,17].

In the tunnel limit $D_j \ll 1$, or for weak backscattering $R_j \ll 1$, Eq. (16) simplifies, respectively, to
\[ D_j(\epsilon) = C_j D(\epsilon) \quad \text{or} \quad R_j(\epsilon) = C_j^{-1} R(\epsilon). \quad (17) \]
These conditions have a simple interpretation. The energy dependence of $D_j(\epsilon)$ or, respectively, $R_j(\epsilon)$ determines the shape and, most importantly, the time delay of the wave packets that correspond to individual electrons (holes) transmitted through the QPC. Equation (17) ensures that no information about the system is contained in the shape or position of these wave packets that would be lost in the measurement process which is sensitive only to the transmission probabilities.

In conclusion, we have analyzed the detector properties of the QPC beyond the linear-response regime and found that both the backaction dephasing rate $\Gamma$ and the measurement rate $W$ are determined by the electron counting statistics. While generally $\Gamma \geq W$, the quantum-limited detector, $\Gamma = W$, can be reached by using tunable detectors that fully utilize the information in the scattering phases, and by optimizing the energy dependence of the transmission probabilities.

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[20] This conclusion may also follow from the consideration of the conditional evolution [11] of the measured system.