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Reference


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CONTINUOUS ANALYSIS OF ONE YEAR OF SCIENCE STUDENTS' WORK, IN LINEAR ALGEBRA, IN FIRST YEAR OF FRENCH UNIVERSITY

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Abstract

Linear algebra is one of the newest fields students discover in their first year at university. Its abstract nature is often a problem for them. We wanted to know if notions in basic logic are prerequisites to succeed in linear algebra, and if yes what kind of previous abilities are needed. We wanted as well to have a better appreciation of what teaching linear algebra consists of and what kind of effects it produces on students. In the following article we describe the methodology employed to analyse results of standard first year science students in a pretest about basic logic and algebra notions, and in all the tests given to them in linear algebra during a year. Then we try to answer the questions raised above with help of this analysis and its statistical results. Finally we will propose a new organisation of teaching linear algebra according to our hypotheses.

1- Introduction

The research presented in this paper is based on the analysis of results of the tests given all the year through, in the field of linear algebra, to students in their first year at a French university.

Our main goals were:
- To better determine what teaching linear algebra consisted of, especially through the analysis of tasks proposed to students, within the questions given in the tests.
- For a standard section of first year science students with a fairly standard teaching of linear algebra, we wanted to determine the methods, procedures and mistakes of students in relation to the tasks proposed to them and relatively to their individual previous abilities in basic logic and algebra notions.
- Being then able to draw a diagnosis on the different effects of this teaching, we may propose some hypotheses for its possible change.

2- The methodology and the hypothesis

We analysed copies of eight different tests.

We first took eighty-four copies from a pretest on basic notions in logic and algebra. This had been given to students in their first weeks at university, before any specific teaching in these fields. The evaluation of this test gave us the individual level of acquisition to what we thought may be prerequisite for linear algebra.

For the analysis itself, we used a methodology introduced by A. Robert and F. Boschet, in their work on the acquisition of real analysis notions in first year at a science university ( [1] et [2] ).
Among the questions on the test, we sorted out four main types: quantification (QA), implication and equivalence (EQ), numerical algebra (AN) and algebraic structure (AS). For the first two types, we distinguished three different levels, in the tasks induced by the questions. The first one is a purely formal setting, but seen from the outside, since it is asked to say whether a proposition expressed in formalised language is true or false (QA1 and EQ1). The second one is formal as well, but the task is this time internal, since it is asked to give the negation of a formalised proposition. Only the QA-type of questions appeared at this level (QA2). The last level corresponds to an interplay between the formal setting and another setting, in the meaning introduced by R. Douady [3]. The questions, this time, consist in translating a proposition from a formalised language into an every-day or a graphic formulation, or vice-versa (QA3 and EQ3).

So we obtained seven different types of questions, which we can associate to seven different bodies or "blocks" of knowledge.

The hypothesis we made and which is induced by Piaget's work, is, in outline, that the acquisition of new knowledge is usually made possible by the destabilisation of old knowledge followed by its reorganisation through complex cognitive mechanisms of destabilisation/reequilibration. The necessary destabilisation is not usually part of the explicit teaching, and the process described above is then of course unconscious. Yet, R. Douady [3] showed that (at least for primary school pupils), didactical situations, in which a notion, meant to be taught, may be seen in at least two different settings, in which the pupils have different levels of ability, is suitable to start this dialectical process in good conditions.

After A. Robert's and F. Boschet's work ([1] and [2]), we think that former knowledge, efficient in different settings, may be a better guarantee for the acquisition of a new notion. More precisely we may raise such questions as: will a student who is very good at formal logic (EQ1, QA1 and QA2) but not very good at dealing with the interaction between formal and natural languages, learn linear algebra less well than a student, who is globally of the same level in logic, but having more homogeneous abilities? For every prerequisite block of knowledge, is there a minimal threshold of acquisition beyond which the probability of success is much higher?

To be able to answer these questions, we have defined for every block, three different states: full (2), half-full (1), empty (0), according to a mark given to the questions related to it. We have also considered the parameter B, giving the number of empty blocks, which mesure the number of gaps in previous abilities.

We then obtained nine different variables (the global mark of the test, the seven blocks, and the number of empty blocks), which evaluate the level of acquisition of basic logic and algebra notions, for each student. A statistical study of the results of the tested population, led us to build a new block Q, with the QAi, to summarise the level of different abilities in questions dealing with quantification.

We finally kept only the following nine variables, which we show to be the most significant ones: the global mark N, the number of empty blocks B and seven variables
(being 1 or 0) for the blocks: EQ1 (2) (full), EQ3 (2), EQ3(0) (empty), Q(2), Q(0), AN(2), AS(2). This seemed to be, with minimal loss, the best way to keep information compact enough and suitable to our further purpose.

Nevertheless in addition to the specific methodology developed here, some restrictions about the test itself, are to be considered, to give the real value of this evaluation, which is of course only a partial way of considering the contents as well as the level of acquisition of preriquired notions in basic logic and algebra. Indeed, if the questions about logic, in the test, seem to be suitable, although necessarily incomplete, the ones about algebra appeared to be less satisfying: numerical questions were a bit too imprecise to give a good evaluation and the ones about structure were too "cultural" to give a real idea of the level of acquisition (for instance: asking someone to give an example of a group is not enough to evaluate his knowledge about groups).

The seven other tests were: four "ordinary" two-weekly tests, the mid-term and the final exam, and a special mid-course true/false-test. Except for the last one all these tests included questions on real analysis subjects.

For each of these tests, we made an a-priori analysis, which includes an explanation of the tasks induced by the questions and the different procedures that could possibly be developed by students. We then gave the statistical results, with marks given to every question and codes to identify special procedures, which we gathered in a table, whose arrays represent the students. We also obtained a global mark for each test. We divided every sample into three categories, according to these marks, we managed to balance the distribution numerically.

We analysed, in this order, thirty-nine papers from the first ordinary test (T1), seventy-four from the mid-term exam (E1), forty-six from T2, fifty-eight from the true/false-test (TF), fifty-eight from T3, fifty from T4 and seventy-three from the final exam (E2). Apart from the mid-term and final exams, none of these tests were compulsory, besides we had to photocopy the papers in the short time while the correctors had them; these two material reasons explain the difference in numbers of papers analysed for each test. In the end, we got unfortunately only nineteen complete sets of papers of the eight tests. Each paper analysed corresponds to a student whose pretest we have analysed anyway, so that the students analysed at each test form a sample of the main population analysed for the pretest.

For each test, we made a short analysis of the new data obtained for the pretest with the new sample. We compared the mean-value of marks, their standard deviation, the percentages of students having EQ1(2), EQ3(2), EQ3(0), Q(2), Q(0), AN(2), AS(2), B=0, B≤1; B≤2, with the equivalent data for the whole population. In each case, we noticed only little variations, which always have rather obvious explanations. The samples imposed on us under material circumstances seem then to be representative enough of the whole population, to give a certain validation to our general conclusions.

For every test we finally gave a crossed table, giving for each of the three different groups of students defined by the mark of the test, the mean value and standard deviation.
from the pretest, and the distribution of the same ten variables as above. We gave a table with percentage on the line and one with percentage on the column, which gave an easily read representation of the correlations between each test and the pretest.

Finally, we analysed more precisely the results of all the tests (including the pretest) for the nineteen students, whose eight papers we had. We made several factorial analyses (Analyse en Composantes Principales) of some of the different characters defined on the sample, although the small number of students did not allow us to make a real statistical analysis. Nevertheless, we got quite a lot of information on every students, which would not have been possible with too many students. More over, we took the results as they appeared in a real teaching situation, with all its complexity. This kind of analysis, for linear algebra had not been made before, as far as we know, in France. So we claim that our work, was a necessary step before carrying out a statistical analysis over many more students. To be able to look at the correlations between the different components of the knowledge in linear algebra over a large statistical population (a few hundred), we have to be more familiar with the contents of the teaching, the different tasks and procedures involved in linear algebra, and we must be able to draw some hypotheses that will help us to build tests according to them. We hope that the kind of analysis, we propose, meets these aims.

3 - The results

a - Global analysis of the contents of the teaching

In most French universities, first year students in science classes follow a two-hour-a-week course over one semester, which represents more or less a fourth of their annual teaching in mathematics. The course usually starts with the axiomatic definition of a vector space, and finishes with the results about diagonalisation of matrices. This is of course an average estimation. In fact linear algebra having completely disappeared from secondary teaching, even for geometry, a new tendency consists, in first year at university, of teaching a bit less abstract linear algebra and a bit more linear algebra for geometry.

The abstract part of this teaching is usually feared by students, because of its esoteric nature and by teachers, because of the bare obviousness of most reasonings, which leaves them without arguments faced with their students' incomprehension.

On another hand, a historical study (cf also [4]), has confirmed us in the idea that linear algebra is a simplifying and unifying concept. For this reason, it is usually very difficult, if not impossible (?), to find "the suitable problem" to introduce a notion related to it, as we would like to do according to G. Brousseau's "Théorie des situations" [5] or R. Douady's "Dialectique outil/objet" [3]. There is no problem, except a few, far too complicated for students, for which linear algebra is an absolute necessity. Besides, even if linear notions give a more elaborated or a more general answer to a problem, it is often too subtle for students to realise, because they already have many difficulties in using concepts, which they are not familiar with, to be able to have a critical look at their work.

This nature, quite specific to linear algebra, leads to a dichotomous attitude in teaching, which is reflected in two different kinds of problems.
The problems of the first kind present applications of linear notions to questions about polynomials, functions or series... They include interplay between different settings, and change of point of view. Most of them are both real problems and good illustrations of the simplification and generalisation given by solutions using linear algebra, but only to someone who has first no difficulty in using linear notions and who is secondly quite familiar with the subject involved. For instance most of the problems of interpolation with polynomials have very elegant and generalisable solutions with use of the theory of vector spaces, but one needs to have quite a lot of calculations to do, to see the simplification given. Besides, in those problems one usually needs to obtain a lot of results, before being able to reach the first questions really concerning linear algebra. So if such problems are given to students, one may have to deal with the following two difficulties:

1) The use of linear algebra will be only an effect of the didactical contract, as it is not absolutely necessary to solve the problem and the students cannot appreciate the simplification it provides. Students will follow the process of resolution induced by the questions even if they see a solution not using linear algebra.

2) The first questions necessary to approach the linear questions may need so many abilities in different fields that only a few students will manage to answer the questions dealing with the notions of linear algebra. The evaluation of the result of such problems is then more on these questions than on linear algebra.

In the second kind of problems, linear concepts are used in a formal setting without interplay with any other setting. Those might be either very formal and difficult questions about subtle notions, such as supplementary spaces..., or on the contrary mechanical use of algorithms such as the search of eigenvalues and eigenvectors of a matrix... In the first case they give useful results, although very hard to obtain, in the second case they are only training for calculation and easily evaluated contents for tests!... These problems do not use "real" vector space, but very general ones, mostly \( \mathbb{R}^n \).

In our analysis, tests T1, E1 and T2, are of the first kind.

E1 is a typical example. The goal of the problem was to obtain Gregory's formula, which gives a polynomial in terms of the values of the \( P(n+1) - P(n) \) \( n = 0 \) to \( \text{deg}(P) \). There is a very attractive solution, through the study of the operator \( D : P \rightarrow Q \) s.t. \( Q(X) = P(X+1) - P(X) \). Yet it is quite long and difficult, it is then really useful only for theoretical reasons or if you need to calculate quite a lot of polynomials. In fact, most of the students didn't succeed in proving all the steps leading to the formula, mostly through lack of technical ability in algebraic calculations. But when they were asked in the last question, to find the polynomials of degree less than three, whose values in 0, 1, 2 and 3 were given, although Gregory's formula had been given, they used a direct method and solved a system of four linear equations with the four coefficients of the polynomial as unknown!

In T1, the question was to find the polynomials of degree less than four, whose values in 0 and 1, as well as the ones of the derivated polynomial were given. The solution induced by the test, was to first find the polynomials, whose all four known values are 0, and then to
deduce the general solution by addition of any solution, for instance the one of the third degree. Of course the first question is obvious, for such polynomials can be divided both by $X^2$ and $(X-1)^2$, but most of the students did not realise that, and again solved a system of four equations with five unknowns! As they have to solve another system to find the solution to the third degree, they ended with more calculations, plus a theoretical proof, than if they had directly solved the system of four linear equations given by the conditions.

In T2, the questions preparing the linear solution were so technical (they used polynomials with two variables), that hardly no students had a chance to answer any question about linear algebra.

It is clear that there is a real difficulty here. We think that such problems should be introduced by explicit metamathematical approach and that the "technical" points they raise in the field of algebraic calculation or logical reasoning should not be underestimated.

TF, T3, T4 and E2 are of the second kind. T3, T4 and E2 are mostly applications of numerical algorithms about the search for eigenvalues and eigenvectors, diagonalisation or reduction to a triangle form of matrices ... But in T3 and E2, we find also some very theoretical questions. The true-false test is typically about abstract notions, although nearly all of them refer to $\mathbb{R}^3$. It would be too long here to develop all the results to this quite specific test, it shows in outlines that formal questions about basic notions of linear algebra such as linear independance, generating subsets, supplementary etc... bring to light some sharp misunderstandings from students.

Generally one of the most obvious difficulties for students in all tasks about linear algebra is to be able to keep control of what they are doing. This goes from the confusion between variables and parameters in the resolution of linear systems and leads to one of the best illustrations of it: in E2, students were asked to find an orthogonal basis of eigenvectors, after they found three eigenvectors of two different eigenvalues, they proved, in all details that they were independent, without shortening the proof to the independance of the only two of same eigenvalue, and then that they were orthogonal.

**b - The main statistical results**

The factorial analysis on the seven series of marks (all but the pretest’s) for the nineteen students reveals two sets of tests: T1, E2 and T4 on one side and T2, TF and T3 on the other side, E2 being just between those two groups.

This is a different distribution to above. These two groups separate the calculating tasks from the more conceptual tasks. Indeed in the first group of tests there were quite a lot of resolutions of linear systems, asked explicitly (T1 and E2) or appearing as the suitable ways to solve questions (determinations of polynomials, eigenvalues or eigenvectors...). T4 consists mainly in the use of algorithms for calculations with matrices. T1 and E2 use also algebraic calculation notions for polynomial or integral. On the other hand T2 and T3 and mostly TF deal with more conceptual problems. The final exam seems to be quite a well-balanced compromise of the two.
This separation is given by the second factorial axis of the analysis, the first one separates the globally successful students from the ones who failed. The distribution of students on the first factorial plan is quite harmoniously spread out, which seems to induce that both numerical and conceptual abilities are useful, but independent, to succeed in linear algebra. For instance, it shows that students can globally succeed in linear algebra, without having a good conceptual basis. For instance they can find the triangle form of a matrix without having a good knowledge of the concept of supplementary subspace, although it is a basic notion for the theory of matrices' reduction. This points out a contradiction in teaching linear algebra. The choice made in most French universities' curricula to teach linear algebra from the definition of a vector space to the diagonalisation of matrices all in one year, induces a restriction in the teaching of basic concepts to the benefits of more easily taught and evaluated notions such as reduction of matrices. This is of course the effect of the difficulties and the failure encountered in the teaching of abstract notions.

**c - Correlations with the pretest**

The correlations with the pretest are globally quite strong. The main correlation appears with the number of empty blocks. This confirms our hypothesis about the existence of a minimal threshold in the acquisition of previous abilities beyond which chances of success in linear algebra are greater. The AN and AS blocks are not very correlated, and among the blocks related to logic, Q is the most correlated of all. These results seem to show that a certain level of previous abilities in basic logic, mainly abilities in the use of quantification, is required to reach a minimal success in linear algebra.

But some results of the more detailed correlations are a bit surprising.

For instance in the true/false test, there were the two following propositions given for any linear map \( f : \mathbb{R}^3 \rightarrow \mathbb{R}^3 \):

- If \((U, V)\) are two independent vectors of \( \mathbb{R}^3 \), then \((f(U), f(V))\) are also independent.
- If \((f(U), f(V))\) are independent, then \((U, V)\) are also independent.

Many students got mixed up in the use of the definition of independent vectors so that they say exactly the contrary of what was true. It first seemed to be a difficulty related to logical notions about quantification and implication. But it appeared that it is only slightly correlated with the results to the pretest. Other similar phenomena may be noticed in our analysis. This leads us to think that logical difficulties specific to linear algebra might exist, and cannot be solved by any former teaching in logic.

The last step in our analysis was to reorganise our data in terms of several tasks as well as procedures in the different fields of linear algebra. We defined 23 variables and made factorial analyses on several groups of them. This gave us answers or enlargement to local hypotheses and the results, which could not be easily summarized, and would take too long to be developed here. We'll try to fill in this gap during the oral presentation.

**4 - Conclusions - Outlook**

Presented in so few pages, this work may seem very disorganised and partial. It compiled quite a large amount of data and had to deal with a field, which was nearly
unexplored by didacticians, so if the conclusions it drew are incomplete, it is nearly by necessity.

We have now to answer our last goal. If notions in basic logic seem to be prerequisite for linear algebra, it seems that prerequisites extend to more general abilities in different areas of algebraic calculations such as polynomials, integral or differential calculus etc... which are not necessarily part of standard mathematical teaching for first year science students, and may have specific aspects in linear questions. As some logical problems seem to be specific to linear algebra as well, we propose the following reorganisation of the teaching of linear algebra.

The first step would be to teach only basic notions but over a longer period and quite separated in time from the calculations with matrices, which could be only a further part of the teaching, not necessarily in the same year.

This first part should be illustrated through the solving of several problems dealing with varied vector spaces. In those problems, there should be an explicit metamathematical approach, in which the student should have an active participation (like comparing two or more different ways to obtain the same solution with or without linear algebra). It should include as well the explicit teaching of logic and algebraic calculation notions useful to solve the problems.

We think that this could help to reduce the difficulties raised by abstraction as logic would be part of the explicit teaching. Finally it seems to be a more satisfying approach from the epistemological point of view, as basic concepts would really appear as unifying and simplifying notions used in various fields in which the students will be given sufficient abilities.

The content of this paper is developed in our doctoral thesis, that should be defended and published by the end of the year. This thesis will also include a historical presentation of the emergence of linear algebra basic concepts and some elements for a new teaching approach.

References and bibliography:

[1] F. BOSCHET and A. ROBERT : " Acquisition des premiers concepts d'analyse sur R dans une section ordinaire de première année de DEUG." Cahier de didactique des mathématiques n°7. IREM de Paris VII.


