Local bias approach to the clustering of discrete density peaks

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Abstract

Maxima of the linear density field form a point process that can be used to understand the spatial distribution of virialized halos that collapsed from initially overdense regions. However, owing to the peak constraint, clustering statistics of discrete density peaks are difficult to evaluate. For this reason, local bias schemes have received considerably more attention in the literature thus far. In this paper, we show that the two-point correlation function of maxima of a homogeneous and isotropic Gaussian random field can be thought of, up to second order at least, as arising from a local bias expansion formulated in terms of rotationally invariant variables. This expansion relies on a unique smoothing scale, which is the Lagrangian radius of dark matter halos. The great advantage of this local bias approach is that it circumvents the difficult computation of joint probability distributions. We demonstrate that the bias factors associated with these rotational invariants can be computed using a peak-background split argument, in which the background perturbation shifts the corresponding probability distribution [...]
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I. INTRODUCTION

In the biasing scenario introduced by Kaiser [1], virialized halos form out of initially overdense regions with a linear density (extrapolated to the redshift of interest) equal to $\delta_c = 1.686$. Since then, this picture has received considerable support from observational data. Even though dark matter halos are extended objects, they form a spatial point process as far as their clustering is concerned. However, this essential feature has remained elusive in most theoretical descriptions of halo clustering, which assume that halos are a Poisson sampling of a more fundamental, continuous halo density field $\delta_\chi(x)$. The peak formalism first proposed by [2,3] in a cosmological context is interesting because it is a well-behaved point process. In this approach, virialized halos are associated with maxima of the initial density field. The displacement from their initial (Lagrangian) to final (Eulerian) position can be computed upon assuming phase space conservation [4]. Clustering statistics of these discrete density peaks display many of the features present in measurements of halo clustering extracted from N-body simulations. In particular, discrete density peaks exhibit a $k$-dependent linear bias factor [5,6], small-scale exclusion [7,8], and a linear velocity bias [9]. Some of these predictions have recently been tested in numerical simulations [10,11]: peaks of the linear density field appear to provide a good approximation to the formation sites of dark matter halos with $M \approx M_*$.

However, despite recent progress towards the computation of peak clustering statistics [4] and a formulation of peak theory within the excursion set formalism [12,13], discrete density peaks lack a clear connection with the more conventional local bias schemes [14], in which halos are approximated as a continuous field. Furthermore, while in the local bias model the computation of halo correlation functions is straightforward (though there are ambiguities regarding the filtering scale etc.), in the peak formalism calculations are particularly tedious owing to the peak constraint [3,4,6,15]. In the most comprehensive analysis thus far, Ref. [4] succeeded in computing the peak two-point correlation $\xi_{pk}(r)$ up to second order, including the Zel’dovich displacement. They showed that some of the first- and second-order contributions could be obtained from a peak-background split formulated in terms of conditional mass functions. In contrast to most analytic models of halo clustering, which assume that the ($k$-independent) bias coefficients are the peak-background split biases, they derived this equivalence from first principles. However, they could not determine the physical origin of the other second-order contributions. Moreover, the peak constraint is clearly too simplistic to describe the clustering of low mass halos. In this mass range, one should consider more elaborated constraints involving the tidal shear, etc. In this regard, it would be very desirable to find a simpler way of computing the correlation functions of generic point processes of a (Gaussian) random field.
In this paper, we suggest a simple, physically motivated prescription based on the peak-background split to compute the correlation functions of generic point processes driven by homogeneous and isotropic Gaussian random fields. We argue that clustering statistics of such point processes can be reduced to the evaluation of correlators of an effective continuous overdensity which, in the case of discrete peaks, is a function of the local (smoothed) mass density field and its derivatives. Our approach combines in a single coherent picture peak theory, peak-background split, local bias, and the excursion set framework. For sake of clarity, we will focus on the two-point correlation function of initial density peaks as computed in Ref. [4] to explain the fundamentals of our approach.

The paper is organized as follows. Section II furnishes a brief summary of clustering in peak theory. Section III is the central section of the paper, where we present the connection between rotational invariants, peak-background split, local bias, and the excursion set framework. For sake of clarity, we will focus on the two-point correlation function of initial density peaks as computed in Ref. [4] to explain the fundamentals of our approach.

II. CORRELATION FUNCTIONS FOR DENSITY PEAKS

We begin with a short recapitulation of the computation of correlation functions in the peak formalism. Let \( \delta_s \) be the linear mass density field smoothed on scale \( R_s \), with a spherically symmetric filter. For convenience, we work with the normalized variables \( \nu(x) = \frac{1}{\sigma_0} \delta_s(x), \eta_i(x) = \frac{1}{\sigma_0} \partial_i \delta_s(x), \) and \( \xi_{ij}(x) = \frac{1}{\sigma_0^2} \partial_i \partial_j \delta_s(x). \) Here, \( \nu \) is the peak height or significance, and

\[
\sigma_n^2(R_s) = \frac{1}{2\pi} \int_0^\infty dk k^{2n+1} P_s(k)
\]

are moments of the power spectrum \( P_s(k) = \langle |\delta_s(k)|^2 \rangle \). A Gaussian filter is frequently adopted to ensure convergence of all the spectral moments \( \sigma_n \), but one should bear in mind that the peak height associated with dark matter halos is always computed with a tophat filter (see Ref. [13] for details). The first few spectral moments \( \sigma_n \) can be combined into a dimensionless spectral width \( \gamma_1 = \sigma_1^2/(\sigma_0^2 \sigma_2) \) that takes values between zero and unity. \( \gamma_1 \) reflects the range over which the smoothed power spectrum \( P_s(k) \) is significant, i.e., \( \gamma_1 = 1 \) for a sharply peaked power spectrum whereas \( \gamma_1 = 0 \) for a power spectrum that covers a wide range of wave numbers.

Correlations of density maxima of \( \delta_s \) can be evaluated using the Kac-Rice formula [16,17]. The trick is to Taylor-expand \( \eta_i(x) \) around the position \( x_{pk} \) of a local maximum. As a result, the number density of (so-called BBKS [3]) peaks of height \( \nu' \) at position \( x \) in the smoothed density field \( \delta_s \) can be expressed in terms of the field \( \delta_s \) and its derivatives:

\[
n_{pk}(\nu', R_s, x) = \frac{3^{3/2}}{R_s^3} \det \zeta(x) |\delta_D[\eta(x)]| \theta[\lambda_3(x)] |\delta_D \nu(x) - \nu'|. \tag{2}
\]

where \( R_s = (\sigma_1/\sigma_3) \) is the characteristic radius of a peak (and not the interpeak distance). The three-dimensional Dirac distribution \( \delta_D(y) \) ensures that all extrema are included. The factors of theta function \( \theta(\lambda_3) \), where \( \lambda_3 \) is the lowest eigenvalue of the shear tensor \( \xi_{ij} \), and the Dirac delta \( \delta_D(\nu - \nu') \) further restrict the set to density maxima of the desired significance \( \nu' \).

The (disconnected) \( N \)-point correlations \( \rho_{pk}^{(N)} \) (or joint intensities) of density maxima are defined as the ensemble averages of products of \( n_{pk}(\nu, R_s, x) \),

\[
\rho_{pk}^{(N)}(\nu, R_s, x_1, \ldots, x_N) = \langle n_{pk}(\nu, R_s, x_1) \times \cdots \times n_{pk}(\nu, R_s, x_N) \rangle.
\]

For the Gaussian initial conditions considered here, multivariate normal distribution are assumed to perform the ensemble average. In the particular case \( N = 1 \), \( \langle n_{pk}(\nu, R_s, x) \rangle = \bar{n}_{pk}(\nu, R_s) \) is the average, differential number density of peaks of height \( \nu \) identified on the filtering scale \( R_s \) [3],

\[
\bar{n}_{pk}(\nu, R_s) = \frac{1}{(2\pi)^3 R_s^3} e^{-\nu^2/2} G_0^{(1)}(\gamma_1, \gamma_1 \nu)
\]

\[
= \frac{e^{-\nu^2/2}}{\sqrt{2\pi}} \left( \frac{1}{V_s} \right) G_0^{(1)}(\gamma_1, \gamma_1 \nu).
\]

In the last equality, \( V_s = (2\pi)^3/2 R_s^3 \) is the typical three-dimensional extent of a density peak [12]. The functions \( G_n^{(a)}(\gamma_1, \gamma_1 \nu) \) are defined in Appendix B. In particular, the ratio \( G_1^{(k)}/G_0^{(1)} \) is equal to the \( k \)th moment \( \mathbf{u}^k \) of the peak curvature \( u \). Similarly, the reduced two-point correlation function for maxima of a given significance separated by a distance \( r = |x| = |x_2 - x_1| \) is

\[
\xi_{pk}(\nu, R_s, r) = \frac{\rho_{pk}^{(2)}(\nu, R_s, r)}{\bar{n}_{pk}^{(2)}(\nu, R_s)} - 1. \tag{5}
\]

Notice that, in \( \rho_{pk}^{(2)} \), we have ignored the shot-noise term \( \bar{n}_{pk} \delta_D(x_2 - x_1) \) that arises from the self-pairs as it matters only at zero-lag (in the peak power spectrum, however, this contributes a constant Poisson noise \( 1/\bar{n}_{pk} \) at all wave numbers).

The calculation of Eq. (5) at second order in the mass correlation and its derivatives is quite tedious [3,4,15] because one must evaluate the joint probability distribution for the ten-dimensional vector of variables \( \mathbf{y}_n = (\eta(x_a), \nu(x_a), \xi_A(x_a)) \) at two different spatial locations \( x_a = x_1 \) and \( x_2 \), i.e., a total of 20 variables. Here, the components \( \xi_A, A = 1, \ldots, 6 \) symbolize the independent entries \( ij = 11, 22, 33, 12, 13, 23 \) of \( \xi_{ij} \). Fortunately, as was shown in Ref. [4], most of the terms nicely combine together, so that the final result can be recast into the compact expression

\[
\bar{n}_{pk}^{(2)}(\nu, R_s) = \frac{1}{(2\pi)^3 R_s^3} e^{-\nu^2/2} G_0^{(2)}(\gamma_1, \gamma_1 \nu),
\]

\[
= \frac{e^{-\nu^2/2}}{\sqrt{2\pi}} \left( \frac{1}{V_s} \right) G_0^{(2)}(\gamma_1, \gamma_1 \nu).
\]

\[
\xi_{pk}(\nu, R_s, r) = \frac{\rho_{pk}^{(2)}(\nu, R_s, r)}{\bar{n}_{pk}^{(2)}(\nu, R_s)} - 1. \tag{5}
\]
The functions $\xi^{(n)}_{\ell}(r)$ are quantities analogous to $\sigma_n^2$ but defined for a finite separation $r$,

$$\xi^{(n)}_{\ell}(R_s, r) = \frac{1}{2\pi} \int_0^\infty dk k^{2(n+1)} P_s(k) j_r(kr),$$  

(7)

where $j_r(x)$ are spherical Bessel functions. On the right-hand side of Eq. (6), all the correlations depend on the filtering scale and the separation. However, the first line contains terms involving the first- and second-order peak bias parameters $b_1$ and $b_\|$, and, therefore, vanish at zero lag. Clearly, they arise because the peak correlation also depends on the statistical properties of $\eta_i$ and $\xi_{ij}$.

Reference [4] also showed that the Lagrangian peak bias factors $b_{n}(k_1, \ldots, k_N)$ can be constructed upon averaging over the peak curvatures products of $b_\nu$ and $b_u$, where

$$b_{\nu}(\nu, R_s) = \frac{1}{\sigma^2_0} \left( \nu - \gamma_1 \right)$$

(8)

$$b_u(\nu, R_s) = \frac{1}{\sigma_1} \left( \frac{u - \gamma_1 \nu}{1 - \gamma_1} \right).$$

(9)

For peak of significance $\nu$ on the smoothing scale $R_s$, the first-order bias $b_1$ is defined as the Fourier space multiplication (we omit the dependence on $R_s$ and $\nu$ for shorthand convenience) [6]

$$b_1(k) = b_{10} + b_{01} k^2 \quad \text{where } b_{10} = \bar{b}_\nu \quad \text{and} \quad b_{01} = \bar{b}_u.$$  

(10)

The overline designates the average over the peak curvature. $b_{01}$ can be quite large for moderate peak heights. In the high peak limit $\nu \gg 1$; however, it is negligible so that $\bar{b}_1$ is nearly scale independent (like in local bias models). Similarly, the Fourier space expression of the second-order peak bias $\bar{b}_\|_{\nu}$ is [4]

$$\bar{b}_\|_{\nu}(k_1, k_2) = b_{20} + b_{11}(k_1^2 + k_2^2) + b_{02}k_1^2 k_2^2. \quad (11)$$

where $k_1$ and $k_2$ are wave modes and the $k$-independent coefficients $b_{20}$, $b_{11}$, and $b_{02}$ are

$$b_{20}(\nu, R_s) = \frac{1}{\sigma_0^2(1 - \gamma_1^2)} \left[ \frac{\gamma_1^2}{(1 - \gamma_1^2)^2} - \frac{1}{(1 - \gamma_1^2)^3} \right].$$

(12)

$$b_{11}(\nu, R_s) = \frac{1}{\sigma_0^2} \left( \frac{1}{(1 - \gamma_1^2)^2} - \frac{1}{(1 - \gamma_1^2)^3} \right) \left[ \gamma_1^2 \Gamma_2 (1 + \gamma_1 \nu)^2 + \gamma_1 (1 - 2\gamma_1 \nu) \right].$$

(13)

and

$$b_{02}(\nu, R_s) = \frac{1}{\sigma_0^2} \left[ \left( \frac{\gamma!^2}{(1 - \gamma_1^2)^2} - \frac{1}{(1 - \gamma_1^2)^3} \right) \gamma_1 \right].$$

(14)

By definition, $\bar{b}_{\nu}$ acts on the functions $\xi^{(n)}_1(r)$ and $\xi^{(n)}_2(r)$ as follows:

$$(\xi^{(n)}_{1} \xi^{(n)}_{2})_{\|_{\nu}} \propto b_{\|_{\nu}}(k_1, k_2) \mathcal{P}_{s}(k_1) \mathcal{P}_{s}(k_2) j_{\nu}(k_1 r) j_{\nu}(k_2 r).$$

(15)

As pointed out by Desjacques et al. [4], the piece $\bar{b}_{\nu}^2 \xi^{(0)}_0 + (1/2)\xi^{(0)}_0 \xi^{(0)}_0 \xi^{(0)}_0$ can be thought of as arising from the continuous, deterministic, local bias relation

$$\delta_{\nu}(x) = b_{10} \delta_s(x) - b_{01} \nabla^2 \delta_s(x) + \frac{1}{2} b_{20} \delta^2_s(x) - b_{11} \delta_s(x) \nabla^2 \delta_s(x) + \frac{1}{2} b_{02} (\nabla^2 \delta_s(x))^2,$$

(16)

where the bias factors $b_{ij}$ are peak-background split bias factors that follow from expanding the conditional peak number density in a series in the small background density perturbation $\delta_s$. This expansion is local in the sense that, except for the filtering, it involves quantities evaluated at $x$ solely. However, an essential difference with the
widespread local bias model [14] is the fact that, when computing the ensemble average \( \langle \delta_{pk}(x_1)\delta_{pk}(x_2) \rangle \), we must ignore all powers of zero-lag moments [such as, e.g., \( \sigma^0_0 \) in \( \langle \delta_{pk}(x_1)\delta_{pk}(x_2) \rangle \)] to recover \( \xi_{pk}(r) \) since the latter does not exhibit such contributions (this “no zero-lag requirement” also arises in the derivation of the “renormalized” bias parameters of Schmidt et al. [18]). All the terms in Eq. (16) are of course invariant under rotations since \( \delta_{pk}(x) \) transforms as a scalar under rotations. Clearly, however, this series expansion is not the most generic Lagrangian expansion we may conceive of (see, e.g., Ref. [19] for nonlocal Lagrangian bias).

Notwithstanding these results, Desjacques et al. [4] did not succeed in finding a physical interpretation of the other second-order terms on the right-hand side of Eq. (6), even though it was pretty clear that they—at least partially—arise from coupling involving the components of the gradient \( \eta_i \) and the Hessian \( \xi_{ij} \).

III. AN INTUITIVE INTERPRETATION OF \( \xi_{pk}(r) \)

In this section, we propose an intuitive, physically motivated explanation of Eq. (6) that is grounded in the peak-background split argument [1]. We begin with a brief introduction to the helicity basis, which was used in Ref. [4] to compute probability distributions of the field and its derivatives at two different spatial locations.

A. Probability density in the helicity basis

The two-point correlation function of initial density peaks is the ensemble average of \( n_{pk}(y_1)n_{pk}(y_2) \) over the joint probability density \( P_2(y_1,y_2;r) \), where \( y_a = y(x_a) \) are the values of the field and its derivatives at position \( x_a \).

In what follows, \( n_{pk}(y) \) will also designate Eq. (2).

Following [4], we can decompose the variables \( y^T \equiv (\eta_i(x_a),\nu(x_a),\eta_s(x_a)) \) that appear in the joint probability density \( P_2(y_1,y_2;r) \) in the helicity basis \((\epsilon_+,\epsilon_-,\ldots)\), where

\[
\epsilon_+ = \frac{i\hat{e}_\phi - \hat{e}_\theta}{\sqrt{2}}, \quad \epsilon_0 = \frac{i\hat{e}_\phi + \hat{e}_\theta}{\sqrt{2}},
\]

and \( \epsilon_\phi, \epsilon_\theta, \) and \( \hat{e}_\phi \) are orthonormal vectors in spherical coordinates \((r,\theta,\phi)\). The orthogonality relations between these vectors are \( \epsilon_+ \cdot \epsilon_+ = \epsilon_0 \cdot \epsilon_0 = 1 \) and \( \epsilon_+ \cdot \epsilon_- = \epsilon_0 \cdot \epsilon_- = 0 \), where the inner product between two vectors \( u \) and \( v \) is defined as \( u \cdot v = u_i\bar{v}_i = u_i v_i \).

Explicit expressions for these variables are

\[
\xi_{ij} = \frac{1}{3} u \delta_{ij} + \tilde{\xi}_{ij},
\]

\[
\tilde{\xi}_{ij} = \sqrt{3} S_{ij} e^{(S)} + \sqrt{3} V^i_{\perp} \sigma^i_{\perp} \; \epsilon_+ - \sqrt{3} V^i_{\perp} \sigma^i_{\perp} \; \epsilon_0.
\]

Here and henceforth, the subscripts “1” and “2” will denote variables evaluated at position \( x_1 \) and \( x_2 \) for shorthand convenience. Similarly, the symmetric tensor \( \xi_{ij} \) can be decomposed into its trace and traceless components,

\[
\xi_{ij} = \epsilon_0 \delta_{ij} + \tilde{\xi}_{ij},
\]

\[
\tilde{\xi}_{ij} = \sqrt{3} S_{ij} e^{(S)} + \sqrt{3} V^i_{\perp} \sigma^i_{\perp} \; \epsilon_+ - \sqrt{3} V^i_{\perp} \sigma^i_{\perp} \; \epsilon_0.
\]
and $\langle \mathbf{\eta}^{(\pm 1)} \mathbf{\eta}^{(\pm 1)} \rangle = \langle \mathbf{\eta}^{(\pm i)} \mathbf{\eta}^{(\pm i)} \rangle$. All the other correlations vanish. Note that the covariances are real despite the fact that the helicity $-1$ and $-2$ variables are complex.

While the average peak number density only depends on the matrix $M$ of covariances at the same location, the matrix $\mathbf{C}(r)$ of covariances at different locations. For $\mathbf{\xi}_{pk}(r)$, the covariance matrix $\mathbf{C}(r) = \langle \mathbf{y} \mathbf{y}^\dagger \rangle$, where $\mathbf{y} = (y_1, y_2)$, is a 20-dimensional matrix that may be partitioned into four $10 \times 10$ block matrices: the zero-point contribution $M$ in the top left and bottom right corners and the cross-correlation matrix $B(r)$ in the top right and bottom left corners, respectively. Expressions for $M$ and $B(r)$ in the helicity basis can be found in the Appendix of Ref. [4].

### B. Rotational invariants

Translational and rotational invariance implies that $\mathbf{\eta}$ does not depend on spatial position, and that $\mathbf{\xi}_{pk}(r)$ is a function of the distance $r$ solely. In this regard, Desjacques et al. [4] noted that, although the covariance matrix $\mathbf{C}(r)$ in the helicity basis (17) does not depend on the direction $\mathbf{r}$ of the separation vector $\mathbf{r}$, it is not equal to the angular average covariance matrix $\bar{\mathbf{C}}(r) = \langle \mathbf{y} \mathbf{y}^\dagger \rangle$. The latter is obtained upon setting $\mathbf{\xi}_{\ell}^{(\mu)} = 0$ whenever $\ell \neq 0$ in the expression of $B(r)$. As a consequence, $\bar{\mathbf{C}}(r)$ retains the correlations $\langle \nu_1 \nu_2 \rangle, \langle \nu_1 \nu_2 \rangle, \langle \mu_1 \mu_2 \rangle$, by only parts of the covariances $\langle \mathbf{\eta}_{\ell_1}^{(\mu)} \mathbf{\eta}_{\ell_2}^{(\mu)} \rangle$ and $\langle \mathbf{\xi}_{\ell_1}^{(\mu)} \mathbf{\xi}_{\ell_2}^{(\mu)} \rangle$. However, Ref. [4] did not provide a convincing explanation for their observation. We shall do it now.

First, it is pretty clear that, since the peak two-point correlation is invariant under rotations of the reference frame, it should be possible to express it in terms of rotational invariants constructed from the variables $\nu, \eta_i$, and $\xi_{ij}$. The peak significance $\nu$ and the trace $u$ are two obvious candidates, but they are not the only ones. The vector $\mathbf{\eta}$ of first derivatives and the traceless matrix $\bar{\mathbf{\xi}}$ yield two additional invariants, i.e., the square modulus $\mathbf{\eta} \cdot \mathbf{\eta}$ and the trace $\text{tr}(\bar{\mathbf{\xi}}^2)$. In the helicity basis, these invariants can be written

$$\mathbf{\eta} \cdot \mathbf{\eta} = \eta^{(0)} \eta^{(0)} + \eta^{(+1)} \eta^{(+1)} + \eta^{(-1)} \eta^{(-1)},$$

and

$$\text{tr}(\bar{\mathbf{\xi}}^2) = \frac{2}{3} [\zeta^{(0)} \zeta^{(0)} + \sum_{j=1,2} (\zeta^{(+j)} \zeta^{(+j)} + \zeta^{(-j)} \zeta^{(-j)})] = \frac{2}{3} [\zeta^{(i)} \zeta^{(i)} + \sum_{j=1,2} (\zeta^{+(i+j)} \zeta^{+(i+j)} + \zeta^{-(i-j)} \zeta^{-(i-j)})].$$

The $3 \times 3$ symmetric matrix $\zeta_{ij}$ actually provides a third invariant with respect to rotations: the determinant $\text{det} \zeta$. However, as we shall see below in the discussion of the peak-background split, because this determinant only enters the peak number density $n_{pk}(y)$ and not the one-point multivariate normal distribution $P_1(y)$ (the argument of the exponential is quadratic in the variables), it does not contribute directly to the peak bias. This suggests that we look at the covariances of $\eta^2(x)$ and $\xi^2(x)$, where these are defined as

$$\eta^2(x) = \mathbf{\eta}(x) \cdot \mathbf{\eta}(x)$$

and

$$\xi^2(x) = \frac{1}{2} \text{tr} [\bar{\mathbf{\xi}}^2(x)].$$

where $\delta_{i}(k)$ are the Fourier modes of the smoothed density field. As we shall see in Sec. III C, the variables $3 \eta^2$ and $5 \xi^2$ are distributed as chi-squared ($\chi^2$) variables with 3 and 5 degrees of freedom, respectively. Using either the Fourier space expression of $\eta^2(x)$, (26) not to be confused here with a Cartesian component of the vector $\mathbf{\eta}$ or the fact that only components of identical helicity correlate, it is straightforward to compute the following correlators (for illustrative purposes, Appendix A furnishes details of the calculation of $\langle \eta_1^2 \eta_2^2 \rangle$):

$$\langle \eta_1^2 \eta_2^2 \rangle = 1 + \sum_{j=1,2} \left[ (\zeta_0^{(j)})^2 + 2 (\zeta_+^{(j)})^2 \right],$$

and

$$\langle \mathbf{\xi}_1 \zeta_2 \rangle = 1 + \sum_{j=1,2} \left[ (\zeta_0^{(j)})^2 + 2 (\zeta_+^{(j)})^2 \right].$$

Ignoring the zero-lag contributions, these terms correspond exactly (up to a sign factor) to some of those entering the second-order contribution of $\mathbf{\xi}_{pk}(r)$ in Eq. (6), with $\langle \zeta_1^{(1/2)} \zeta_2^{(1/2)} \rangle$ being proportional to $\langle \eta_1^2 \eta_2^2 \rangle$, in particular. The computations of correlators involving $\zeta_{ij}^2$ proceeds in a similar way although, in this case, it is much easier to sum the correlations among equal helicity components. For instance,

$$\langle \zeta_1 \zeta_2 \rangle = 1 + \sum_{j=1,2} \left[ (\zeta_0^{(j)})^2 + 2 (\zeta_+^{(j)})^2 \right].$$
where we used the fact that \(\langle \xi_1^{(s)} \xi_2^{(-s)} \rangle = \langle \xi_1^{(s)} \xi_2^{(s)} \rangle\). After some algebra, we find

\[
\langle \xi_1^2 \xi_2^2 \rangle = 1 + \frac{2}{5\sigma_1^2} \left[ (\xi_0^{(2)})^2 + \frac{10}{7} (\xi_2^{(2)})^2 + \frac{18}{7} (\xi_4^{(2)})^2 \right].
\]

and

\[
\langle \xi_1^2 \eta_2^2 \rangle = 1 + \frac{2}{5\sigma_1^2\sigma_2^2} \left[ 3(\xi_3^{(3/2)})^2 + 2(\xi_1^{(3/2)})^2 \right].
\]

The cross-correlations of \(\eta_1^2\) and \(\zeta_1^2\) with \(u_2\) in place of \(\nu_2\) are identical except for the superscript \(s\), which should be replaced by \((n+1)\). Again, all these terms can be found among the second-order contributions on the right-hand side of Eq. (6).

Therefore, the actual dependence of \(\xi_{\nu k}(r)\) on the invariants \(\eta^2(x)\) and \(\zeta^2(x)\), whose covariances involve the correlation functions \(\xi_{\ell}(\gamma)\) with \(\ell \neq 0\), is the fundamental reason for \(C(r)\) being different from the angle average \(\bar{C}(r)\). Those correlations, which arise upon expanding the joint probability density at second order, eventually all nicely combine together [and with terms proportional to \((\xi_0^{(s)})^2\) and \((\xi_0^{(2)})^2\)] to yield the second-order correlators \(\langle \eta_1^2 \nu_2^2 \rangle, \langle \zeta_1^2 \eta_2^2 \rangle\) etc.

The question then arises of the calculation of the coefficients of these quadratic terms in the peak two-point correlation function. We already know that the coefficients multiplying products of the form \(\xi_0^{(s)} \xi_0^{(n)}\) are the quadratic peak-background split biases associated with the scalars \(\nu\) and \(u\). Does this hold also for the coefficients multiplying \(\langle \eta_1^2 \zeta_2^2 \rangle, \langle \xi_1^2 \nu_2^2 \rangle\), etc.? 

C. Generalizing the peak-background split

The probability distribution \(P_1(y)\) that is needed to compute \(\bar{n}_{pk}\) is a multivariate Gaussian of covariance matrix \(M\),

\[
P_1(y) d^{10}y = \frac{1}{(2\pi)^5 |\det M|^{1/2}} e^{-Q_1(y)} d^{10}y.
\]

Owing to rotational invariance, this probability density is a function of \(\nu, u, \eta^2\), and \(\zeta^2\) solely (see, e.g., Ref. [21] for a systematic analysis of distribution functions of homogeneous and isotropic random fields). The quadratic form \(Q_1(y)\) that appears in the exponential factor reads

\[
Q_1(y) = \frac{\nu^2 + u^2 - 2 \gamma_1 \nu u}{2(1 - \gamma_1^2)} + 3 \frac{\eta^2}{2} + \frac{5}{2} \zeta^2.
\]

so that \(\exp[-Q_1(y)]\) retains factorization with respect to \((\nu, u), \eta^2\), and \(\zeta^2\). Furthermore, since \(\eta^{(s)}\) (with \(s = \pm 1\)) and \(\zeta^{(s)}\) (with \(s = \pm 1, \pm 2\)) are complex random variables with mean 0 and variance 1/3 and 1/5, respectively, and since \(\langle \nu \eta^{(0)} \rangle = \langle \nu \eta^{(0)} \rangle = 0\) and \(\langle \eta^{(0)} \nu^{(0)} \rangle = 0\) at the same spatial location, the quantities \(3 \eta^2(x)\) and \(5 \zeta^2(x)\) are independent \(\chi^2\)-distributed variables with 3 and 5 degrees of freedom, respectively (similar conclusions can be drawn for the distribution of the components of the deformation tensor; see Refs. [19,22]). Therefore, the one-point probability density can also be written as

\[
P_1(y) d^{10}y = N(v, u) d\nu d u \times \chi_1^2(3 \eta^2) \chi_1^2(5 \zeta^2) d(3 \eta^2),
\]

where \(N(v, u)\) is the bivariate normal

\[
N(v, u) = \frac{1}{2\pi\sqrt{1 - \gamma_1^2}} \exp\left[ -\frac{\nu^2 + u^2 - 2 \gamma_1 \nu u}{2(1 - \gamma_1^2)} \right].
\]

and

\[
\chi_1^2(x) = \frac{1}{2^{k/2}\Gamma(k/2)} x^{k/2-1} e^{-x/2}
\]

is a \(\chi^2\) distribution with \(k\) degrees of freedom. Note that the distribution of \(\zeta^2\) is coupled with the last rotational invariant \(\det \zeta\) [23]. However, it can be easily checked that, upon integrating over the (uniform) distribution of \(\det \zeta\), we obtain the \(\chi^2\) distribution \(\chi_1^2(5 \zeta^2)\).

Reference [4] discussed how the peak bias factors \(b_{ij}\) can be derived from a peak-background split. They argued that, while the \(k\)-dependent piece \(b_{n0}\) is related to the \(n\)th order derivative of the differential number density \(\bar{n}_{pk}\), derivatives cannot produce the bias factors \(b_{01}, b_{11}, \text{etc.}\). multiplying the \(k\)-dependent terms. For this reason, they considered a second implementation of the peak-background split [24,25] in which the dependence of the mass function on the overdensity of the background is derived explicitly. However, it is possible to write all the peak bias factors \(b_{ij}\) as derivatives of \(N(v, u)\) rather than \(\bar{n}_{pk}\). More precisely, the \(b_{ij}\) are the bivariate Hermite polynomials

\[
H_{ij}(v, u) = N(v, u)^{-1} \left( -\frac{\partial}{\partial v} \right)^i \left( -\frac{\partial}{\partial u} \right)^j N(v, u)
\]

relative to the weight \(N(v, u)\), further averaged over the peak curvature \(u\). Therefore, they are peak-background split biases in the sense that they can be derived from the transformation \(v \rightarrow v + e_1\) and \(u \rightarrow u + e_2\), where \(e_1\) and \(e_2\) are long-wavelength background perturbations uncorrelated with the (small-scale) fields \(v(x)\) and \(u(x)\). Reference [4] did not express the peak-background split this way because they considered the effect of a background perturbation after the integration over the peak curvature. In terms of the rotational invariants introduced above, we can write the \(b_{ij}\) as

\[
\sigma_0^2 \sigma_2^2 b_{ij} = \frac{1}{\bar{n}_{pk}} \int d^{10}y \bar{n}_{pk}(y) H_{ij}(v, u) P_1(y).
\]
where it is understood that $P_1(y)$ takes the form of Eq. (37) and $d^{10}y = dvdu(d3\eta^3dS_5^2)$. Factors of $1/\sigma_1$ and $1/\sigma_2$ are introduced because bias factors are ordinarily defined relative to the physical field $\delta_1(x)$ and its derivatives rather than the normalized variables. In practice, the integral is most easily performed upon transforming the 5 degrees of freedom attached to $S_5^2$ to the shape parameters $v$ and $w$ and the three Euler angles that describe the orientation of the principal axis frame (see Appendix B for details).

In Ref. [23], it was noticed, in the presence of non-Gaussianity, the one-point probability density $P_1(y)$ can be expanded in the set of orthogonal polynomials associated with the weight provided by $P_1(y)$ in the Gaussian limit. The same logic applies to the peak bias factors. Namely, the $b_{ij}$ are drawn from the orthogonal polynomials associated with $\mathcal{N}(\nu, u)$, i.e., bivariate Hermite polynomials. Therefore, we expect that $\eta^2(x)$ and $\zeta^2(x)$ also generate bias parameters, and that these are drawn from the orthogonal polynomials associated with $\chi^2$ distributions, i.e., generalized Laguerre polynomials. These are defined as

$$L_n^{(\alpha)}(x) = \frac{x^{-\alpha} e^x}{n!} \frac{d^n}{dx^n}(e^{-x} x^{n+\alpha})$$

(42)

and are orthogonal over $[0, \infty]$ with respect to the $\chi^2$ distribution with $k = 2(\alpha + 1)$ degrees of freedom. The orthogonality relation can be expressed as

$$\int_0^\infty dx x^n e^{-x} L_n^{(\alpha)}(x)L_m^{(\alpha)}(x) = \frac{\Gamma(n + \alpha + 1)}{n!} \delta_{nm}.$$  

(43)

The first generalized Laguerre polynomials are $L_0^{(\alpha)}(x) = 1$ and $L_1^{(\alpha)}(x) = -x + \alpha + 1$.

Given the correlator Eq. (29), the term proportional to $1/\sigma_1^4$ on the right-hand side of Eq. (6) indicates that the first-order bias parameter $\chi_{10}$ associated with the invariant $\sigma_1^2 \eta^2(x) = (\nabla \delta_1)^2(x)$ is $\chi_{10} = -3/(2\sigma_1^2)$. The aforementioned considerations suggest that we define the $k$th-order bias factor as the Laguerre polynomial $(-1)^k L_k^{(1/2)}(3\eta^2/2)$ averaged over all the possible peak configurations, i.e.,

$$\sigma_1^{2k} \chi_{10} = \frac{(-1)^k}{\bar{n}_{pk}} \int d^{10}y n_{pk}(y)L_k^{(1/2)}(3\eta^2/2)P_1(y).$$

(44)

Taking into account the peak constraint, the first-order bias factor thus is

$$\chi_{10} = \frac{1}{\sigma_1^2 \bar{n}_{pk}} \int d^{10}y n_{pk}(y)\left(\frac{3}{2} \eta^2 - \frac{3}{2}\right)P_1(y)$$

$$= -\frac{3}{2\sigma_1^2},$$

(46)

which is precisely what we were expecting. Similarly, we define the bias parameter $\chi_{0k}$ associated with the invariant $\zeta^2(x)$ as the ensemble average of the Laguerre polynomial $L_k^{(3/2)}(5\zeta^2/2)$ orthogonal with respect to the weight $\chi_3^2(5\zeta^2)$. Namely,

$$\sigma_2^{2k} \chi_{0k} = \frac{(-1)^k}{\bar{n}_{pk}} \int d^{10}y n_{pk}(y)L_k^{(3/2)}(5\zeta^2/2)P_1(y).$$

(47)

Note that, although we use a single symbol $\chi_{ij}$ to designate the bias factors derived from the $\chi^2$ distributions, the variables $\eta^2$ and $\zeta^2$, unlike $\nu$ and $u$, are uncorrelated. The first-order bias factor thus is

$$\chi_{01} = \frac{1}{\sigma_2^2 \bar{n}_{pk}} \int d^{10}y n_{pk}(y)\left(\frac{5}{2} \zeta^2 - \frac{5}{2}\right)P_1(y).$$

(48)

To evaluate the integral, we first express the measure $d(5\zeta^2)$ in terms of the ellipticity $\nu$ and prolateness $w$, so that $\zeta^2$ can be written as $\zeta^2 = 3v^2 + w^2$ (see Appendix B for details). A multiplicative factor of $\zeta^2$ will arise upon, e.g., taking the derivative of $\exp(-5\alpha \zeta^2/2)$ with respect to $\alpha$. In the notation of Desjacques et al. [4], our factor of $\zeta^2$ precisely corresponds to their derivative term $-(2/5)\partial_w \ln G_0^{(a)}(\gamma_1, \gamma_1, \nu)$ evaluated at $\alpha = 1$ (see Appendix B). Taking into account the factor of $G_0^{(1)}(\gamma_1, \gamma_1, \nu)$ in the denominator, $\chi_{01}$ can eventually be written

$$\chi_{01} = -\frac{5}{2\sigma_2^2} \left(1 + \frac{2}{5} \partial_w \ln G_0^{(a)}(\gamma_1, \gamma_1, \nu)\right)_{\nu=1}.$$ (49)

The physical interpretation of this result is straightforward: $\zeta^2(x)$ is a scalar that describes the asymmetry of the peak density profile. In the high peak limit, $\chi_{01} \to 0$ reflecting the fact that the most prominent peaks are nearly spherical (see Fig. 9 of Ref. [4]).

The physical origin for the appearance of these orthogonal polynomials can be found in the peak-background split. Long-wavelength background perturbations locally modulate the mean of the distributions $\mathcal{N}(\nu, u)$, $\chi_2^2(3\eta^2)$ and $\chi_3^2(5\zeta^2)$. The resulting noncentral distributions can then be expanded in the appropriate set of orthogonal polynomials. In practice, it is convenient to introduce a shift or translation operator $\hat{T}_\epsilon$ to describe the action of a background perturbation on the distribution of rotational invariants. For the scalars $\nu$ and $u$, we define the shift operator as

$$\hat{T}_\epsilon \equiv \exp(-\epsilon_1 \partial_\nu - \epsilon_2 \partial_u),$$

(50)

where $\epsilon_1$ and $\epsilon_2$ are small perturbations to the peak significance and the peak curvature, i.e., $\nu \to \nu + \epsilon_1$ and $u \to u + \epsilon_2$. The action of $\hat{T}_\epsilon$ on the probability density $\mathcal{N}(\nu, u)$ is to shift the (zero) mean of $\nu$ and $u$ by $-\epsilon_1$ and $-\epsilon_2$, respectively [the reason for the minus sign is that Hermite polynomials include a factor of $(-1)^{\nu+1}$]. A straightforward calculation gives

$$\mathcal{N}(\nu, u)^{-1} \hat{T}_\epsilon \mathcal{N}(\nu, u) = \frac{\mathcal{N}(\nu - \epsilon_1, u - \epsilon_2)}{\mathcal{N}(\nu, u)}$$

$$= f(\epsilon_1, \epsilon_2)e^{\epsilon_1 \sigma_0 b u + \epsilon_2 \sigma_1 b u},$$

(51)

where $b_u$ and $b_v$ are given in Eqs. (8) and (9), and $f(\epsilon_1, \epsilon_2)$ is the exponential factor in $\mathcal{N}(\nu, u)$ with the replacement...
we recover the bias factors $b_ij$ once the results are averaged over all locations that satisfy the peak constraint. Note that the bias parameter $b_{ij}$ defined in Ref. [18] bears the same physical meaning as our $b_{01}$: both represent the leading-order response of the tracer abundance to a uniform shift in the curvature of the density field.

For the quadratic variables $\eta^2$ and $\zeta^2$, Eq. (42) suggests that we express the shift operator in terms of both $x$ and $\partial_x$. The definition is somewhat cumbersome because we must take into account not only the ordering of $x$ and $\partial_x$, but also the factor of $\Gamma(n + \alpha + 1)/n!$ in the orthogonality relation Eq. (43). A sensible definition of $\hat{T}_e$ for the variable $x = 3\eta^2$ and $5\zeta^2$ is

\[
\hat{T}_e =: \sum_{j=0}^{\infty} \frac{\Gamma(\frac{5}{2} k)}{j! \Gamma(\frac{3}{2} k + j)} \left( -\frac{\epsilon}{2} \partial_x x \right)^j \Gamma\left(1 - \frac{3}{2} k, \frac{1}{2} \partial_x x \right),
\]

where $I_{\alpha}(x)$ is a modified Bessel function of the first kind and the symbol $::$ of normal ordering is borrowed from quantum field theory. In the present discussion, the normal ordering is defined as

\[
(\partial_x x)^n f(x) \equiv \partial^n_x (x^n f(x)),
\]

where $f(x)$ is some test function. With this definition, the action of $\hat{T}_e$ on a $\chi^2$ distribution with $k = 2(\alpha + 1)$ degrees of freedom is

\[
\hat{T}_e \chi^2_k(x) = \frac{1}{2} \left[ \sum_{j=0}^{\infty} \frac{(-\frac{5}{2})^j}{j! \Gamma(\frac{1}{2} k + j)} (\partial_x x)^j \chi^2_k(x) \chi^{\alpha - x/2}
\]

\[
= \frac{1}{2} \sum_{j=0}^{\infty} \frac{(-\frac{5}{2})^j}{j! \Gamma(\frac{1}{2} k + j)} \partial_x \left[ \left( \chi^2_k \right)^{\alpha + j} e^{-x/2} \right]
\]

\[
= e^{-x/2} \frac{\chi^2_k}{2} \sum_{j=0}^{\infty} \frac{(-\frac{5}{2})^j}{\Gamma(\frac{1}{2} k + j)} L_{\alpha+j}(\chi^2_k)
\]

\[
= \chi^2_k \sum_{j=0}^{\infty} \frac{\Gamma(\frac{1}{2} k)}{\Gamma(\frac{3}{2} k + j)} \left( -\frac{\epsilon}{2} \partial_x \right)^j L_{\alpha+j}(\chi^2_k).
\]

This is precisely the Laguerre series expansion of a noncentral $\chi^2$ variate derived in Ref. [26] (see Appendix C). Therefore,

\[
[\chi^2_k(x)]^{-1} \hat{T}_e \chi^2_k(x) = \frac{\chi^2_k(x; \epsilon)}{\chi^2_k(x)}.
\]

Note that the more common generating function

\[
(1 - e)^{-\alpha-1} \exp\left[ -\frac{xe}{2(1 - e)} \right] = \sum_{n=0}^{\infty} e^x L_n^{(\alpha)}(x)
\]

appears to bear little connection with the noncentral $\chi^2$ distribution.

To better understand the reason why the peak-background split generates a noncentral $\chi^2$ distribution, we note that, owing to the relation $H_{2\alpha}(x) \sim L_\alpha(x^2)$ between Hermite and Laguerre polynomials, we could also have defined $\chi_{10}$ and $\chi_{01}$ as second derivatives of $P_1(y)$, where $y$ is now the vector $(\nu, u, \eta_1, \xi_1)$ of independent normal random variables such that $\eta^2 = \sum_{i=1}^{3} \eta_i^2$ and $\zeta^2 = \sum_{i=1}^{3} \xi_i^2$. A little algebra shows that

\[
\chi_{10} = \frac{1}{n_{pk}} \int d^{10} y P_{pk}(y) \sum_{j=1}^{3} \frac{1}{\sigma_j^2} \partial_x \partial_x \chi_{10} P_{1}(y).
\]

Another way of writing this formula would be to absorb a factor of $1/\sqrt{2}$ in the definition of $\eta^{(1)}$ (and thus explicitly deal with complex normal distributions). Analogously, we have

\[
\chi_{01} = \frac{1}{n_{pk}} \int d^{10} y P_{pk}(y) \sum_{j=A}^{5} \frac{1}{\sigma_j^2} \partial_x \partial_x \chi_{01} P_{1}(y).
\]

This suggests that the effect of a background perturbation on $\eta^2$ and $\zeta^2$ can also be thought of as shifting the components of the first derivatives according to $\eta_i \rightarrow \eta_i + \epsilon_{3i}$, and those of $\zeta$ according to $\zeta_A \rightarrow \zeta_A + \epsilon_{5A}$. The small perturbations $\epsilon_{3i}$ and $\epsilon_{5A}$ need not be the same for distinct $i$ and/or $A$. However, owing to invariance under rotations, only the length of the vector $\sum_{i=1}^{3} (\epsilon_{3i})^2 = \epsilon_3$ and $\sum_{A=1}^{5} (\epsilon_{5A})^2 = \epsilon_5$ matter. This is the reason why the background perturbation effectively shifts the respective $\chi^2$ distributions to noncentral $\chi^2$ distributions, with noncentrality parameter $\lambda = \epsilon_3$ and $\lambda = \epsilon_5$. Note that it should be possible to formulate this peak-background split with the conditional peak number density $n_{pk}(\nu, R_j, \delta_j, R_i)$ in a large-scale region of overdensity $\delta_j$, like in Ref. [4]. However, one should then consider two long-wavelength
perturbations $\eta^2$ and $\xi^2$ in order to describe the effect of the background perturbation on $\eta^2(x)$ and $\xi^2(x)$, in addition to $\delta_1$ (which suffices to describe the effect of the background wave for both $\nu$ and $u$ since these variables are correlated).

To conclude, the author of Ref. [27] also pointed out that, even though there is no functional relation $n_{pk} = F(\delta)$ for discrete density peaks, it is nevertheless possible to define renormalized bias parameters as the expectation values $c_n = \langle F \rangle$. However, the values were not computed explicitly, nor was it specified what is $F$. Here, we demonstrated explicitly that each of the combinations $(\nu, u, \eta^2$ and $\xi^2$ of rotational invariants generates a set of orthogonal polynomials which, upon taking the ensemble average over all the possible peak configurations, yields a set of bias factors. Furthermore, we showed that these bias parameters can be constructed from a suitable application of the peak-background split to the probability densities characterizing the invariants. We will now demonstrate that we can interpret the peak two-point correlation as arising from a functional relation of the form $\delta_{pk} = F(\delta_\nu, \ldots)$.

**D. A local bias approach to $\xi_{pk}(r)$**

First, let us make sure that [4] obtained the correct expression for $\xi_{pk}(r)$. Adding all the second-order contributions induced by $\eta^2$, $\xi^2$ and their cross-correlations with $\nu$ and $u$, our result differs from that of Ref. [4] in that the last term on the right-hand side of Eq. (6) appears to miss a multiplicative factor of $1 + (2/5)\partial_\alpha \ln G_0^{(a)}(\gamma_1, \gamma_1 \nu)|_{\alpha-1}$. Checking the calculation of Desjacques et al. [4], we found the missing multiplicative factor in Eqs. (A50) and (A51) of Ref. [4], in the form of $q(r)\text{tr}(\xi^2)$. This term was fortuitously omitted in that work’s final expression of $\xi_{pk}(r)$. Consequently, the correct answer is

$$\xi_{pk}(\nu, R, r) = \left( b_{10}^2 \delta_1(x) + b_{01}^2 \delta_2(x) + b_{20}^2 \delta_1(x) + b_{11}^2 \delta_2(x) + 1/2 b_{02}^2 \delta_2(x) \right) + \chi_{10} \left( 3 \delta_1 \delta_2 + \delta_1 \delta_2 + \delta_2 \delta_2 \right) + \chi_{10} \left( 3 \delta_1 \delta_2 + \delta_1 \delta_2 + \delta_2 \delta_2 \right),$$

(62)

We note that this omission has an impact only on the small-scale ($r \approx 20 h^{-1}$ Mpc) peak correlation displayed in Fig. 2 of Ref. [4]. Their results concerning the peak-background split, the gravitational evolution of $\xi_{pk}(r)$, or the scale dependence of bias around the baryon acoustic oscillation are unaffected.

Even though we cannot write down a relation of the form $n_{pk} = F(\delta)$, the peak correlation function up to second order can nonetheless be thought of as arising from a local bias expansion $\delta_{pk} = F(\delta_s, \ldots)$, i.e.,

$$\delta_{pk}(x) = b_{10} \delta_1(x) - b_{01} \nabla^2 \delta_1(x) + \frac{1}{2} b_{20} \nabla^2 \delta_2(x) - b_{11} \delta_1(x) \nabla^2 \delta_1(x) + \frac{1}{2} b_{02} \nabla^2 \delta_2(x) + \chi_{10} \nabla^2 \delta_2(x) + \frac{1}{2} \chi_{10} \delta_2 \delta_1 + \delta_1 \delta_2 \nabla^2 \delta_2(x),$$

(62)

provided that we ignore all the contributions involving moments at zero lag. Nonlocality enters through the filtering solely (which is the reason why we still call it a local expansion). As emphasized in Ref. [4], it is important to realize that $\delta_{pk}$ is not a count-in-cell quantity. Counts-in-cells can generally be constructed using the void generating function, see e.g., Refs. [28,29], but it is beyond the scope of this paper to compute moments of the peak frequency distribution function. Since all the bias factors are peak-background split biases obtained from a suitable average of orthogonal polynomials, one could try to write down an expansion in terms of orthogonal polynomials in the variables $\delta_s$, $\nabla \delta_s$, etc. such that all the contributions involving moments at zero lag cancel out. We will explore this possibility in future work. Note that this idea was put forward for the first time by Szalay [30], who considered correlations of regions above threshold as a proxy for luminous tracers. More recently, Musso et al. [31] proposed an algorithm based on Hermite polynomials to extract $k$-dependent bias factors from cross-correlations between the halo and a Hermite-transformed mass density field.

To make a connection with the formalism of Matsubara [27] (see also Ref. [32]), we define the Fourier space peak bias parameters $c_{\nu}(k_1, \ldots, k_n)$ as the sum over all the contributions to $\delta_{pk}(x)$ from a given order. We thus have

$$c_1(k) \equiv (b_{10}^2 + b_{01}^2)W(kR_s)$$

(63)

and

$$c_2(k_1, k_2) \equiv (b_{20}^2 + b_{11}^2 + b_{02}^2)W(kR_s)$$

(64)
where $W(kR_s)$ is the smoothing kernel. These definitions are consistent with those of the “renormalized” bias parameters introduced by Matsubara [27] who argued that, owing to rotational symmetry, the peak bias factors should take the above functional form. In particular, the correspondence between the bias factors associated with $\eta^2$ and $\xi^2$ and those of Matsubara [27] is $E_2 = 3 \chi_{01}$, $C_2 = -2 \chi_{10}$, and $D_2 = b_{02} - \chi_{01}$. We stress, however, that the peak bias factors discussed in this work have not been obtained by means of a renormalization procedure. We speculate that the local bias expansion Eq. (62) can be extended to all orders to match the exact peak two-point correlation function at all separations. Namely,

$$\delta_{pk}(x) = \sum_{n=1}^{\infty} \frac{1}{n!} \int \frac{d^3k_1}{(2\pi)^3} \cdots \frac{d^3k_n}{(2\pi)^3} c_n(k_1, \ldots, k_n) \times \delta(k_1) \cdots \delta(k_n) e^{i(k_1 + \cdots + k_n)\cdot x},$$ \hspace{1cm} (65)

where $c_n(k_1, \ldots, k_n)$ is a sum over all the possible combinations of rotational invariants involving exactly $n$ powers of the linear density field $\delta_\chi$ and/or its derivatives.

Our peak-background split approach provides a simple way of predicting the bias coefficients associated with any rotational invariant quantity. The Hermite weighting scheme introduced by Musso et al. [31] furnishes a practical way of measuring the biases $b_{ij}$ from simulations. Clearly, their scheme could be extended to also measure the biases $\chi_{ij}$. However, because discrete density maxima require a somewhat more sophisticated treatment of counts-in-cells, we leave this for future work. Here, we merely establish a recursion relation between the $b_{ij}$ by considering either the property

$$\left( \frac{\partial}{\partial \nu} + \gamma_1 \frac{\partial}{\partial u} \right) e^{-\xi_1(y)} = -\nu e^{-\xi_1(y)},$$ \hspace{1cm} (66)

or the generating function in Eq. (51). In the latter case, upon substituting $\nu \to (\sigma_0/\sigma_1)^2 (\nu/\sigma_0 - b_\nu)$ in the exponential factor, we find

$$\delta_\nu b_{01} = \left( \frac{\sigma_0}{\sigma_1} \right)^2 (\nu^2 - \delta_\nu b_{10}).$$ \hspace{1cm} (67)

at the first order whereas, at the second order, we obtain

$$\delta_\nu^2 b_{11} = \delta_\nu^2 b_{10} + \frac{\delta_\nu^2 \gamma_1^2}{(1 - \gamma_1^2)}$$

$$= \left( \frac{\sigma_0}{\sigma_1} \right)^2 \delta_\nu^2 \left[ \frac{\nu}{\sigma_0} \delta_\nu + \frac{\gamma_1^2 \nu^2}{(1 - \gamma_1^2)} \right]$$

$$= \left( \frac{\sigma_0}{\sigma_1} \right)^2 \left[ -\nu^2 + \nu^2 \delta_\nu b_{10} - \delta_\nu^2 b_{20} \right].$$ \hspace{1cm} (68)

At least for $n = 1, 2$, $b_{kl}$ with $1 \leq l \leq k \leq n$ can be expressed as a linear combination of $b_{0l}$, $1 \leq k \leq n$, plus a polynomial in $\nu$. This agrees with the findings of Musso et al. [31] obtained within the excursion set approach [31] [except for the multiplicative factors of $(\sigma_0/\sigma_1)^2]$. Similar relations should hold at any order in the peak bias parameters $b_{ij}$. This structure arises from the fact that the peak-background split acts on probability densities, which are continuous functions of space. For discrete tracers such as density maxima, the background perturbation affects the mean of the Gaussian multivariate $P_1(y)$, but not the fields entering the expression of the peak number density $n_{pk}(y)$. The peak constraint weights the peak-background split series expansion such that the peak bias factors are recovered.

**E. Correlation functions for excursion set peaks**

To make connection with the clustering of dark matter halos, we must ensure that the density in a tophat region centered at the peak location never reaches the collapse threshold $(\delta_c)$ on any smoothing scale $R > R_s$. Reference [33] showed that enforcing the conditions $\delta(R_s) > \delta_c$ and $\delta(R_s + \Delta R_s) < \delta_c$ as in Ref. [34] provides a very good approximation to the first-crossing distribution when the stochastic walks generated from the variation of $R_s$ are strongly correlated. An important consequence of this result is the possibility of restricting the excursion set to those locations that meet the peak constraint [12]. The number density of dark matter halos per unit mass and volume is usually written

$$\tilde{n}(M) = \frac{\bar{\rho}}{M} f(\nu) \frac{d\nu}{dM},$$ \hspace{1cm} (70)

where $f(\nu)$ is the multiplicity function. Following the approach of [12,34], the number density of peaks identified on the filtering scale $R_s$ and satisfying the aforementioned conditions is

$$n_{ESP}(R_s)dR_s = \frac{3^{3/2}}{R_s^4} \left( \frac{\sigma_2}{\sigma_0} \right)^2 \int d^{10} y n_{pk}(y) u P_1(y) R_s dR_s,$$ \hspace{1cm} (71)

where, for simplicity, we have assumed that the smoothing kernel is Gaussian, but it is straightforward to generalize these results to arbitrary filters. Therefore, $d\nu/dR_s = \nu R_s (\sigma_1/\sigma_0)^2$ and we can write the excursion set peaks multiplicity function as
Here, $V \equiv M/\bar{\rho}$ is the Lagrangian volume associated with the filter (usually tophat). $f_{\text{ESP}}$ is the fundamental ingredient in the mass function prediction of Paranjape et al. [13] since it can be interpreted as a multiplicity function. It is pretty clear that, in the peak two-point correlation, the constraint $\delta(R_s) > \delta_c$ and $\delta(R_s + \Delta R_s) < \delta_c$ will translate into an extra multiplicative factor of $(\sigma^2/\sigma_0)uR_s dR_s$ in the integrand. Therefore, the ESP correlation functions can also be obtained in the exact same way as that of the BBKS peaks, but with a peak number density

$$n_{\text{ESP}}(\nu', R_s, \mathbf{x}) = \frac{3/2}{R_s^2 - \frac{\nu - \nu'}{\gamma_1}} \left| \text{det} \frac{\partial}{\partial R_s}(\mathbf{x}) \text{det} \frac{\partial}{\partial \nu'}(\mathbf{x}) \right| \delta_D(\mathbf{y}) \delta_H(\lambda_3)$$

(73)

The bias parameters for excursion set peaks are thus given by

$$b_{ij} = \frac{1}{\sigma_i^2 \sigma_j^2 n_{\text{ESP}}} \int d^5 y n_{\text{ESP}}(y) H_{ij}(\nu, u) P_1(y),$$

$$X_{10} = \frac{(-1)^k}{\sigma_1^2 n_{\text{ESP}}} \int d^5 y n_{\text{ESP}}(y) L_{12}(\nu, u) P_1(y),$$

$$X_{0k} = \frac{(-1)^k}{\sigma_2^2 n_{\text{ESP}}} \int d^5 y n_{\text{ESP}}(y) L_{32}(\nu, u) P_1(y).$$

(74)

They are similar to the bias factors of BBKS peaks, except for the fact that the $k$th order moment $\bar{u}_k$ of the peak curvature must be replaced by $\bar{u}^{k+1}_k/\bar{u}$ in Eqs. (12)–(14), and $G^{(a)}_0$ must be replaced by $\tilde{G}^{(a)}_1$ in Eq. (49). For large values of $\omega = \gamma_1 \nu$, $\tilde{G}^{(a)}_1$ asymptotes to $\tilde{G}^{(a)}_1 = \alpha^{-5/2} \omega^5$. This implies that $\bar{u}^{k+1}_k/\bar{u}$ converges towards $-5/2$ in the limit $\omega \to \infty$. Hence, the second-order bias induced by the asymmetry of the peak profile also vanishes in the high peak limit for the ESP multiplicity function, as it should be.

To illustrate the behavior of the ESP bias parameters, we follow the methodology adopted in Ref. [13]. Namely, we smooth the density field with a tophat filter, while sticking to the Gaussian filter to define $\eta$ and $\zeta$ (so that the second spectral moment $\sigma_2$ remains finite). Therefore, we have $\gamma_1 = \sigma^2_\eta/(\sigma_0 \sigma_2 G)$, where the subscript $T$ and $G$ refer to tophat and Gaussian filtering, respectively, and $X$ denotes a mixed filtering, i.e., one filter is Gaussian and the other is tophat. Next, we construct the mapping between $R_T$ and $R_G$ by finding the $R_G(R_T)$ for which $\langle \delta_G \delta_T \rangle = \langle \delta^2_T \rangle$. Finally, to account for departures from the spherical collapse approximation, we consider a moving barrier of the (square-root) form $B = \delta_m + \beta \sigma_0$, where $\beta = 0.43$ [35]. For simplicity, we will ignore the scatter around $B$ even though it is quite substantial in the range of $\nu$ in which we are interested. Figure 1 shows the first- and second-order peak bias factors at $z = 0$ for a $\Lambda$CDM cosmology with $\sigma_8 = 0.82$. Note that while $M = 10^{15} M_\odot/h$ translates into

FIG. 1 (color online). First-order (left) and second-order (right) Lagrangian peak bias factors derived from the excursion set peaks mass function Eq. (72) assuming a moving barrier $B(\sigma_0) = \delta_m + 0.43 \sigma_0$ (see text). The bias parameters have been multiplied by the appropriate factors of $\sigma_i$, so that they all are dimensionless and multiply combinations of normalized variables in the effective local bias expansion Eq. (62). In particular, the second-order bias factors $\sigma_{10}^2 b_{10}, \sigma_0 \sigma_1^2 b_{11}$ and $\sigma_0^2 b_{02}$ are associated with $\nu(x)$ and $u(x)$, whereas $\sigma_{10}^2 \chi_0$ and $\sigma_{11}^2 \chi_{10}$ weight the contributions from $\eta^2(x)$ and $\zeta^2(x)$, respectively. Vertical lines mark the peak significance at which the halo mass is $M = 10^{12}, 10^{13}, 10^{14}$, and $10^{15} M_\odot/h$ (from left to right). Dashed curves indicate negative values. Results are shown at $z = 0$ for a $\Lambda$CDM cosmology with normalization $\sigma_8 = 0.82$. 

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a significance of $\nu = 1.2$, the actual height of $\nu = 1.2$
density peaks is $B/\sigma_0 = 1.6$.

These results can be generalized to arbitrary filtering of
the mass density field. For non-Gaussian initial conditions,
there are a couple of subtleties that will be discussed
elsewhere [36].

IV. DISCUSSION AND CONCLUSIONS

We have shown that the two-point correlation function
$\xi_{pk}(r)$ of discrete density peaks can be computed, up to
second order at least, from an effective local bias expansion
in continuous fields that are invariant under rotations of the
coordinate frame. This local expansion is not a count-in-cell
relation in the sense that $\delta_{pk}(x)$ is merely an effective
overdensity that can be used to recover the true $\xi_{pk}(r)$ from
a trivial evaluation of $\langle \delta_{pk}(x_1) \delta_{pk}(x_2) \rangle$. One of the con-
sequences is that there only is one physically motivated
smoothing scale: the Lagrangian radius $R_s$ of the halos. Yet
another important difference with the widespread local bias
model is that one shall ignore all the contributions from
zero-lag moments in order to obtain the correct $\xi_{pk}(r)$.

All the bias coefficients can be derived from a peak-
background split argument in which the background per-
turbation shifts the zero mean of the one-point probability
distribution functions of the rotationally invariant fields,
unlike essentially all the other peak-background split for-
mulations that consider a change in the mass density field
solely. Consequently, it is possible to derive bias factors
from a peak-background split argument even if the varia-
bles are integrated over. The resulting probability densities
can then be expanded in orthogonal polynomial bases. For
the normally distributed peak height $r(x)$ and curvature
$u(x)$, these are bivariate Hermite polynomials whereas, for
the chi-squared distributed $\eta^2(x)$ and $\xi^2(x)$, these are
generalized Laguerre polynomials. The peak bias factors
are then obtained upon averaging the appropriate orthogo-
nal polynomials over all the spatial locations that satisfy
the peak constraint.

We have demonstrated that our simple local expansion
reproduces the two-point peak correlation function $\xi_{pk}(r)$
computed at second order by Desjacques et al. [4] after a
tedious expansion of the joint probability density
$P_2(\mathbf{y}_1, \mathbf{y}_2; r)$. We believe that it should remain valid at
higher orders. Furthermore, because discreteness enters
the calculation only when averaging the orthogonal poly-
nomials, we speculate that this local bias expansion com-
bined with the peak-background split approach presented
here can be generalized to describe the clustering of any
point process of a Gaussian random field. The great advan-
tage of our approach is that it circumvents the computation
of $P_2(\mathbf{y}_1, \mathbf{y}_2; r)$ and requires only the evaluation of $P_1(\mathbf{y})$.

Our approach can be easily generalized to more sophis-
ticated constraints involving, for instance, the tidal shear
$\partial_i \partial_j \Phi(\mathbf{x})$, where $\Phi(\mathbf{x})$ is the gravitational potential. As
noted in Refs. [19,22], the quadratic invariant $x^2(x)$ [the
equivalent of our $\xi^2(x)$ but with $\delta$ replaced by $\Phi$] follows
a $\chi^2$ distribution with 5 degrees of freedom. Therefore,
we expect that its associated bias parameters are given by
some suitable average of the Laguerre polynomials
$L^{(3/2)}_k(x)$. If the (nonspherical) collapse occurs at the spatial
location of density peaks or includes the dependence on the
large scale environment, then the $\chi^2$ distribution will be
replaced by the appropriate conditional probability density
[37,38], to which we shall apply the peak-background split
in order to read off the new bias parameters.

Corrections induced by nonlinear gravitational evolution
can also be decomposed into rotational invariants [39–41].
Therefore, if one ignores the diffusion kernels (i.e., the
propagators introduced by Catelan et al. [42]), then it is
straightforward to find explicit expressions for the Eulerian
bias parameters in terms of local and nonlocal Lagrangian
bias factors [19,43–45]. This procedure can clearly be
applied to our effective bias expansion Eq. (62), with the
important caveat that discrete density peaks exhibit a sta-
tistical velocity bias [9]. Notwithstanding this, we expect
from the structure of the $F_2$ kernel that the bias factors $\chi_{ij}$
remain constant with time, in agreement with the findings
of Desjacques et al. [4]. For a more realistic treatment of
gravitational motions, it should be possible to compute the
evolved two-point peak correlation $\xi_{pk}(r, z)$ in the frame-
work of the integrated perturbation theory proposed by
Matsubara [27]. We leave all this to future work.

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APPENDIX A: COMPUTING CORRELATORS

For illustration, we evaluate the cross-covariance $\langle \eta^2_1 \eta^2_2 \rangle$
of the square modulus of the gradient $\eta^2(x)$ at two different
spatial locations $\mathbf{x}_1$ and $\mathbf{x}_2$. We have

$$\langle \eta^2_1 \eta^2_2 \rangle = \frac{1}{\sigma_1^2} \left\{ \prod_{i=1}^{d} \int \frac{dk_i}{(2\pi)^3} \right\} (k_1 \cdot k_2)(k_3 \cdot k_4) \delta_s(k_i)$$
$$\times \delta_s(k_3) \delta_s(k_4) e^{(k_1+k_2)x_1 + i(k_1+k_3)x_2}\right.$$  
$$= 1 + 2 \frac{1}{\sigma_1^4} \int \frac{dk_1}{(2\pi)^3} \int \frac{dk_2}{(2\pi)^3} (k_1 \cdot k_2)^2 P_1(k_1)$$
$$\times P_1(k_2) e^{i(k_1+k_2)x}$$
$$= 1 + \frac{2}{\sigma_1^4} \delta_{ij} \delta_{lm} J_{ij}(r) J_{lm}(r),$$

where

$$J_{ij}(r) = \int \frac{dk}{(2\pi)^3} k_i k_j P_1(k) e^{ik \cdot r}. $$
To evaluate $J_{ij}(r)$, we express $k_i$ in terms of the components $\hat{k}_i = k_i/k$ of the unit vector and take advantage of the fact that the integral over the angular variables is

$$
\frac{1}{4\pi} \int d\Omega \hat{k}_i \hat{k}_j e^{i\hat{k}_r \hat{r}} = \frac{1}{3} (j_1(kr) + j_2(kr)) \delta_{ij} - j_2(kr) \hat{r}_i \hat{r}_j.
$$

(A3)

Therefore, the product $\delta_{ij} \delta_{lm} J_{ij}(r) J_{lm}(r)$ becomes (we omit the $r$ dependence for conciseness)

$$
\delta_{ij} \delta_{lm} J_{ij} J_{lm} = \delta_{ij} \delta_{lm} \left[ \frac{1}{3} (\xi_0^{(1)} + \xi_2^{(1)}) \delta_{il} - \xi_2^{(1)} \hat{r}_i \hat{r}_l \right] 
\times \left[ \frac{1}{3} (\xi_0^{(1)} + \xi_2^{(1)}) \delta_{jm} - \xi_2^{(1)} \hat{r}_j \hat{r}_m \right]
= \frac{1}{3} \left( \xi_0^{(1)} + \xi_2^{(1)} \right)^2 - \frac{2}{3} \xi_2^{(1)} (\xi_0^{(1)} + \xi_2^{(1)}) + (\xi_2^{(1)})^2
= \frac{1}{3} \left[ (\xi_0^{(1)})^2 + 2(\xi_2^{(1)})^2 \right],
$$

(A4)

which yields Eq. (29) once the multiplicative factor of $(2/\sigma_1^2)$ and the additive zero-lag contributions are accounted for.

**APPENDIX B: SHAPE FACTOR FOR PEAKS**

In terms of the ordered eigenvalues $\lambda_1 \geq \lambda_2 \geq \lambda_3$ of the Hessian matrix $\xi_{ij}$, the asymmetry parameters that quantify the departure from a spherically symmetric peak density profile are $\nu = (\lambda_1 - \lambda_3)/2$ and $w = (\lambda_1 - 2\lambda_2 + \lambda_3)/2$. The peak constraint together, with our choice of ordering, impose the four conditions $\nu \geq 0$, $-\nu \leq w \leq \nu$, $(u + w) \geq 3\nu$, and $u \geq 0$. Following [3], we also introduce an auxiliary function that measures the degree of asphericity expected for a peak,

$$
F(u, \nu, w) = (u - 2w)[(u + w)^2 - 9nu^2]v(u^2 - w^2).
$$

(B1)

This function scales as $\propto u^3$ in the limit $u \gg 1$.

In Ref. [4], the peak two-point correlation up to second order (i.e., terms quadratic in the correlation of the density field and its derivatives) is written as the sum of the linear contribution and three second-order terms $\xi^{(2i)}_{ik}$, $i = 1, 2, 3$.

In particular, $\xi^{(2i)}_{ik}$ contains all the terms for which the $\nu$ dependence cannot be expressed as a polynomial in the linear and quadratic bias parameters $b_{ij}$. Their expression is phrased in terms of

**APPENDIX C: NONCENTRAL CHI-SQUARED DISTRIBUTIONS**

The probability density of a noncentral chi-squared distribution $\chi^2_k(x; \lambda)$ with $k$ degrees of freedom and noncentrality parameter $\lambda$ is given by

$$
\chi^2_k(x; \lambda) = \frac{e^{-(x+\lambda)/2}}{2^{k/2} \Gamma(k/2)} L_n^k\left(\frac{x}{2}\right),
$$

(C1)

where $\alpha = k/2 - 1$ and $I_n(x)$ is a modified Bessel function of the first kind. Reference [26] proposed the following Laguerre polynomial expansion,

$$
\chi^2_k(x; \lambda) = \frac{e^{-x/2}}{2} \left(\frac{x}{2}\right)^\alpha \sum_{j=0}^{\infty} \frac{(-\lambda/2)^j}{\Gamma(k/2 + j)} L_j^k\left(\frac{x}{2}\right).
$$

(C2)

The noncentral $\chi^2$ distribution can also be represented as a Poisson-weighted mixture of central $\chi^2$ distributions (this was used by, e.g., Ref. [19] to estimate the nonlocal Lagrangian bias induced by ellipsoidal collapse). Note, however, that this representation does not make apparent the connection with the bias parameters $\chi_{ij}$. 

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