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Fourier transform of the $2k_F$ Luttinger liquid density correlation function with different spin and charge velocities

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I. INTRODUCTION

Interacting one-dimensional systems have been proven to exhibit exceptionally rich physics such as power-law decay of the correlation functions with nonuniversal exponents depending on the interaction and exotic phenomena such as spin-charge separation. Some of these phenomena have been identified experimentally in the various realizations of one-dimensional systems such as organic conductors, nanotubes, or quantum wires. The standard paradigm describing the low-energy properties of these systems is known as the Luttinger liquid (LL) theory. While the correlation functions in LL theory are most conveniently represented in space-time variables, often experimental results are interpreted more directly in terms of the Fourier transform into momentum and frequency space. Because of the branch-cut singularity structure of the real-space zero-temperature correlation functions, obtaining the Fourier space representation is not always straightforward and requires care in evaluating the integrals. Perhaps this is why explicit forms of the momentum and frequency dependent response functions have been comparatively slow in entering the literature.

The Fourier transforms of the zero-temperature single-particle Green’s function were computed in the early 1990s, while their finite-temperature versions followed a few years later. Whereas in these references closed analytical forms for the spinless Luttinger liquid are provided, the spinful case appears to be much more difficult, and besides numerical results, only a few analytical results are known. Moreover, even at zero temperature, an exact closed-form expression for the $2k_F$ part of the density-density correlation function for the spinful case appears to still be absent from the literature; some special cases of a Luther-Emery liquid with two gapped modes are studied in Ref. 16. In this paper, we provide an exact expression for the Fourier transform of the density-density correlation function in the realistic case of different spin and charge velocities and essentially arbitrary coupling constants in the spin and charge sectors. These results have potential implications for several experiments in one-dimensional systems. Let us mention, for example, Coulomb drag between quantum wires and measurements of the voltage noise on a metallic gate in close proximity to quantum wire. In the case of strong interactions where the spin-incoherent regime can be obtained, the finite-temperature Fourier transform of the $2k_F$ density correlations contains important information about the LL to spin-incoherent LL crossover.

This paper is organized as follows. In Sec. II, we introduce the notation and conventions we will use to describe the spin and charge sectors (including the correlation functions) of the LL. In Sec. III, we obtain an exact, closed-form expression for the Fourier transform of the zero temperature density-density correlation function in the important case of different spin and charge velocities. We present approximate zero-temperature results near the spin and charge singularities in Sec. IV and finite-temperature results in Sec. V. We summarize our main points in Sec. VI. A few useful results and expressions are relegated to the appendixes.

II. SPINFUL TOMONAGA-LUTTINGER MODEL

The low-energy properties of a one-dimensional system of spinful fermions can be studied with the following Hamiltonian in bosonized form:

$$\hat{H} = \sum_{\nu=p,o} \frac{v_\nu}{2\pi} \int dx \left( \frac{1}{K_\nu} \left( \partial_x \phi_\nu \right)^2 + K_\nu \left( \partial_x \theta_\nu \right)^2 \right)$$

where $\phi_\nu$ (or $\phi_o$) is a bosonic field representing charge (spin) collective mode oscillations, $\theta_\nu$ is the dual field satisfying $[\phi_\nu(x), \theta_\nu(x')] = i\pi \delta_{\nu o} \delta(x-x')$, $v_\nu$ are the propagation velocities of these modes, and $K_\nu$ are their stiffness constants (in this paper, we take $\hbar = 1$). The Hamiltonian (1) becomes SU(2) invariant for the special value $K_o = 1$. The fact that the charge and spin fields commute leads to the well-known effect of spin-charge separation, a consequence of which is the factorization of certain correlation functions into a product of spin and charge components when expressed as a function of space and time. It turns out that for the line $K_p + K_o = 1$ in parameter space, this factorization also occurs in Fourier space.
In the bosonic language, the density operator has the representation

\[
\rho = \rho_0 - \frac{\sqrt{2}}{\pi} \partial_x \phi_p + 2\rho_0 \cos(2k_Fx - \sqrt{2}\phi_p) \cos \sqrt{2}\phi_p \\
+ 2\rho_0 \cos(4k_Fx - 2\sqrt{2}\phi_p).
\]

(2)

The first term is the average density \( \rho_0 \), the gradient term represents the density oscillations with zero momentum, and the third and four terms are the \( 2k_F \) and \( 4k_F \) parts of the density fluctuations, respectively. The decomposition (2), in

\[
(T_\tau \rho_{2k_F}(x,\tau)\rho_{2k_F}(0,0)) = \rho_0^2 \cos(2k_Fx) \prod_{\nu=\sigma} \left\{ \left. \sin \left( \frac{\pi}{\beta v_\nu} (x + iv_\nu \tau - i\alpha \text{sgn} \tau) \right) \right| \sin \left( \frac{\pi}{\beta v_\nu} (x - iv_\nu \tau + i\alpha \text{sgn} \tau) \right) \right\} \left( \frac{\pi \alpha}{\beta v_\nu} \right)^{k_F/2},
\]

(4)

where \( \beta \) is the inverse temperature and \( \alpha \) is a short-distance cutoff of the order of the inverse bandwidth.\(^{30}\) We shall follow the standard procedure to find \( \chi_{2k_F}(q,\omega) \), the Fourier transform of Eq. (4), and then analytically continue the result to get the retarded function. In spite of the simple factorized expression (4), a closed analytical form of its Fourier transform has been obtained only for the case where both spin and charge velocities are equal, or both Luttinger parameters are equal to 1.\(^{23}\) For the SU(2) invariant case with arbitrary \( K_p \), the double integral resulting from the Fourier transform can be reduced to a single integral, but it cannot be evaluated in closed form. However, if we set \( i\alpha = 0 \) in the denominator of Eq. (4), the double integral can be performed at zero temperature for arbitrary values of \( v_\nu \) and \( K_p \) inside the parameter regime \( K_p + K_\sigma < 2 \). Last, these results can be extended to the region \( 2 < K_p + K_\sigma < 4 \) where the Fourier transform bears the same functional form than for \( K_p + K_\sigma < 2 \) plus a constant that depends on \( \alpha \). In particular, the singular behavior near \( k = v_\nu, \omega \) is given in both regimes by Eqs. (15) and (16).

III. EVALUATION OF \( \chi_{2k_F}(Q,\omega) \) AT ZERO TEMPERATURE

The Fourier transform of Eq. (4) at zero temperature can be written as

\[
\chi_{2k_F}(q,i\nu) = \frac{1}{2} \left[ \chi^0_{2k_F}(q+2k_F,i\nu) + \chi^0_{2k_F}(q-2k_F,i\nu) \right],
\]

(5)

with

\[
\chi(q,\omega) = \chi_0(q,\omega) + \chi_{2k_F}(q,\omega) + \chi_{4k_F}(q,\omega),
\]

(3)

where we have neglected higher-order subdominant contributions. Our main interest is in the computation of the \( 2k_F \) part for arbitrary values of the collective mode velocities \( v_\nu \) and of the Luttinger parameters \( K_p \).

In coordinate space and imaginary time, the time-ordered density-density correlation function is given by

\[
\frac{1}{A^a B^b} = \frac{\Gamma(a+b)}{\Gamma(a) \Gamma(b)} \int_0^1 dw \frac{w^{a-1}(1-w)^{b-1}}{[wA + (1-w)B]^{a+b}},
\]

(7)

where we have taken \( i\alpha \to 0 \) in the denominators.\(^{31}\) To proceed, the denominators must be combined. One way to do this is through the introduction of another integral by making use of Feynman’s parametrization formula,\(^{24}\)

\[
\chi^0_{2k_F}(q,i\nu) = -\rho_0^2 \frac{\delta^{K_p+K_\sigma}}{2} \int_{-\infty}^{\infty} dx d\tau \frac{x^K e^{-i(q-\nu\tau)}}{(x^2 + v_\sigma^2 \tau^2)^{(K_p+K_\sigma)/2}}.
\]

(6)

By interchanging the integration orders, one can evaluate the integrals in \( x \) and \( \tau \) as

\[
\int_0^1 dw \frac{w^{K_p+K_\sigma-1}(1-w)^{K_\sigma-1}e^{-(q-\nu\tau)}}{[x^2 + v_\sigma^2 \tau^2 + w(v_\sigma^2 - v_\sigma^2 \tau^2)^2]^{(K_p+K_\sigma)/2}}.
\]

(8)
where the limit in the last step the restriction $K_\rho + K_\sigma < 2$ holds in order to obtain a finite result. This restriction is an artifact of letting $i\alpha \to 0$ in the denominators. Had $i\alpha$ been kept finite, we would have obtained a more general result, but we would have been unable to perform the integrals so simply. However, it is easy to extend these results to the region $2 < K_\rho + K_\sigma < 4$. This can be done by noting that the piece that diverges in the limit $\alpha \to 0$ does not depend on $k$ and $\nu$. To extract this constant, we write the imaginary exponential factor in the integrand of Eq. (8) as the sum $[e^{-i(\nu - i\rho)(w - \nu)}] + 1$; in the first term in square brackets and since the divergence has been subtracted, one can safely take the limit $\alpha \to 0$, which leads to Eq. (9) but now restricted to $2 < K_\rho + K_\sigma < 4$. The second term just adds a cutoff dependent constant. Finally, the integral in $w$ can be performed, and after some algebra we find

$$\chi^0_{2k_F}(q, i\nu) = -\frac{4\pi \rho_0^2 (\alpha/2)^{K_\rho + K_\sigma} \Gamma \left(1 - \frac{K_\rho + K_\sigma}{2}\right)}{\Gamma \left(\frac{K_\rho + K_\sigma}{2}\right)} \times \int_0^1 \frac{\rho^2 + \rho^2_{\sigma} q^2 (v^2 + v^2_{\sigma} q^2)^{K_\rho + K_\sigma - 1}}{(v^2 + v^2_{\sigma} q^2)^{K_\rho + K_\sigma - 1}} \times F_1 \left(\frac{K_\rho + K_\sigma - 1}{2}, 2 - \frac{K_\rho + K_\sigma}{2}, r, 1 - \frac{\rho^2 + \rho^2_{\sigma} q^2}{\rho^2 + \rho^2_{\sigma} q^2}\right),$$

(10)

where $F_1$ is Appell’s hypergeometric function of two variables and $r = 1 - \rho^2 / \rho^2_{\sigma}$. Interestingly, when $K_\rho + K_\sigma = 1$ these expressions greatly simplify. The denominator in the integrand of Eq. (9) equals unity, and the integral reduces to a standard Gauss hypergeometric function. Thus, with $K_\rho + K_\sigma = 1$ one arrives at the simple result

$$\chi^0_{2k_F}(q, i\nu) = -2\pi \rho_0^2 \alpha (v^2 + v^2_{\sigma} q^2)^{-K_\rho / 2} (v^2 + v^2_{\sigma} q^2)^{-K_\rho / 2},$$

(11)

which shows that for this line in parameter space, the factorization of the correlation function that represents spin-charge separation is also obtained in Fourier space. However, the latter factorization is nontrivial in the sense that each factor in $\chi^0_{2k_F}(q, i\nu)$ is not the Fourier transform of the corresponding factor in Eq. (4).
\[
\chi_{2k_F}^0(q, \omega) = -4 \pi \rho^2 \left( K_F - K_F \right) \left( K_F + K_F \right) A_{\sigma} \left| \frac{\omega^2 - v_{\rho q}^2}{\omega^2 - v_{\rho q}^2} \right|^{K_F/2} \\
\times e^{-i\pi(K_F+K_F)} e^{i\pi(K_F+K_F)/2} \\
+ B_{\sigma} \left| \frac{\omega^2 - v_{\rho q}^2}{\omega^2 - v_{\rho q}^2} \right|^{K_F/2} e^{-i\pi(K_F+K_F)/2} \\
\text{for } \omega = \pm v_{\rho q}, 
\]

(13)

where \(A_{\rho\sigma}\) and \(B_{\rho\sigma}\) are given in Appendix A.

More schematically, the singular behavior is given by

\[
\chi_{2k_F}^0(q, \omega) \sim \left| \frac{\omega^2 - v_{\rho q}^2}{\omega^2 - v_{\rho q}^2} \right|^{K_F/2} + \text{const}, 
\]

(15)

\[
\chi_{2k_F}^0(q, \omega) \sim \left| \frac{\omega^2 - v_{\rho q}^2}{\omega^2 - v_{\rho q}^2} \right|^{K_F/2} + \text{const} 
\]

(16)

for \(\omega = \pm v_{\rho q}\) and \(\omega = \pm v_{\rho q}\), respectively. In Fig. 1, we show plots of the real and imaginary parts of \(\chi_{2k_F}(q, \omega)\) [as defined in Eq. (5)] as a function of \(\omega\), for a fixed value of \(q=0\) and for a ratio of velocities \(v_{\rho}/v_{\rho} = 0.4\). The frequency is expressed in units of the charge energy \(E_{\rho} = u_{\rho} k_F\). The singular behavior near charge and spin singularities (13) and (14) is also shown. There will be a divergence near the charge singularity for \(\frac{K_F}{2} + K_\sigma < 1\) and a dip otherwise. In cases where a divergence is present, it can be accompanied by a change of sign when the frequency goes through the singularity \((\omega < v_{\rho q} < \omega > v_{\rho q})\) as shown in Fig. 1(b) (real part) [contrary to Fig. 1(a)]. This is due to the imaginary exponential factors in Eq. (13), and it is easy to see that the condition to be fulfilled for a change of sign is \(\frac{K_F}{2} + K_\sigma < 1\). There are analogous results for the spin singularity at \(\omega \approx \pm v_{\rho q}\). Notice that the presence of prefactors and additional constants in Eqs. (13) and (14) may change dips into peaks of finite height, as is the case of the singular behavior near the spin singularity of the real part of \(\chi_{2k_F}\) in Fig. 1(a) and near the charge singularity of the imaginary part of \(\chi_{2k_F}\) in Figs. 1(c) and 1(d).

In the figure, the four possible cases are presented. In Fig. 1(a), we observe a divergence in the charge singularity in the imaginary part; in (b) both singularities are divergent; in (c) the divergence is in the spin singularity, and in (d) there are no divergences. One especially important case is the SU(2) symmetric line \(K_F = 1\), where the charge singularity is always finite, and the spin singularity can be divergent for \(K_F < 1/2\). In particular, Eq. (1) represents the low-energy effective Hamiltonian for the repulsive Hubbard model away from half filling, in which case the spin parameter flows to the fixed point \(K_\sigma = 1\) and \(\frac{1}{2} < K_F < 1\). In this situation, \(\chi_{2k_F}\) shows no divergences in the spin and charge singularities, as depicted in Fig. 1(d).

V. \(\chi_{2k_F}(Q, \omega)\) AT FINITE TEMPERATURE

The finite-temperature case is important as it tells how the singularities at \(\omega = \pm v_{\rho q}, \pm v_{\rho q}\) will be rounded with temperature. Also, if \(v_{\rho}/v_{\rho} \ll 1\) the spin-incoherent regime (defined as \(v_{\rho} k_F \ll k_B T \ll v_{\rho} k_F\)) can be approached from tempera-
FIG. 2. (Color online) Imaginary parts of $\chi_{2k_F}(q,\omega)$ as a function of frequency for $q=0$, $v_\sigma/v_F=0.1$, $K_\rho=0.5$, $K_\sigma=1$. In the inset, we observe the suppression of the correlation function under the effect of temperature (we show its behavior for fixed values of $\omega$).

FIG. 3. (Color online) Imaginary parts of $\chi_{2k_F}(q,\omega)$ as a function of frequency for $q=k_F/2$, $v_\rho/v_F=0.2$, $K_\rho=0.4$, $K_\sigma=0.2$. The four peak structure is the effect of taking $q \neq 0$.

VI. SUMMARY

We have given an exact closed-form expression for the zero-temperature Fourier transform of the $2k_F$ component of the retarded density-density correlation function in a Luttinger liquid with different velocities of spin and charge oscillations and arbitrary stiffness constants. Additionally, we have found approximate expressions near the collective spin and charge mode singularities that essentially take the form of power laws, whose exponents depend in a simple manner on $K_\rho$ and $K_\sigma$. We also compared these approximations directly with the exact result.

We were not able to find an exact result for the finite-temperature case, but we were able to evaluate the Fourier transform numerically and determine some approximate results. One important result of the analysis is that the $2k_F$ oscillations are dramatically (exponentially) suppressed with temperature when the spin and charge velocities are very different. This has implications for observable quantities that depend on the $2k_F$ density correlations, such as Coulomb drag or voltage fluctuations on a metallic gate proximate to a quantum wire.

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APPENDIX A: COEFFICIENTS OF EXPANSIONS NEAR CHARGE AND SPIN SINGULARITIES

The coefficients $A_{p\sigma}$ and $B_{p\sigma}$ appearing in Eqs. (13) and (14) are given by
\[ \Gamma \left( 1 - K_{\alpha} - \frac{K_{\rho}}{2} \right) \]
\[ \left/ \Gamma \left( \frac{K_{\rho}}{2} \right) \right. \]
\[ \times F \left( K_{\rho} + K_{\alpha} - 1, \frac{K_{\rho} + K_{\alpha} - 1}{2}, K_{\rho} + K_{\alpha} - 1; r \right), \tag{A1} \]

\[ \Gamma \left( 1 - K_{\alpha} - \frac{K_{\rho}}{2} \right) \]
\[ \left/ \Gamma \left( \frac{K_{\rho}}{2} \right) \right. \]
\[ \times F \left( K_{\rho} + K_{\alpha} - 1, \frac{K_{\rho} + K_{\alpha} - 1}{2}, K_{\rho} + K_{\alpha} - 1; r \right), \tag{A2} \]

\[ \Gamma \left( 1 - K_{\alpha} - \frac{K_{\rho}}{2} \right) \]
\[ \left/ \Gamma \left( \frac{K_{\rho}}{2} \right) \right. \]
\[ \times F \left( K_{\rho} + K_{\alpha} - 1, \frac{K_{\rho} + K_{\alpha} - 1}{2}, K_{\rho} + K_{\alpha} - 1; r \right), \tag{A3} \]

where \( F \) is the Gauss hypergeometric function (sometimes written as \(_2F_1\)).

**APPENDIX B: GENERAL FORMULA FOR FOURIER TRANSFORM OF 2K\textsubscript{F} RESPONSE FUNCTION**

The \( 2k_F \) part of the density-density correlation function (4) can be expressed as

\[ \chi(q, iv) = \int_{-\infty}^{\infty} dx \int_{0}^{B} d\tau e^{i(q - q')x} \chi(x, \tau) \]  

(B2)

to write \( \chi(q, iv) \) as a convolution of terms,

\[ \chi(q, iv) = \frac{1}{\beta} \sum_{\omega_n} \int dq' \frac{1}{2\pi} \chi(q - q', iv - i\omega_n) \chi(q', i\omega_n). \]  

(B3)

Using the spectral representation

\[ \chi(q, z) = -\frac{1}{\pi} \int dz \text{Im} \chi_{\text{ret}}(q, \epsilon), \]  

(B4)

one finds

\[ \chi(q, iv) = \frac{1}{\beta} \sum_{\omega_n} \int dq' \frac{1}{2\pi} \int d\epsilon_1 \int d\epsilon_2 \frac{d\epsilon}{\pi} \frac{\text{Im} \chi_{\text{ret}}(q - q', \epsilon_1) \text{Im} \chi_{\text{ret}}^{*}(q', \epsilon_2)}{iv - i\omega_n - \epsilon_1 - i\omega_n - \epsilon_2}, \]  

(B5)

where the outer sum can be evaluated with standard complex integration:

\[ \frac{1}{\beta} \sum_{\omega_n} \int \frac{1}{iv - i\omega_n - \epsilon_1} \frac{1}{iv - \epsilon_2 - \epsilon_1} = -\frac{n_B(\epsilon_2) - n_B(-\epsilon_1)}{iv - \epsilon_2 - \epsilon_1}. \]  

(B6)

Therefore, one finds

\[ \chi(q, iv) = -\int dq' \frac{1}{2\pi} \int d\epsilon_1 \int d\epsilon_2 \frac{d\epsilon}{\pi} \frac{n_B(\epsilon_2) - n_B(-\epsilon_1)}{iv - \epsilon_1 - \epsilon_2}, \]  

(B7)

from which the analytical continuation \( iv \rightarrow \omega + i\delta \) can readily be done to yield

\[ \text{Im} \chi(q, \omega) = -\int dq' \frac{1}{2\pi} \int d\epsilon' \frac{d\epsilon'}{\pi} \frac{n_B(\epsilon' - \omega) - n_B(\epsilon')}{iv - \epsilon' - \epsilon_2}, \]  

(B8)

where we use definition of the Fourier transform

\[ \chi(q, iv) = \int_{-\infty}^{\infty} dx \int_{0}^{B} d\tau e^{i(q - q')x} \chi(x, \tau) \]  

(B2)

\[ \text{Im} \chi(q, \omega) = -\int dq' \frac{1}{2\pi} \int d\epsilon' \frac{d\epsilon'}{\pi} \frac{n_B(\epsilon' - \omega) - n_B(\epsilon')}{iv - \epsilon' - \epsilon_2}. \]  

(B8)
24 M. E. Peskin and D. V. Schroeder, An Introduction to Quantum Field Theory (Addison-Wesley, Reading, MA, 1995).
29 Expressions for a few special cases appear in Ref. 23.
30 Strictly speaking, expression (4) is valid for $(x, \tau) \gg \alpha$, and therefore, the expression of $\chi_{2k_F}(q, \omega)$ will be valid for $(q, \omega) \ll \alpha^{-1}$. On the other hand, neglecting $\alpha$ will only restrict the values of $K_p$ and $K_p$ for which our results are meaningful to the parameter region where $K_p + K_p < 2$.
31 Notice that in Eq. (6) we defined the Fourier transform integral to run from $-\infty$ to $\infty$, instead of running from 0 to $\infty$ as one naively would do in the zero-temperature limit. In the latter case, the equivalence between the retarded and the imaginary time-ordered functions cannot be established.