Variable-Range Hopping and Quantum Creep in One Dimension

NATTERMANN, Thomas, GIAMARCHI, Thierry, LE DOUSSAL, Pierre

Abstract

We study the quantum nonlinear response to an applied electric field $E$ of a one-dimensional pinned charge-density wave or Luttinger liquid in the presence of disorder. From an explicit construction of low-lying metastable states and of bounce instanton solutions between them, we demonstrate quantum creep $v = e^{-c/E^{1/2}}$ as well as a sharp crossover at $E = E^*$ towards a linear response form consistent with variable-range hopping arguments, but dependent only on electronic degrees of freedom.

Reference


DOI: 10.1103/PhysRevLett.91.056603
Variable-Range Hopping and Quantum Creep in One Dimension

Thomas Nattermann,1 Thierry Giamarchi,2 and Pierre Le Doussal3

1Institut für Theoretische Physik, Universität zu Köln, Zülpicher Strasse 77 D-50937 Köln, Germany
2Université de Genève, DPMC, 24 Quai Ernest Ansermet, CH-1211 Genève 4, Switzerland
3CNRS-Laboratoire de Physique Théorique de l’École Normale Supérieure, 24 Rue Lhomond, Paris 75231, France

(Received 12 March 2003; published 31 July 2003)

We study the quantum nonlinear response to an applied electric field $E$ of a one-dimensional pinned charge-density wave or Luttinger liquid in the presence of disorder. From an explicit construction of low-lying metastable states and of bounce instanton solutions between them, we demonstrate quantum creep $v = e^{-c/E^{1/2}}$ as well as a sharp crossover at $E = E^*$ towards a linear response form consistent with variable-range hopping arguments, but dependent only on electronic degrees of freedom.

Computing the response of a disordered elastic system to an external driving force is a long-standing problem. This is of theoretical importance and also relevant for a host of experimental systems, both classical and quantum. For classical systems, typical experimental realizations are domain walls [1,2] and vortex lattice in type II superconductors [3,4]. Pinned quantum crystals are charge- or spin-density waves [5], Wigner crystal in two-dimensional electron gas [6,7], and disordered Luttinger liquids [8]. In the absence of quantum or thermal fluctuations, disorder leads to pinning or localization. It was initially believed that thermal activation over barriers between pinned states would result [9] in a simple exponential decay $v(F) = \sigma F$, albeit with an exponentially small mobility $\sigma$. However, the glassy nature of such disordered elastic systems leads instead to divergent barriers and to a nonlinear response [10,11] of the form $v = \exp(-\beta F^{-\mu})$ known as creep [3].

In quantum disordered systems, barriers between the many metastable states can be overcome by thermal and quantum activation. Determination of the relation $v(F)$ is thus an even more difficult and mostly open question. Two main issues arise: (i) does one recover a quantum creep formula at $T = 0$ when the system can unpin via quantum tunneling over barriers; (ii) does one recover linear response at $T > 0$, $v(F)/F \rightarrow \sigma$ and what is the $T$ dependence of the conductivity $\sigma$? Although these questions have been answered in detail via controlled instanton calculations for pure systems such as the sine-Gordon model [12–14], with and without dissipation, no controlled method has been found for the disordered problem. Results were obtained using physical arguments for very disordered electronic systems [15]. The renormalization method used for creep in classical systems [16] was extended to quantum problems [17], but suffers from the same limitations [18]. The conductivity of charge-density waves was studied by Larkin and Lee [19], but only in a strong pinning regime considering tunneling around single impurities.

In this Letter we study the driven quantum dynamics of a pinned 1D charge-density wave or of 1D interacting electrons [Luttinger liquid (LL)] in the localized phase, performing a controlled calculation of the tunneling rates. It is known that this system renormalizes to strong disorder where the (classical) ground state can be found exactly and low-lying kinklike excited states constructed. We then study instanton (bounce) solutions and estimate the semiclassical tunneling rate between these states, in the presence of an applied (electric) field. This demonstrates a quantum creep law $v = e^{-c/E^{1/2}}$ at zero temperature. At small nonzero temperature, we show that a sharp crossover occurs between quantum creep for $E > E^*$ and linear response for $E < E^*$. The temperature dependence of the conductivity is of the form $\sigma \propto e^{-c/T^{\nu/2}}$ consistent with Mott’s variable-range hopping (VRH) arguments [20]. Applied to the Luttinger liquid, this extends in a precise way the validity of the VRH formula to interacting electrons in $d = 1$. Note that here, contrary to standard VRH arguments, this result is determined by the electronic degrees of freedom and is not dependent in an essential way on coupling to other degrees of freedom such as phonons. This leads to quite different energy scales than the standard VRH mechanism.

We consider the Hamiltonian of a charge-density wave where the density has a sinusoidal modulation

$$\rho(x) = \rho_0 \cos(Qx - \phi(x)), \quad (1)$$

where $\phi$ is the phase of the charge-density wave. The phase $\phi$ obeys the standard phase action [21]

$$S \frac{\hbar}{\hbar} = \int_0^L dx \int_0^{\beta \hbar u} dy \frac{1}{2\pi K} [(\partial_y \phi)^2 + (\partial_x \phi)^2], \quad (2)$$

where $u$ is the velocity, $y = u\tau$, and $\beta$ is the inverse of the temperature. Furthermore, the system has a short distance cutoff (lattice spacing) $\alpha$. The disorder is modeled by a random potential $V(x)$ coupled to the density by $H = -\int dx V(x) \rho(x)$. Assuming that $\phi$ varies slowly at the scale $Q^{-1}$, we can retain only the Fourier components of $V(x)$ close to $Q$ [22]. This leads to the action
where we represent the disorder with a random amplitude $A$ and phase $\phi$, which are both slowly varying variables. For a Gaussian disorder initially, the disorder $\xi(x) = A(x)\epsilon_i(x)$ obeys $\xi(x)\xi(x') = D\delta(x - x')$, other averages are zero. Adding an external electric field $E$ to the system adds to the action

$$S_E/\hbar = \int dxdy E\phi(x,y), \tag{4}$$

with $E = E\rho_0/(Qu\hbar)$. The action (2)–(4) also describes a LL in the presence of disorder [8,22]. In that case, $Q = 2\pi\rho_0 = 2K_F$, where $K_F$ is the Fermi wave vector for fermions. $K$ is the standard Luttinger parameter that describes the interaction effects ($K = 1$ for noninteracting electrons and $K < 1$ for repulsive interactions). Our study thus directly gives the conductivity of disordered LLs. In that case, the pinning of the phase variable $\phi$ corresponds to the Anderson localization of the system.

At $T = 0$ the disorder is a relevant variable. It pins the phase $\phi$. In the ground state the phase $\phi$ varies by a quantity of order $2\pi$ over a distance $\xi_{loc}$, which is the pinning length of the charge-density wave [21] or the localization length in the presence of interactions for the interacting particles [8,22]. To determine the dynamics of this model, we renormalize the system up to a point where the disorder is of the order of one. Since we are interested in the limit of very low temperatures and fields, we can renormalize the action in the absence of $E$ and at $T = 0$. The flow in that case is well known [22,23] and we do not reproduce it here. The disorder $D$ scales to strong coupling, and the parameter $K$ decreases. We stop the flow at the length scale $l^*$ for which $A = 1$. At that length scale the disorder being of the order of one, the pinning length is of the order of the lattice spacing $\alpha$. The original localization length of the system is thus $\xi_{loc} = \alpha e^{l^*}$. The electric field is also renormalized and becomes $2e/\hbar K' = E_l = Ec^{l^*}$ and time and space are rescaled by a factor $e^{-l^*}$. In what follows, we denote with a star the renormalized quantities at the scale $\xi_{loc}$. Since $u$ also renormalizes, one can absorb this renormalization by rescaling the time by $u^*/u$, which changes $u \to u^*$ in all the above expressions.

To study the dynamics we consider (2)–(4) with the renormalized parameters. Although we stopped the flow when the disorder is of the order of one, we assume that we are truly at strong disorder and can thus consider that the amplitude of the disorder is very large. The main effects thus come from the fluctuations of the random phase of the disorder. In order to perform a semiclassical approximation for the dynamics, one must first determine the (classical) ground state of the renormalized system for $E = 0$. The disorder being time independent, the action is minimized by $\phi(x) = 0$. It is convenient to go back to a lattice description. The energy on the lattice is

$$S_{dis}/\hbar = -\frac{1}{2} \int dxdy A(x) \frac{e^{i\phi(x,y) - \xi(x)}}{2\pi Ka^2} + H.c., \tag{3}$$

with $N = L/\xi_{loc}$ and we take $-\pi < \xi_i \leq \pi$. Since the renormalized disorder $A^*$ in (3) can be considered to be large ($A^* = 1$), to minimize the cosine term one needs to take $\phi_i = \xi_i + 2\pi n_i$, where $n_i$ are integers. The energy becomes

$$\frac{H}{\hbar} = \frac{1}{2\pi K^*\alpha} \sum_{i=1}^{N} [(\phi_{i+1} - \phi_i)^2 - A^* \cos(\phi_i - \xi_i)], \tag{5}$$

where $m_0$ is an arbitrary integer and $x$ denotes the closest integer to $x$. $[x] = 0$ if $-1/2 < x < 1/2$ and $[x] = 1 ([x] = -1)$ for $x > 1/2$ ($x < -1/2$). The values $[f_i]$ thus completely characterize the ground state. Here one takes the $\xi_i$ uniformly distributed; hence, the $n_i$ perform a random walk and the ground state has roughness exponent 1/2 [i.e., $\phi(L)$ scales as $\phi(L) \sim L^{1/2}$] in agreement with other calculations [25].

In the presence of the electric field $E$, any one of these ground states (with $m_0$ fixed) becomes metastable since the phase $\phi$ wants to increase to gain energy from the field. We estimate the tunneling rate out of these metastable states if the electric field $E$ is weak. They are given by $P \sim e^{-S_0/\hbar}$ to exponential accuracy, where $S_0^*$ is the action of a bounce. This is the instanton solution that corresponds to the minimal action needed to go between the two minima and back [12,26]. Such an instanton has the shape of a bubble of typical size $L_x$ in the space direction and $L_{\tau}$ in the time direction and is schematically represented in Fig. 1. If we denote $i, j$ the coordinate in space and time, respectively, then

$$\phi_{ij} = \phi_{ij}^0 + \delta \phi_{ij} = 2\pi(n_i + m_{ij}) + \xi_i, \tag{8}$$

where $\delta \phi_{ij}$ is the deviation from the ground state. We consider unit instantons with $m_{ij} = 1$ inside the bubble and $m_{ij} = 0$ outside. The region where $\phi_{ij}$ interpolates between these two values is the wall which encircles the bubble, which is very thin in the large $A^* = 1$ limit considered here. It is useful to recall that in the pure sine-Gordon model (obtained here taking all $\xi_i = 0$, $n_i = m_i$) the bounce instanton solution (with zero friction coefficient) is a circle in the $x, \tau$ plane, since the theory is Lorentz invariant. Here the surface tension of the instanton walls is highly anisotropic. For a “time-like” wall parallel to the $x$ axis, the surface tension is the same as pure sine-Gordon (with renormalized
parameters) since the disorder is time independent. The corresponding cost in the action is \( \sigma_x L_x \), where the line tension of such walls is \( \sigma_x = 2\pi/(K^*\alpha) \). For a “space-like” wall parallel to the \( \tau \) axis, the surface tension \( \sigma_s(i) \) is a random variable. In first approximation the typical instanton now has a rectangular shape, bounded in \( x \) by two vertical segments parallel to the \( \tau \) axis at coordinates \( x = i_0 \) and \( x = i_0 + L_x \), chosen as places where \( \sigma_s(i) \) is small. The rectangle is closed by two timelike segments at \( \tau = \tau_0 \) and \( \tau = \tau_0 + L_\tau \).

Let us consider a segment of length \( L_x \) of the wall parallel to the \( \tau \) direction between sites \( i \) and \( i + 1 \). The extra action due to the presence of the instanton is

\[
\Delta S = \frac{2\pi}{K^*\alpha} \sum_f [m^2_{i+1,j} + 2m_{i+1,j}(n_{i+1}^0 - n_i^0 - f_i)].
\]

(9)

Thus, for a unit instanton \( m_{i+1,j} = 1 \), the line tension \( \Delta S/\hbar = \sigma_s(i)L_x \) depends on space position \( i \):

\[
\sigma_s(i) = \frac{4\pi}{K^*\alpha} g_i,
\]

(10)

with \( g_i = [f_i] - f_i + 1/2 \). One easily sees that \( g_i \) is uniformly distributed on the interval \([0, 1] \). In particular, there is a definite weight around \( g_i = 0 \) which corresponds to “weak points” in the construction of the ground state where one can bifurcate from point \( i \) up to the boundary at low energy cost to a state where the phase is shifted by \( 2\pi \) on the right of \( i \) (or conversely \(-2\pi\) on the left of \( i \) for the wall on the right). Although these states can be close in energy, the tunneling rate to them is zero. To obtain a nonzero tunneling rate, one must consider “a kink,” i.e., tunneling to a neighboring state where the phase is shifted by \( 2\pi \) between two walls. This is the tunneling process described by the above instanton.

The total action cost of the above rectangular instanton is thus

\[
\Delta S/\hbar = \left[ \sigma_x(i_0) + \sigma_x(i_0 + L_x) \right] L_x + 2\sigma_x L_x \frac{4\pi\epsilon L_x L_\tau}{K^*\alpha}.
\]

(11)

Since the two smallest numbers in a set of \( L_x/\alpha \) random numbers are typically of order \( \alpha/L_x \), one can estimate \( \sigma_x(i_0) + \sigma_x(i_0 + L_x) \approx (4\pi/K^*\alpha)(\alpha/L_x) \). One can then easily estimate the line tension (10) and by minimizing the action (11) get the optimal size for the instanton (for small \( \epsilon \))

\[
L_x^{\text{opt}} = \sqrt{\alpha/\epsilon}, \quad L_\tau^{\text{opt}} = 1/(2\epsilon).
\]

(12)

This yields a decay rate:

\[
P \sim e^{-(4\pi/K^*\alpha)\sqrt{\alpha/\epsilon}} = \exp \left[ -\frac{4\sqrt{2\pi}}{(K^*)^{3/2}} \sqrt{\frac{Q}{\rho_0 E\xi_{\text{loc}}}} \right],
\]

(13)

where we have introduced a characteristic energy scale \( \Delta = h u^*/\xi_{\text{loc}} \) associated with the localization length. Note that \( u^*/\xi_{\text{loc}} \) is the pinning frequency [21]. For a simple sine-Gordon theory, the dependence is \( e^{-\Delta/\xi} \) and \( \Delta_M \) is the Mott gap. This expression corresponds to Zener tunneling across the gap.

Although the above analysis is expected to give correctly the electric field dependence, the precise prefactor in the exponential might be modified by additional physical effects, and its precise determination, beyond the crude estimate given here, is delicate. First, strictly speaking, in order to have a stationary state some amount of dissipation should be introduced in the model. This dissipation changes the cost of the time variation of the phase and thus \( \sigma_x \), but does not affect \( \sigma_s(i) \). It thus slightly changes the prefactor which could, in principle, be studied as in [13]. Next, since \( \sigma_x(i_0) \neq \sigma_x(i_0 + L_x) \), the instanton has a lozenge shape and the spacelike portion can improve its action by taking advantage locally of favorable pins. That may slightly renormalize downwards \( \sigma_x \). Let us also point out that to obtain the response of the system we have computed here a typical instanton, which can occur repeatedly in the volume of the system. There are rarer events that correspond to faster tunneling. Let us divide the system in intervals of scale \( R_x \gg L_x^{\text{opt}} \) (up to the system size). Within each interval there is typically one place to put two walls separated by \( L_x^{\text{opt}} \) and for which \( \sigma_x(i_0) + \sigma_x(i_0 + L_x^{\text{opt}}) \approx 1/R_x \). Thus, the ground state tunnels (back and forth) with these states at a much faster rate. However, since these tunneling events correspond to special places, the density of such atypical kinks being \( O(L_x^{\text{opt}}/R_x) \) they cannot lead to a macroscopic current. The system thus stays essentially blocked until the tunneling events due to the typical instantons can take place. It would, however, be interesting to see whether such rare events could serve as nucleation centers for “quantum avalanches,” which could only increase the creep rate. Although this clearly goes beyond the present study, the explicit construction of the low-lying states presented here may allow for a precise study of this faster dynamics, at least numerically.
where \( S (11) \) that the action decreases linearly with the size of the bounce at zero temperatures (dashed line) as shown in Fig. 2. Because of the presence of a nonzero temperature, this means that the above analysis which was done at \( \beta = \infty \) remains valid as long as the size of the bounce in the time direction is smaller than \( L x \). When the size of the bounce reaches the boundary, the instanton opens (there are periodic boundary conditions in imaginary time) as shown in Fig. 2. Because there is now no contribution coming from instantons parallel to the space direction, it is easy to see from (11) that the action decreases linearly with the size \( L x \) of the instanton. The tunneling rate is thus fixed by the maximum barrier, i.e., the value of the action when the bounce reaches \( L x = \tau M \). Because now the maximum barrier is not fixed by the electric field any more, one has to consider both the forward and backward jumps as for the standard thermally assisted flux flow argument [9]. The net probability current is thus proportional to

\[
J \propto e^{-S'/\beta} - e^{-S' + e(4\pi L x /K^* \alpha)}
\]

\[
\propto e^{-S'/\beta} 2 \sinh \left( \frac{4\pi L x}{K^* \alpha} \right).
\]

where \( S' \) is the action of the bounce as given by the saddle point (12) when \( L x = \tau M \). Thus one recovers below the crossover field \( \epsilon = \xi_{\text{loc}}/(2\beta \hbar \Delta) \) a linear response, with a conductivity proportional to

\[
\sigma(T) \propto e^{-(S'/\beta)} = \exp \left[ -\frac{4\pi}{K^*} \sqrt{2\beta \Delta} \right]. \tag{15}
\]

Quite remarkably, the temperature dependence of the conductivity as obtained by the present formula is identical to Mott's variable-range hopping [20], where the transition between localized states close in energy is provided by an external source of inelastic scattering such as the electron-phonon interaction. The important difference between our result and the standard VRH law is that here, inelastic processes are coming from the electron-electron interaction itself (hidden in the existence of the Luttinger liquid parameter \( K \)). Thus, the prefactor in the exponential contains electronic energy scales. No phonons are needed at variance with the VRH derivation. Our result also leads to a quite different energy scale in the exponential. Although our calculation is done in one dimension only, it is most likely that in higher dimensions as well one can obtain similar formulas. Let us note that in one dimension the above instanton picture is very similar physically to the VRH picture, if one remembers that in a Luttinger liquid a kink in \( \phi \) is related to the presence of a charge though the formula \( \rho = -\nabla \phi / \pi \). Shifting the ground state by one unit is equivalent to moving an electron.

T.G. thanks B. L. Altshuler for interesting discussions and FNS-MANEP for support. T.N. acknowledges financial support by the Volkswagen foundation as well as hospitality of ESPCI Paris and University Paris-Sud.