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SOME GROUPS WHOSE REDUCED C*-ALGEBRAS HAVE STABLE RANK ONE

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ABSTRACT.

Un groupe $\Gamma$ de l'un des types suivants possède une C*-algèbre réduite $C_\alpha^*(\Gamma)$ qui est de rang stable 1:

(i) un groupe hyperbolique qui est ou bien sans torsion et non élémentaire ou bien un réseau cocompact dans un groupe de Lie réel connexe simple non compact de rang réel 1 et de centre réduit à un élément;

(ii) une somme amalgamée $\Gamma_1 \ast_H \Gamma_2$ selon un sous-groupe $H$ fini telle qu'il existe $\gamma \in \Gamma$ avec $\gamma^{-1}H\gamma \cap H = \{1\}$.

Les preuves utilisent une propriété de semi-groupe libre, reflétant une abondance de sous semi-groupes libres dans $\Gamma$, et une propriété de $\ell^2$-rayon spectral, qui permet des calculs de rayons spectraux de certains éléments de $C_\alpha^*(\Gamma)$ en termes de $2$-normes.

ENGLISH ABSTRACT

It is proved that, for the following classes of groups, $\Gamma$, the reduced group C*-algebra $C_\alpha^*(\Gamma)$ has stable rank 1:

(i) hyperbolic groups which are either torsion-free and non-elementary or which are cocompact lattices in a real, noncompact, simple, connected Lie group of real rank 1 having trivial center;

(ii) amalgamated free products of groups, $\Gamma = G_1 \ast_H G_2$, where $H$ is finite and there is $\gamma \in \Gamma$ such that $\gamma^{-1}H\gamma \cap H = \{1\}$.

The proofs involve some analysis of the free semigroup property, which is one way of saying that a group $\Gamma$ has an abundance of free sub-semigroups, and of the $\ell^2$-spectral radius property, which says that spectral radius of appropriate elements in $C_\alpha^*(\Gamma)$ may be computed with the 2-norm.

§1. INTRODUCTION AND STATEMENT OF MAIN RESULTS.

Let $A$ be a C*-algebra with unit. We denote by $sr(A) \in \{1, 2, \ldots, \infty\}$ the stable rank of $A$ and by $RR(A) \in \{0, 1, \ldots, \infty\}$ the real rank of $A$, defined respectively in [Ri83] and [BP91]. We recall that one has

(i) $RR(A) \leq 2sr(A) - 1$

(ii) $sr(A) = 1 \iff GL(A)$ is dense in $A$

(iii) $RR(A) = 0 \iff GL(A_{\text{sa}})$ is dense in $A_{\text{sa}}$.

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where $\mathrm{GL}(A)$ [respectively $A_{\text{n.a.}}, \mathrm{GL}(A_{\text{n.a.}})$] denotes the group of invertible elements [resp. the real subspace of self-adjoint elements, the subset $\mathrm{GL}(A) \cap A_{\text{n.a.}}$] in $A$. Let us also recall that $	ext{sr}(C(\Omega)) = \left[ \frac{1}{2} \dim(\Omega) \right] + 1$ and $\text{RR}(C(\Omega)) = \dim(\Omega)$ for the abelian $C^*$-algebra of continuous functions on a compact space $\Omega$ of dimension $\dim(\Omega)$; thus, in general, $\text{sr}(A)$ and $\text{RR}(A)$ may be thought of as “dimensions” of $A$ (or of some dual of $A$).

Several properties follow from the stable rank of $A$ being 1. On the one hand, there are nonstable $K$–theory properties [Ri83], [Ri87], [Br95]: if $p$ and $q$ are projections in $A$ such that $[p]_0 = [q]_0$ in $K_0(A)$, then $p$ and $q$ are both Murray–von Neumann equivalent and homotopic in $A$; furthermore, letting $U(A)$ be the unitary group of $A$ and $U(A)_0$ its connected component containing 1, the natural group homomorphism $U(A)/U(A)_0 \rightarrow K_1(A)$ is an isomorphism. On the other hand, a result of Friis and Rørdam [FR96] shows that almost normal elements of $A$ can be approximated by normal elements of $A$.

For more on these ranks, see among others [Br91], [Br95], [FR96], [HV84], [Ri87], [Vi].

In this paper, $\Gamma$ or $\Gamma_\iota$, etc., will always denote a group taken with discrete topology.

Given a group $\Gamma$, one lets it act on $l^2(\Gamma)$ by left translation: $(\lambda(x)\xi)(y) = \xi(x^{-1}y)$, for $\xi \in l^2(\Gamma)$, $x, y \in \Gamma$. This extends by linearity to a representation of the group algebra, $\lambda: \mathbb{C} \Gamma \rightarrow B(l^2(\Gamma))$, and the reduced group $C^*$–algebra $C^*_r(\Gamma)$ of $\Gamma$ is the closure of $\lambda(\mathbb{C} \Gamma)$ in the norm topology of $B(l^2(\Gamma))$. This $C^*$–algebra has a canonical, faithful trace, $\tau(a) = \langle a\delta_e, \delta_e \rangle$, where $\delta_e$ is the characteristic function of the identity element of $\Gamma$. Any $a \in C^*_r(\Gamma)$ may be written as a sum $\sum_{\gamma \in \Gamma} a_{\gamma} \lambda(\gamma)$, and one has then $\tau(a) = a_1$. We will identify the algebra $\mathbb{C} \Gamma$ with the dense $*$-subalgebra of $C^*_r(\Gamma)$ of finite sums $\sum_{\gamma \in \Gamma}^\text{finite} a_{\gamma} \lambda(\gamma)$.

As a straightforward consequence of the facts recalled above, one has $\text{sr}(C^*_r(\Gamma)) = 1$ and $\text{RR}(C^*_r(\Gamma)) = 0$ for a group $\Gamma$ which is (finite or) locally finite; one has also

$$\text{sr}(C^*_r(\mathbb{Z}^n)) = \left[ \frac{n}{2} \right] + 1 \quad \text{and} \quad \text{RR}(C^*_r(\mathbb{Z}^n)) = n$$

for all $n \geq 1$. The starting point of the present work is the following, where $F_2$ denotes the free group on two generators.

**Theorem 1.1** ([DHR97], [PV82]). One has

$$\text{sr}(C^*_r(F_2)) = 1 \quad \text{and} \quad \text{RR}(C^*_r(F_2)) = 1.$$
The first equality is Theorem 1.1 in [DHR97]; see also [Ro97]. The second follows from
the inequality (i) above and from the fact that $\mathcal{C}_\lambda^*(F_2)$ has no projection besides 0 and 1
([PV82], see also the proof of Section 1.1 in [Co86]), so that $\text{RR} (\mathcal{C}_\lambda^*(F_2)) \neq 0$.

The result in [DHR97] applies more generally to free products $\Gamma = \Gamma_1 \star \Gamma_2$ with $|\Gamma_1| \geq 2$
and $|\Gamma_2| \geq 3$, and this is extended in [Dy] to tensor products of such free products. Our
purpose is to generalize this further, as in Theorems 1.5 and 1.6 below.

Consider a pair $(A, \tau)$ where $A$ is a $C^*$-algebra with unit and where $\tau$ is a faithful tracial
state. For $a \in A$, we define the 2-norm $\|a\|_2 = \sqrt{\tau(a^*a)}$ in the usual way, and the $\ell^2$-spectral
radius

$$r_2(a) = \limsup_{n \to \infty} \sqrt[n]{\|a^n\|_2}.$$

Recall that the usual spectral radius of $a$ is $r(a) = \lim_{n \to \infty} \sqrt[n]{\|a^n\|}$; as $\|x\|_2 \leq \|x\|$ for all
$x \in A$, one has clearly $0 \leq r_2(a) \leq r(a)$.

Recall that a subset $F$ of a group $\Gamma$ is free if the subgroup of $\Gamma$ generated by $F$ is free
over $F$, i.e. if all reduced words in elements of $F$ and their inverses are nontrivial.

**Definition 1.2.** Let $\Gamma$ be a group and let $F$ be a subset of $\Gamma$.

(i) $F$ is semifree if the subsemigroup of $\Gamma$ generated by $F$ is free over $F$, i.e. if $n, m \in \mathbb{N}$,

$$x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_m \in F \text{ and } x_1x_2\cdots x_n = y_1y_2\cdots y_m \text{ imply that } n = m \text{ and }$$

$$x_i = y_i \text{ for every } 1 \leq i \leq n.$$

(ii) $F$ has the $\ell^2$-spectral radius property if, for every $a \in \text{span} F \subseteq \mathbb{C}\Gamma$, $r(a) = r_2(a)$.

In §4, we will indicate easy examples of subsets of groups which are semifree but not free.
Here is a related notion which we will use.

**Definition 1.3.** A group $\Gamma$ has the free semigroup property if for every finite subset, $F \subseteq \Gamma$,
there is $\gamma \in \Gamma$ such that $\gamma F \overset{\text{def}}{=} \{ \gamma x \mid x \in F \}$ is semifree.

**Theorem 1.4.** Let $\Gamma$ be a group and suppose that for every finite subset $F \subseteq \Gamma$, there is
$\gamma \in \Gamma$ such that $\gamma F$ is semifree and has the $\ell^2$-spectral radius property. Then its reduced group
$C^*$-algebra, $C^*_\lambda(\Gamma)$, has stable rank equal to 1.

The proof amounts to extracting some essential features of the proof of [DHR97, Theorem 3.8]; see §2. Theorem 1.1 follows from Theorem 1.4 because non-abelian free groups have the
free semigroup property (a straightforward fact to check) and the $\ell^2$-spectral radius property
(a consequence of [Haa79]).
A straightforward consequence of Theorem 1.4 is that, if $\Gamma$ is a group with both the $\ell^2$–spectral radius property and the free semigroup property, then $C^*_\lambda(\Gamma)$ has stable rank one. After our study of the $\ell^2$-spectral radius property in §3 and of the free semigroup property in §4, we obtain the following, which is one of our main results, about the hyperbolic groups of Gromov [Gr87].

**Theorem 1.5.** Let $\Gamma$ be a hyperbolic group and suppose that $\Gamma$ is either torsion free and non–elementary or that $\Gamma$ is a cocompact lattice in a real, noncompact, simple, connected Lie group, $G$, where $G$ is of real rank 1 and has trivial center. Then

$$sr(C^*_\lambda(\Gamma)) = 1.$$  

In §5 we apply Theorem 1.4 to some groups, $\Gamma = G_1 *_H G_2$, which are free products with amalgamation. (We always take $H \subset G_i$, $H \neq G_i$; see §5 for definitions.) This yields our other main result, which is the following theorem.

**Theorem 1.6.** Let $\Gamma = G_1 *_H G_2$ be an amalgamated free product of groups with $H$ finite. Suppose that there is $g_0 \in \Gamma$ such that

$$(3) \quad g_0^{-1}Hg_0 \cap H = \{1\}.$$  

Then

$$sr(C^*_\lambda(\Gamma)) = 1.$$  

Note that if $\Gamma$ is an amalgamated free product of groups satisfying the hypotheses of Theorem 1.6 and if $H$ is nontrivial, then the $C^*$–algebra $C^*_\lambda(\Gamma)$ is simple and has unique tracial state, by [Ha85] and Proposition 5.1. Of course, there are amalgamated free products of groups $\Gamma = G_1 *_H G_2$ with $H$ finite, where no $g_0 \in \Gamma$ makes (3) hold, and yet $C^*_\lambda(\Gamma)$ has stable rank one. For example, if $G_i = G'_i \times H$, then $\Gamma = \Gamma' \times H$, where $\Gamma'$ is the free product of $G'_1$ and $G'_2$, and hence $C^*_\lambda(\Gamma) \cong C^*_\lambda(\Gamma') \otimes C^*(H)$. But $C^*_\lambda(\Gamma')$ has stable rank one by [DHR97], (or using Theorem 4.11 of [Ri83] if $G'_1$ and $G'_2$ each have order two), and hence, since the class of $C^*$–algebras of stable rank one is closed under tensoring with matrix algebras and under direct sums (see [Ri83]) it follows that $C^*_\lambda(\Gamma)$ has stable rank one.

**Open Problem 1.7.** If $\Gamma = G_1 *_H G_2$ is an amalgamated free products of groups with $H$ finite, does $C^*_\lambda(\Gamma)$ necessarily have stable rank one?

Finally, §6 contains remarks and speculations on $RR(C^*_\lambda(\Gamma))$.

We close this introductory section with some more open problems and an observation.
Open Problem 1.8. Let $G$ be a connected real semi-simple Lie group with center reduced to one element and without compact factor, and let $\Gamma$ be a Zariski-dense discrete subgroup of $G$. What is the value of $\text{sr}(C^*_\lambda(\Gamma))$? How does it compare with $\text{sr}(C^*_\lambda(G))$ as computed by Sudo [Su97]?

Open Problem 1.9. Given $n \in \{2, 3, \ldots, \infty\}$, is there a group $\Gamma$ such that $C^*_\lambda(\Gamma)$ is simple and has stable rank $n$? It is presently unknown if any such group exists for any $n \geq 2$. (Compare with [Vi].)

Observation. It is crucial to distinguish the above reduced group C*-algebras from other ones. For example, for the maximal C*-algebra of a nonabelian free group, one has $\text{sr}(C^*_\text{max}(F_2)) = \infty$. (See Theorem 6.7 in [Ri83].)

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§2. Proof of Theorem 1.4.

Proof. Step one. Recall that, for any element $a$ in a C*-algebra $A$ with unit, one has $d(a, \text{GL}(A)) \leq r(a)$, where $d$ denotes the distance defined by $d(x, y) = \|x - y\|$.

Indeed, for any $\epsilon > 0$, one has $b \overset{\text{def}}{=} a - (r(a) + \epsilon) \in \text{GL}(A)$ and $\|a - b\| = r(a) + \epsilon$, so that $d(a, \text{GL}(A)) \leq r(a) + \epsilon$.

Step two. For a group $\Gamma$ and an element $c = \sum_{x \in X} c_x \lambda(x) \in C\Gamma$ where $X$ is a semifree subset of $\Gamma$, one has $r_2(c) = \|c\|_2$.

Indeed, for any integer $n \geq 1$, one has $c^n = \sum_{y \in X^n} c_y \lambda(y)$ with $c_y = c_{x_1} c_{x_2} \ldots c_{x_n}$ whenever $y = x_1 x_2 \ldots x_n \in X^n$, so that $\|c^n\|_2 = (\|c\|_2)^n$, and consequently $r_2(c) = \|c\|_2$.

Step three (a rephrasing of the proof of Theorem 1.1 in [DHR97]). Let $\Gamma$ be as in Theorem 1.3 and set $A = C^*_\lambda(\Gamma)$. Suppose for contradiction that the set $\text{GL}(A)$ of invertible elements is not dense in $A$. By [Ro88] or [Ro97], there is $a \in A$ such that $\|a\| = 1$ and the distance from $a$ to $\text{GL}(A)$ is 1. If one had $\|a\|_2 = 1$, namely $\tau(a^*a) = 1$, one would have $a^*a = 1$ by faithfulness of the trace $\tau$, and $a$ would be unitary; as $a$ is not invertible by hypothesis, one has $\epsilon \overset{\text{def}}{=} 1 - \|a\|_2 > 0$. 
Let $b = \sum_{x \in X} b_x \lambda(x) \in C\Gamma$, where $X \subset \Gamma$ is the support of $b$, be such that $\|b - a\| < \frac{1}{3} \epsilon$. Then $d(b, \text{GL}(A)) \geq d(a, \text{GL}(A)) - \frac{1}{3} \epsilon = 1 - \frac{1}{3} \epsilon$ and

$$\|b\|_2 \leq \|a\|_2 + \|b - a\| \leq 1 - \epsilon + \frac{1}{3} \epsilon < 1 - \frac{1}{3} \epsilon \leq d(b, \text{GL}(A)).$$

By assumption, there exists $\gamma \in \Gamma$ such that $Y \overset{\text{def}}{=} \gamma X$ is semifree and has the $\ell^2$-spectral radius property. If $c \overset{\text{def}}{=} \lambda(\gamma)b \in C\Gamma$, one has $\|c\|_2 = \|b\|_2$, $d(c, \text{GL}(A)) = d(b, \text{GL}(A))$, and it follows from step two that $r_2(c) = \|c\|_2$. Furthermore, by the $\ell^2$-spectral radius property, one has also $r(c) = r_2(c)$. Consequently, using step one, one has

$$\|b\|_2 < d(b, \text{GL}(A)) = d(c, \text{GL}(A)) \leq r(c) = \|c\|_2 = \|b\|_2$$

which is preposterous. This shows that

$$\text{sr}(A) = 1.$$

\[\square\]

§3. Groups with the $\ell^2$-spectral radius property.

P. Jolissaint has defined a property (RD) for groups; according to [Jo90, 1.2.2], a group $\Gamma$ has property (RD) if, for some length function $L$ on $\Gamma$, for some positive constants $c$ and $s$ and for every $a = \sum_{\gamma \in \Gamma} a_\gamma \lambda(\gamma) \in C\Gamma$, we have

$$\|a\| \leq c \|a\|_{2,s,L} \quad \text{where} \quad \|a\|_{2,s,L} \overset{\text{def}}{=} \left( \sum_{\gamma \in \Gamma} |a_\gamma|^2 (1 + L(\gamma))^{2s} \right)^{1/2}.$$

It follows from Proposition 8(iii) of [HRV93] that property (RD) implies the $\ell^2$-spectral radius property. However, the direct proof is quite easy, so we give it below.

**Proposition 3.1.** (i) Let $\Gamma$ be a group with property (RD). Then for every $a \in C\Gamma$,

$$r(a) = \lim_{n \to \infty} \|a^n\|_2^{1/n}.$$  

In particular, $\Gamma$ has the $\ell^2$-spectral radius property.
(ii) A group has the $\ell^2$-spectral radius property if and only if all its finitely generated subgroups have it.

Proof. (i) Let $L$ be a length function on $\Gamma$ with respect to which $\Gamma$ has property (RD). Then for some $c$ and $s$ and for every $a \in \mathbf{C}\Gamma$,

$$\|a\| \leq c(1 + L(a))^s \|a\|_2,$$

where $L(a) \overset{\text{def}}{=} \max\{L(g) \mid g \in \text{supp}(a)\}$. Now $\|a^n\|_2 \leq \|a^n\|$ and $L(a^n) \leq nL(a)$ for every $n \in \mathbb{N}$. Therefore

$$r_2(a) \leq r(a) = \lim_{n \to \infty} \|a^n\|^{1/n} \leq \lim_{n \to \infty} \inf \epsilon^{1/n} (1 + nL(a))^{s/n} \|a^n\|^{1/2}_2 = \lim_{n \to \infty} \inf \|a^n\|^{1/2}_2 \leq r_2(a).$$

(ii) As the $\ell^2$-spectral radius property of a group $\Gamma$ is formulated in terms of $a \in \mathbf{C}\Gamma$, and as for any such $a$ there exists a finitely generated subgroup $\Gamma_0$ of $\Gamma$ with $a \in \mathbf{C}\Gamma_0 \subset \mathbf{C}\Gamma$, this claim is straightforward.

\[\square\]

Examples 3.2. The following are classes of groups which have property (RD):

(i) finitely generated groups having polynomial growth;

(ii) hyperbolic groups of Gromov;

(iii) subgroups of groups having property (RD), direct products of finitely many groups having property (RD) and free products of finitely many groups having property (RD);

(iv) free products of groups with amalgamation, $\Gamma = G_1 \ast_H G_2$, where $G_1$ and $G_2$ have property (RD) and where $H$ is finite;

(v) free products of groups with amalgamation, $\Gamma = G_1 \ast_H G_2$, where $H$ has property (RD) and is central and of finite index in each of $G_1$ and $G_2$; this class includes the torus knot groups $\Gamma = \langle s, t \mid s^m = t^n \rangle$ where $m, n \in \mathbb{N}$, $(m, n) = 1$.

Proof. See [Jo90] for these and other examples, as well as [Ha88] for (ii).

\[\square\]

Note that by [Jo90, 3.1.8] (or see [Va97]), amenable groups not of polynomial growth fail to have property (RD).

In addition to groups with property (RD), the following groups have the $\ell^2$-spectral radius property, whether they have property (RD) or not.
Proposition 3.3. The following are classes of groups which have the $\ell^2$–spectral radius property:

(i) abelian groups, and more generally nilpotent groups;
(ii) finitely generated groups of subexponential growth;
(iii) direct products and free products of arbitrarily many groups having (RD), and, more generally, inductive limits of groups having the $\ell^2$–spectral radius property.

Proof. Parts (i) and (iii) follow from Proposition 3.1.ii and Examples 3.2. However, let us now give an additional argument for an abelian group, $\Gamma$. Every element of $C^*_\chi(\Gamma)$ is normal, and the state $\tau_\Gamma$ is faithful. Thus the $C^*$–subalgebra generated by $a$ is isomorphic to $C(\text{Sp}(a))$, and the restriction of $\tau_\Gamma$ is integration with respect to a measure having support equal to $\text{Sp}(a)$. Now $(\|a^n\|_2)^{1/n} = \|a\|_2^n$ tends to $\|a\| = \|a\|_\infty$ as $n$ grows without bound.

For part (ii), we will use the following lemma.

Lemma 3.4. Let $\Gamma$ be a group with a finite generating set $S$. Let $L : \Gamma \to \mathbb{N}_0 \supseteq \mathbb{N} \cup \{0\}$ be the associated length function. For $k \in \mathbb{N}_0$, let $\sigma(k)$ be the number of elements, $\gamma \in \Gamma$, such that $L(\gamma) = k$. Let $c = (\sum_{n=1}^{\infty} n^{-2})^{1/2} = \pi/\sqrt{6}$. Then for every $a \in C\Gamma$,

$$\|a\| \leq c \sigma(L(a))^{1/2} (1 + L(a))\|a\|_2,$$

where $L(a) \overset{\text{def}}{=} \max\{L(\gamma) \mid \gamma \in \text{supp}(a)\}$.

Proof.

$$\|a\| \leq \|a\|_1 = \sum_{k \geq 0} \sum_{\gamma \in \Gamma^{L(\gamma) = k}} |a_\gamma|$$

$$\leq \sum_{k \geq 0} \sigma(k)^{1/2} \left( \sum_{\gamma \in \Gamma^{L(\gamma) = k}} |a_\gamma|^2 \right)^{1/2}$$

$$\leq \sigma(L(a))^{1/2} \sum_{k \geq 0} \frac{1 + L(a)}{1 + k} \left( \sum_{\gamma \in \Gamma^{L(\gamma) = k}} |a_\gamma|^2 \right)^{1/2}$$

$$\leq \sigma(L(a))^{1/2} (1 + L(a)) \left( \sum_{k \geq 0} \frac{1}{(k + 1)^2} \right)^{1/2} \left( \sum_{k \geq 0} \sum_{\gamma \in \Gamma^{L(\gamma) = k}} |a_\gamma|^2 \right)^{1/2}$$

$$= c \sigma(L(a))^{1/2} (1 + L(a))\|a\|_2,$$

where in (4) and (5) we have used the Cauchy–Schwartz inequality.

\qed
Proof of 3.3(ii). By the above lemma, for every \( n \in \mathbb{N} \),
\[
\|a^n\| \leq c \sigma \left( nL(a) \right)^{1/2} \left( 1 + nL(a) \right) \|a^n\|_2.
\]
But if \( \Gamma \) has subexponential growth then \( \limsup \sigma(n)^{1/n} = 1 \), so \( r(a) \leq r_2(a) \), which implies \( r(a) = r_2(a) \).

Now 3.3(ii), together with Jolissaint’s result which we mention after Examples 3.2, provide for examples of finitely generated groups having the \( \ell^2 \)-spectral radius property but not property (RD). Note that the example on page 99 of [HRV93] shows that certain amenable groups, namely finitely generated solvable groups which are not almost nilpotent, do not have the \( \ell^2 \)-spectral radius property.

§4. Groups with the free semigroup property.

We show first examples of groups having subsets which are semifree but not free.

(i) If \( \Gamma = \langle x, a \mid a^2 = 1 \rangle \) is the free product of the infinite cyclic group and the group of order two, then \( \{ xa, x \} \) is semifree but not free.

(ii) If \( \Gamma \) is the group of affine transformations of the real line, then the set, \( F \), consisting of \( a : t \mapsto t - 8 \) and \( b : t \mapsto t + 8 \) is not free (because \( \Gamma \) is solvable) but is semifree. Indeed, set \( X = [-3, -1] \), \( Y = [1, 3] \) and \( z = 0 \). One has
\[
a(X \cup Y \cup \{ z \}) \subseteq X \quad \text{and} \quad b(X \cup Y \cup \{ z \}) \subseteq Y.
\]
For a word \( w \) in \( a \) and \( b \), the left–most letter of \( w \) is consequently \( a \) if \( w(z) \in X \) and is \( b \) if \( w(z) \in Y \). It follows that \( F \) is semifree (see also the proof of Lemma 4.3 below).

We will now show that groups which act on topological spaces in a certain way have the free semigroup property. We start with some standard definitions (compare [Ha85, §3]).

**Definition 4.1.** Let \( X \) be a Hausdorff topological space and let \( \phi \) be a homeomorphism of \( X \). We say that \( \phi \) is hyperbolic (or acts hyperbolically on \( X \)) if it has two fixed points, \( s_\phi \) and \( r_\phi \) (the source and the range), which are distinct, and such that given any neighborhoods, \( S \) and \( R \), of \( s_\phi \), respectively \( r_\phi \), we have \( \phi^n(X \setminus S) \subseteq R \) for every \( n \) large enough. Two hyperbolic homeomorphisms are called transverse if they have no fixed points in common.
Definition 4.2. Let $\Omega$ be a Hausdorff topological space and let $\Gamma$ be a group acting on $\Omega$ by homeomorphisms. We say the action is

- **minimal** if every orbit of the action is dense in $\Omega$;
- **strongly faithful** if, for every finite subset $F \subseteq \Gamma \setminus \{1\}$, there is $\omega \in \Omega$ such that $f\omega \neq \omega$ for every $f \in F$;
- **strongly hyperbolic** if there are $g, h \in \Gamma$, each acting hyperbolically on $\Omega$, which are transverse.

Note that the above definition of strongly hyperbolic actions is, on the face of it, weaker than the definition found in [Ha85]. However, the two definitions are equivalent, because from the two transverse hyperbolic homeomorphisms, $g$ and $h$, one can find arbitrarily many pairwise transverse hyperbolic homeomorphisms, $g, h^{k_1}gh^{-k_1}, \ldots, h^{k_n}gh^{-k_n}$, by taking $0 < k_1 < \cdots < k_n$ growing fast enough.

Lemma 4.3. Let $\Gamma$ be a group acting on a Hausdorff topological space $\Omega$. Suppose that the action is minimal, strongly faithful and strongly hyperbolic. Then $\Gamma$ has the free semigroup property.

Proof. Let $F$ be a finite subset of $\Gamma$. We must find $\gamma \in \Gamma$ so that $\gamma F$ is semifree. From the remark after Definition 4.2, it is clear that $\Gamma$ is an infinite group. Hence, multiplying $F$ on the left by a group element, if necessary, we may assume without loss of generality that the identity does not lie in $F$. Let $F = \{f_1, f_2, \ldots, f_n\}$. We may choose an open subset $U \subseteq \Omega$, such that

$$\forall i \quad f_i(U) \cap U = \emptyset$$

$$\forall i \neq j \quad f_i(U) \cap f_j(U) = \emptyset.$$  \hfill (6)

Indeed, set $F' \overset{\text{def}}{=} F \cup \{f_i^{-1} f_j \mid i < j\}$, use the strong faithfulness of the action to find $\omega \in \Omega$ such that $f'\omega \neq \omega$ for every $f' \in F'$, and let $U$ be a sufficiently small open neighborhood of $\omega$.

Now we will find $\gamma \in \Gamma$, acting hyperbolically on $\Omega$ and having source and range in $U$, and such that

$$\gamma(\Omega \setminus U) \subseteq U.$$  \hfill (7)

Let $\gamma_1$ and $\gamma_2$ be transverse hyperbolic elements of $\Gamma$, having sources $s_1$, respectively $s_2$ and ranges $r_1$, respectively $r_2$. Since the orbit of $r_2$ under $\Gamma$ is dense in $\Omega$, there is $x \in \Gamma$ such that $x(r_2) \in U$. Set $\gamma_3 = x\gamma_1 x^{-1}$ and $\gamma_4 = x\gamma_2 x^{-1}$. Then $\gamma_3$ and $\gamma_4$ are transverse hyperbolic, and $\gamma_4$ has range, say $r_4$, in $U$. Since the source and range, $s_3$ and $r_3$, of $\gamma_3$ are distinct from the source, $s_4$, by the hyperbolicity of $\gamma_4$, for some $n$ large enough we have $\gamma_4^n(\{s_3, r_3\}) \subseteq U$. 
Hence \( \gamma \overset{\text{def}}{=} \gamma_4^n \gamma_3 \gamma_4^{-n} \) is a hyperbolic homeomorphism having source and range in \( U \). Replacing \( \gamma \) by some power of \( \gamma \) if necessary, we may assume without loss of generality that (7) holds.

Now we claim that \( \gamma F \) is semifree. Let \( \omega \) be one of the fixed points of \( \gamma \), so in particular \( \omega \in U \). Using (6) and (7), we see that for every \( p \in \mathbb{N} \) and \( i_1, \ldots, i_p \in \{1, \ldots, n\} \),

\[
\gamma f_{i_1} \gamma f_{i_2} \cdots \gamma f_{i_p} (\omega) \in U.
\]

Let \( p, q \in \mathbb{N} \) and \( i_1, \ldots, i_p, j_1, \ldots, j_q \in \{1, \ldots, n\} \) and suppose

\[
\gamma f_{i_1} \gamma f_{i_2} \cdots \gamma f_{i_p} = \gamma f_{j_1} \gamma f_{j_2} \cdots \gamma f_{j_q}.
\]

We will show that \( p = q \) and that \( i_1 = j_1 \), \( i_2 = j_2 \), \ldots, \( i_p = j_p \), using induction on \( \min(p, q) \).

We have

\[
f_{i_1} \gamma f_{i_2} \cdots \gamma f_{i_p} (\omega) = f_{j_1} \gamma f_{j_2} \cdots \gamma f_{j_q} (\omega),
\]

and the left-hand-side of (8) belongs to \( f_{i_1} (U) \) while the right-hand-side belongs to \( f_{j_1} (U) \), so by (6) we must have \( i_1 = j_1 \). If \( p = 1 \) then we must have \( q = 1 \) too, since otherwise we would have \( \omega = \gamma f_{j_2} \gamma f_{j_3} \cdots \gamma f_{j_q} (\omega) \in f_{j_2} (U) \), so \( \omega = \gamma f_{j_2} \gamma f_{j_3} \cdots \gamma f_{j_q} (\omega) \in f_{j_2} (U) \), which cannot be. If \( \min(p, q) > 1 \) then

\[
\gamma f_{i_1} \gamma f_{i_3} \cdots \gamma f_{i_p} = \gamma f_{j_1} \gamma f_{j_3} \cdots \gamma f_{j_q},
\]

and the induction hypothesis applies.

\( \square \)

**Examples 4.4.** The following classes of groups, \( \Gamma \), have actions on Hausdorff topological spaces which satisfy the hypotheses of Lemma 4.3, hence have the free semigroup property:

(i) torsion–free, non–elementary hyperbolic groups \( \Gamma \);

(ii) lattices \( \Gamma \) in connected real Lie groups \( G \) which are simple, noncompact, of real rank 1 and with trivial center;

(iii) free products \( \Gamma = G_1 * G_2 \) of two groups \( G_1 \) and \( G_2 \), where \( G_1 \) has at least two elements and \( G_2 \) has at least three elements;

(iv) More generally, free products \( \Gamma = G_1 *_H G_2 \) with amalgamation over a common subgroup \( G_1 \supseteq H \subseteq G_2 \), with \( H \neq \{1\} \) and the property that, for every finite subset \( F \subseteq \Gamma \setminus \{1\} \), there is \( \gamma \in \Gamma \) with \( \gamma^{-1} F \gamma \cap H = \emptyset \).

**Citations.** For (i) we consider \( \Gamma \) acting on its Gromov boundary, \( \partial \Gamma \). This action is minimal and strongly hyperbolic (see [GH90, 8.27] and [GH90, 8.37]). Moreover, each element of \( \Gamma \) acts hyperbolically on \( \partial \Gamma \) (see [GH90, 8.28]), and consequently has exactly two fixed points. Since \( \partial \Gamma \) is an infinite set (see [GH90, 7.15]), it follows that the action is strongly faithful.
In case (ii) we use the minimality of the action of $\Gamma$ on the sphere at infinity, $G/P = \partial(G/K)$, of the symmetric space $G/K$, where $K$ (respectively $P$) denotes a maximal compact (respectively, minimal parabolic) subgroup of $G$; (see Lemma 8.5 of [Mo73]). The fixed point set in $G/P$ of each element of $\Gamma \setminus \{1\}$ is a submanifold of strictly positive codimension, and it thus follows that the action is strongly faithful.

Note that in case (ii), if $\Gamma$ is cocompact, then it is hyperbolic and $G/P$ is equal to the Gromov boundary, $\partial \Gamma$. Thus if $\Gamma$ is also torsion free, then it falls into both of cases (i) and (ii).

For (iii) and (iv), let $\Gamma$ act on the boundary of the tree defined in [Se77, I.4.1]. The required properties of this action are proved in [Ha85, Prop. 8].

\[ \square \]

**Remark 4.5.** A group $\Gamma$ satisfying the properties of Lemma 4.3 is a Powers group in the sense of [Ha85], and thus $C^*_\alpha(\Gamma)$ is simple and has unique tracial state.

In the next section, we will show that the condition in (iv) above is satisfied if $H \neq G_1$ and $H \neq G_2$ and if there is $g \in \Gamma$ such that $g^{-1}Hg \cap H = \{1\}$. (See [B684] for related results about the reduced $C^*$–algebras of amalgamated free product groups.)

§5. Free products of groups with amalgamation.

Let $I$ be a set and, for $\iota \in I$, let $G_\iota$ be a group having presentation

$$G_\iota = \langle x_1^{(\iota)}, x_2^{(\iota)}, \ldots ; r_1^{(\iota)}, r_2^{(\iota)}, \ldots \rangle.$$

Let $H_\iota \subseteq G_\iota$ be subgroups ($\iota \in I$) and suppose for some fixed $\iota_0 \in I$ there are isomorphisms $\phi_\iota : H_\iota \to H_{\iota_0}$ (with $\phi_{\iota_0} = \text{id}$). Then the free product of $(G_\iota)_{\iota \in I}$ with amalgamation over $(H_\iota)_{\iota \in I}$ by the isomorphisms $(\phi_\iota)_{\iota \in I}$ is the group with presentation

$$\Gamma = \langle (x_1^{(\iota)}, x_2^{(\iota)}, \ldots)_{\iota \in I}; (r_1^{(\iota)}, r_2^{(\iota)}, \ldots)_{\iota \in I}, (h = \phi_\iota(h))_{h \in H_{\iota_0}, \iota \in I} \rangle.$$

We will abbreviate this by writing

$$\Gamma = (\ast_{\iota \in I} G_\iota), \quad (9)$$

or, if $I = \{1, 2\}$ by $\Gamma = G_1 \ast_H G_2$, where it is understood that $H$ is a group that is identified with the subgroups $H_\iota$ in a way that is compatible with the isomorphisms $\phi_\iota$. We will always assume that $H \neq G_\iota$ for every $\iota \in I$. 
Given an amalgamated free product of groups (9), we fix $X_i \subseteq G_i$ so that $X_i \cup \{e\}$ is a set of coset representatives for the left cosets of $H$ in $G_i$. Then every $g \in \Gamma$ has a unique normal form

$$g = x_1 x_2 \cdots x_n h$$

for $h \in H$, $n \in \mathbb{N} \cup \{0\}$ and $x_j \in X_{i_j}$ where $\iota_1 \neq \iota_2, \iota_2 \neq \iota_3, \ldots, \iota_{n-1} \neq \iota_n$; (see [LS77]). (Actually, this is a bit sloppy: the normal form is more correctly taken to be the list $(x_1, x_2, \ldots, x_n, h)$; but we trust no confusion will result.) We define the length of $g$ to be $L(g) = n$ and for $1 \leq j \leq n$ we let $\iota_j(g) = \iota_j$ where $x_j \in X_{i_j}$.

We now show that the condition in 4.4(iv) is satisfied under mild conditions.

**Proposition 5.1.** Let $\Gamma = G_1 *_H G_2$ be a free product of groups with amalgamation, where $H$ is nontrivial. Suppose that there is $\gamma \in \Gamma$ such that $\gamma^{-1} H \gamma \cap H = \{1\}$. Then for every finite subset, $F$, of $\Gamma \backslash \{1\}$, there is $g_0 \in \Gamma$ such that $g_0^{-1} F g_0 \cap H = \emptyset$. Hence $\Gamma$ has the free semigroup property.

**Proof.** Let $k \in \mathbb{N}$ be such that $2k > \max\{L(g) \mid g \in F\}$. Let $x \in G_1 \backslash H$ and $y \in G_2 \backslash H$. Without loss of generality assume that $\gamma$ has normal form starting with an element of $X_1$, namely that $\iota_1(\gamma) = 1$. Let $F' = (y^{-1}x^{-1})^k F(xy)^k$ and note that every $g \in F'$ either belongs to $H$ or has normal form starting and ending with an element of $X_2$, namely that, $g \in H$ or $\iota_1(g) = 2 = \iota_{L(g)}(g)$. Therefore, since $1 \not\in F'$, we get $\gamma^{-1} F' \gamma \cap H = \emptyset$. Hence letting $g_0 = (xy)^k \gamma$ we get $g_0^{-1} F g_0 \cap H = \emptyset$. Then $\Gamma$ has the free semigroup property by 4.3 and 4.4(iv).

Now we turn to methods that will allow us to apply Theorem 1.4 to amalgamated free products of groups. Our methods owe much to [DHR97], hence also to [Haa79].

Note that $X_i^{-1} \cup \{e\}$ is a set of representatives for the right cosets of $H$ in $G_i$, and there is a bijection from $X_i \times H$ to $H \times X_i$ given by $(x_i^{-1} h)(x_i, x)$ such that $xh = h x^{-1}$. Hence for each $g \in \Gamma$ of length $n$, in addition to the $h, x_j$ and $\iota_j$ appearing in the normal form (10), there are unique $h_0, h_1, \ldots, h_n = h \in H$ and $x_1 \in X_{i_1}, x_2 \in X_{i_2}, \ldots, x_n \in X_{i_n}$ such that for each $p \in \{0, 1, \ldots, n\}$,

$$g = x_1 x_2 \cdots x_p h_p x_{p+1}^{-1} \cdots x_n^{-1}.$$  

We call (11) the $p$-normal form of $g$.

Let $Y_n$ be the set of all elements of $\Gamma$ whose length is $n$. Furthermore, define the (linear) expectation $E_n : \mathbb{C} \Gamma \to \text{span} Y_n$ by, for $g \in \Gamma$, $E_n(g) = g$ if $L(g) = n$ and $E_n(g) = 0$ if
$L(g) \neq n$. Given $a \in \mathbb{C}I$ and $i \in I$, let $F_i(a)$ be the number of all the different elements of $X_i$ that appear in the normal forms of elements of the support of $a$. Furthermore, let $F_0(a)$ be 1 if $E_0(a) \neq 0$ and 0 if $E_0(a) = 0$. Let $K(a)$ be the constant defined by

$$K(a)^2 = \max\{F_0(a), \max_{i \in I} F_i(a)\}.$$  

**Lemma 5.2.** Let $\Gamma = (\ast_H)_{i \in I} G_i$ be an amalgamated free product of groups, where $H$ is a finite group. Let $k, \ell, n \in \mathbb{N} \cup \{0\}$ and let $a \in \text{span}Y_k$ and $b \in \text{span}Y_\ell$. If $n = k + \ell - 2p$ for some $p \in \{0, 1, 2, \ldots, \min(k, \ell)\}$ then

$$\|E_n(ab)\|_2 \leq |H|^{1/2} \|a\|_2 \|b\|_2. \quad (12)$$  

If $n = k + \ell - (2p + 1)$ for some $p \in \{0, 1, 2, \ldots, \min(k, \ell) - 1\}$ then

$$\|E_n(ab)\|_2 \leq |H|^{1/2} K(a) \|a\|_2 \|b\|_2. \quad (13)$$  

If $n > k + \ell$ or $n < |k - \ell|$ then $E_n(ab) = 0$.

**Proof.** Let

$$a = \sum_{g \in Y_k} \alpha_g g \quad \text{and} \quad b = \sum_{g \in Y_\ell} \beta_g g,$$

where $\alpha_g, \beta_g \in \mathbb{C}$. Let

$$Z_n = \{z = x_1 x_2 \cdots x_n \mid x_j \in X_{t_j}, t_1 \neq t_2, \ldots, t_{n-1} \neq t_n\}.$$

Suppose $n = k + \ell - 2p$ for $p \in \{0, 1, 2, \ldots, \min(k, \ell)\}$. Using $(k - p)$–normal forms, respectively $p$–normal forms, we have

$$a = \sum_{z \in Z_{k-p}, \overset{z \in Z_{k-p}}{i \in H}} \alpha_{(z h_1 z^{-1})} z h_1 z^{-1} \quad \text{and} \quad b = \sum_{w \in Z_{\ell-p}, \overset{w \in Z_{\ell-p}}{h_2}} \beta_{(\bar{w} h_2 w^{-1})} \bar{w} h_2 w^{-1},$$

where here and below, the sums are over all $z$ and $\bar{z}$ such that $t_{k-p}(z) \neq t_p(\bar{z})$, respectively over all $\bar{w}$ and $w$ such that $t_p(\bar{w}) \neq t_{\ell-p}(w)$. Then

$$E_n(ab) = \sum_{z \in Z_{k-p}, \overset{z \in Z_{k-p}}{i \in H}} \alpha_{(z h_1 z^{-1})} \beta_{(\bar{z} h_2 w^{-1})} z h_1 h_2 w^{-1},$$

where the sum is over all $z$ and $\bar{z}$ and $w$ such that

$$t_{k-p}(z) \neq t_p(\bar{z}), \quad t_p(\bar{z}) \neq t_{\ell-p}(w), \quad \text{and} \quad t_{k-p}(z) \neq t_{\ell-p}(w).$$
Then
\[
\| E_n (ab) \|_2^2 = \left| \sum_{z \in Z_{k-p}} \sum_{\tilde{z} \in Z_p} \alpha(z_{h_1 \tilde{z}^{-1}}) \beta(\tilde{z} (h^{-1})_{h^{-1}w^{-1}}) \right|^2
\]

\[
\leq \left( \sum_{z \in Z_{k-p}} \left( \sum_{\tilde{z} \in Z_p} |\alpha(z_{h_1 \tilde{z}^{-1}})|^2 \right) \right) \left( \sum_{\tilde{z} \in Z_p} \left( \sum_{h_{1} \in H} |\beta(\tilde{z} (h^{-1})_{h^{-1}w^{-1}})|^2 \right) \right)
\]

\[
= |H| \| a \|_2 \| b \|_2.
\]

This shows (12).

Now suppose \( n = k + \ell - (2p + 1) \) for \( p \in \{0, 1, \ldots, \min(k, \ell) - 1\} \). Then using \((k - p - 1)\)-normal forms, respectively \((p + 1)\)-normal forms, we have

\[
a = \sum_{z \in Z_{k-p-1}, \tilde{z} \in Z_p, h_1 \in H} \alpha(z_{h_1 \tilde{z}^{-1}}) z_{h_1 x_1^{-1} \tilde{z}^{-1}}, \quad \text{and} \quad b = \sum_{\tilde{w} \in Z_p, \tilde{w} \in Z_{l-p-1}, h_2 \in H} \beta(\tilde{w} x_2 h_2 w^{-1}) \tilde{w}_{x_2 h_2 w^{-1}},
\]

where here and below, the sums are over all \( z, \tilde{z} \) and \( \tilde{w} \) such that \( \ell_{k-p-1}(z) \neq \ell \) and \( \ell \neq \ell_{l-p-1}(\tilde{z}) \), respectively over all \( \tilde{w}, \ell \) and \( w \) such that \( \ell_{l-p-1}(\tilde{w}) \neq \ell \) and \( \ell \neq \ell_{l-p-1}(w) \). Hence

\[
E_n (ab) = \sum_{\ell \in I} \sum_{z \in Z_{k-p-1}} \sum_{x_1, x_2, \tilde{z} \in Z_p} \sum_{x_1 \neq x_2, h_1, h_2 \in H} (\alpha(z_{h_1 x_1^{-1} \tilde{z}^{-1}}) \beta(\tilde{z} x_2 h_2 w^{-1}) \tilde{z}(h_1 x_1^{-1} x_2 h_2) w^{-1}).
\]

For fixed \( \ell \in I \), \( z \in Z_{k-p-1} \) and \( w \in Z_{l-p-1} \) let

\[
y = y(\ell, z, w) = \sum_{x_1, x_2, \tilde{z} \in Z_p, x_1 \neq x_2, h_1, h_2 \in H} (\alpha(z_{h_1 x_1^{-1} \tilde{z}^{-1}}) \beta(\tilde{z} x_2 h_2 w^{-1}) \tilde{z}(h_1 x_1^{-1} x_2 h_2) w^{-1}).
\]
Now since $\alpha_{z_1 z_2 = z_1^{-1} z_2} = 0$ unless $x_1 \in F_i(a)$, we have

$$\|y\| \leq \sum_{z, \hat{z} \in Z_p} \left| \sum_{x \in X, \hat{h} \in H} \alpha_{z_1 z_2 = z_1^{-1} z_2} \beta_{\hat{z} x_2 h_2 = \hat{z} x_2 h_2} \right|$$

$$\leq (|H| F_i(a))^{1/2} \left( \sum_{z, \hat{z} \in Z_p} \left| \sum_{x \in X, \hat{h} \in H} \alpha_{z_1 z_2 = z_1^{-1} z_2} \beta_{\hat{z} x_2 h_2 = \hat{z} x_2 h_2} \right|^2 \right)^{1/2}.$$

Hence

$$\|y\|^2 \leq |H| F_i(a) \sum_{z, \hat{z} \in Z_p} \left| \sum_{x \in X, \hat{h}_1, \hat{h}_2 \in H} \alpha_{z_1 z_2 = z_1^{-1} z_2} \beta_{\hat{z} x_2 h_2 = \hat{z} x_2 h_2} \right|^2$$

$$\leq |H| F_i(a) \left( \sum_{z, \hat{z} \in Z_p} \left| \sum_{x \in X, \hat{h}_1, \hat{h}_2 \in H} \alpha_{z_1 z_2 = z_1^{-1} z_2} \beta_{\hat{z} x_2 h_2 = \hat{z} x_2 h_2} \right|^2 \right) \left( \sum_{\hat{w} \in Z_p} \left| \beta_{\hat{w} x_2 h_2 = \hat{w} x_2 h_2} \right|^2 \right)$$

$$= |H| F_i(a) \left( \sum_{z, \hat{z} \in Z_p} \left| \sum_{x \in X, \hat{h}_1, \hat{h}_2 \in H} \alpha_{z_1 z_2 = z_1^{-1} z_2} \beta_{\hat{z} x_2 h_2 = \hat{z} x_2 h_2} \right|^2 \right).$$

Hence

$$\|E_n(a b)^2\|_2 = \sum_{z, \hat{z} \in Z_p} \| y_{z, \hat{z}, w} \|_2^2$$

$$\leq |H| K(a)^2 \sum_{z, \hat{z} \in Z_p} \left( \sum_{x \in X, \hat{h}_1, \hat{h}_2 \in H} \left| \sum_{z, \hat{z} \in Z_p} \left| \sum_{x \in X, \hat{h}_1, \hat{h}_2 \in H} \alpha_{z_1 z_2 = z_1^{-1} z_2} \beta_{\hat{z} x_2 h_2 = \hat{z} x_2 h_2} \right|^2 \right) \right) \left( \sum_{\hat{w} \in Z_p} \left| \beta_{\hat{w} x_2 h_2 = \hat{w} x_2 h_2} \right|^2 \right).$$

This shows (13).

Using the formula $|L(g) - L(g')| \leq L(gg') \leq L(g) + L(g')$, we conclude that the final statement of the lemma is true.
Lemma 5.3. Let \( a \in \text{span} Y_k \). Then

\[
\| a \| \leq (2k + 1)|H|^{1/2} K(a) \| a \|_2.
\]

Proof. Let \( b \in \ell^2(\Gamma) \) have finite support and let \( b_j = E_j(b) \). Then for every \( n \in \mathbb{N} \),

\[
\| E_n(ab) \|_2 \leq \sum_{j=0}^\infty \| E_n(ab_j) \|_2
\]

\[
= \sum_{j=[n-k]}^{n+k} \| E_n(ab_j) \|_2
\]

\[
\leq |H|^{1/2} K(a) \| a \|_2 \sum_{j=[n-k]}^{n+k} \| b_j \|_2
\]

\[
\leq (2k + 1)^{1/2} |H|^{1/2} K(a) \| a \|_2 \left( \sum_{j=[n-k]}^{n+k} \| b_j \|_2^{2} \right)^{1/2}.
\]

Hence

\[
\| ab \|_2^2 = \sum_{n=0}^\infty \| E_n(ab) \|_2^2
\]

\[
\leq (2k + 1) |H| K(a)^2 \| a \|_2^2 \sum_{n=0}^\infty \sum_{j=[n-k]}^{n+k} \| b_j \|_2^2
\]

\[
\leq (2k + 1)^2 |H| K(a)^2 \| a \|_2^2 \sum_{j=0}^\infty \| b_j \|_2^2
\]

\[
= (2k + 1)^2 |H| K(a)^2 \| a \|_2^2 \| b \|_2^2.
\]

\[\square\]

Lemma 5.4. Let

\[ a \in \text{span} \left( \bigcup_{j=0}^k Y_j \right). \]

Then

\[
\| a \| \leq (2k + 1)^{3/2} |H|^{1/2} K(a) \| a \|_2.
\]
Proof. Let \( a_j = E_j(a) \). Then

\[
\|a\| \leq \sum_{j=0}^{k} \|a_j\| \\
\leq |H|^{1/2} K(a) \sum_{j=0}^{k} (2j + 1) \|a_j\|_2 \\
\leq (2k + 1) |H|^{1/2} K(a)(k + 1)^{1/2} \left( \sum_{j=0}^{k} \|a_j\|_2^2 \right)^{1/2} \\
\leq (2k + 1)^{3/2} |H|^{1/2} K(a) \|a\|_2.
\]

\[\square\]

**Lemma 5.5.** Let \( G \) be a group and let \( H \) be a subgroup of \( G \). Suppose \( g \in G \) is such that \( g^{-1} H g \cap H = \{1\} \). Then for every \( h_1, h_2 \in H \),

\[ h_1 g H = h_2 g H \quad \text{implies} \quad h_1 = h_2. \]

**Proof.** If \( h_1 g H = h_2 g H \) then \( h_1 g h_1^{-1} = h_2 g h_2^{-1} \) for some \( h_1, h_2 \in H \). Hence \( g^{-1} h_2^{-1} h_1 g = h_2 (h_1)^{-1} \in H \), so by hypothesis \( h_2^{-1} h_1 = 1 \).

\[\square\]

**Lemma 5.6.** Let \( \Gamma = G_1 *_H G_2 \) be an amalgamated free product of groups with \( H \) finite. Suppose that there are \( y, z \in G_2 \) such that the double cosets \( H y H, Hz H \), and \( H \) are distinct, and suppose there is \( g_0 \in \Gamma \) such that \( g_0^{-1} H g_0 \cap H = \{1\} \). Then \( \Gamma \) satisfies the hypothesis of Theorem 1.4, hence \( C^*_\lambda(\Gamma) \) has stable rank 1.

**Proof.** Let \( F \) be a finite subset of \( \Gamma \). We will find an element \( \gamma \in \Gamma \) so that \( \gamma F \) is semifree and has the \( \ell^2 \)-spectral radius property. Choose \( X_1 \) and \( X_2 \) as in the beginning of this section so that \( y, z \in X_2 \) and \( x \in X_1 \). Let \( \ell \in \mathbb{N} \) be such that \( \ell/2 > N \overset{\text{def}}{=} \max \{ L(g) \mid g \in F \} \) and let \( u' = (xy)\ell \) and \( v = (z^{-1}x)^\ell \). Since \( y H z^{-1} \cap H = \emptyset \), we see that for every \( g \in F \), \( u' g v \notin H \), \( L(u' g v) \leq 2\ell + N \) and the normal form of \( u' g v \) starts and ends with an element of \( X_1 \), i.e. \( \nu_1(u' g v) = 1 = \nu_{L(u' g v)}(u' g v) \). Let \( g_0 = d_1 d_2 \cdots d_q h \) be the normal form of \( g_0 \). We may without loss of generality assume that \( h = 1 \). Let

\[
\begin{align*}
g_0' &= \begin{cases} 
y 0 y & \text{if } d_1 \in X_1, \ d_q \in X_1 \\
y 0 & \text{if } d_1 \in X_1, \ d_q \in X_2 \\
y 0 y & \text{if } d_1 \in X_2, \ d_q \in X_1 \\
y 0 & \text{if } d_1 \in X_2, \ d_q \in X_2,
\end{cases}
\end{align*}
\]

(14)

Let \( m \in \mathbb{N} \) be such that \( m/2 > 2\ell + N + L(g_0) + 3 \), let \( r = g_0' x y (x z)^m x y \) and let \( u = ru' \).
Claim 5.6a. $uFv$ is semifree.

Proof. Let $p \in \mathbb{N}$ and let $g_1, g_2, \ldots, g_p \in F$. Let $u'g_jv$ have normal form

$$e_{j,1}e_{j,2} \cdots e_{j,k(j)}h_j.$$ 

Recall that $e_{j,1}, e_{j,k(j)} \in X_1$. We will show that $g_1$ can be recovered from knowing only the normal form of

$$(ug_1v)(ug_2v) \cdots (ug_pv). \quad (15)$$

This will suffice to prove the claim. The details of the argument vary slightly in the different cases (14). We will give the argument in the case $g_0' = g_0$; the variations for the other cases will be apparent. If $p = 1$ we are done, so assume $p > 1$. Let $h_jg_0$ have normal form

$$h_jg_0 = d_1^{(j)}d_2^{(j)} \cdots d_q^{(j)}h_j.$$ 

Note that, by Lemma 5.5, $h_j$ is determined by the coset representative $d_1^{(j)}d_2^{(j)} \cdots d_q^{(j)}$. We see that the normal form of (15) is

$$d_1 \cdots d_qy(xz)^mxyz_1 \cdots c_{1,k(1)}d_1^{(1)}d_2^{(1)} \cdots d_q^{(1)}x_0y_1(x_1z_1x_2z_2 \cdots x_mz_m)x_{m+1}y_2 \cdots \quad (16)$$

where every $x_i \in X_1$, $y_1, y_2 \in HyH \cap X_2$ and every $z_i \in HzH \cap X_2$. Now $m$ was chosen large enough so that in the part

$$xyz_1 \cdots c_{1,k(1)}d_1^{(1)}d_2^{(1)} \cdots d_q^{(1)}$$

of (16), there is no sequence of $m$ or more elements of $HzH \cap X_2$ separated by single elements of $X_1$. Hence the first such sequence occurring in the normal form of (15) is the sequence $(xz)^m$ found in (16), and the second such sequence is $(x_1z_1x_2z_2 \cdots x_mz_m)$ found in (16). Hence after identifying these sequences, we can recover $d_1^{(1)}d_2^{(1)} \cdots d_q^{(1)}$ and also $c_{1,1} \cdots c_{1,k(1)}$. As mentioned earlier, $h_1$ is determined by $d_1^{(1)}d_2^{(1)} \cdots d_q^{(1)}$. Hence we have found $u'g_1v = c_{1,1} \cdots c_{1,k(1)}h_1$, which allows us to find $g_1$. Now proceeding by induction, we can find $g_2, g_3, \ldots, g_p$ one after the other. This completes the proof of Claim 5.6a.

Claim 5.6b. $uFv$ has the $\ell^2$-spectral radius property.

Proof. Let $D$ be the set of all elements of $X_1 \cup X_2$ that appear in the normal forms of group elements belonging to $\bigcup_{n=1}^{\infty} (uFv)^n$. We first show that $D$ is finite. Let $D_1$ be the set of all elements of $X_1 \cup X_2$ that appear in the normal forms of group elements belonging to $uFv$. 
Then $D_1$ is finite. Moreover, since no cancellation occurs when we multiply elements of $uFv$, i.e. since
\[
L((g_1v)(ug_2v)\cdots(ug_pv)) = \sum_{j=1}^{p} L(ug_jv)
\]
for every $p \in \mathbb{N}$ and $g_1, \ldots, g_p \in F$, we see that $c' \in D$ only if there are $h, h' \in H$ and $c \in D_1$ such that $hc = c'h'$. Since $H$ is finite, it follows that $D$ is finite.

Let $M$ be the maximum of the lengths of the elements of $uFv$. Now let $a \in \text{span}(uFv)$. Then by Lemma 5.4, for every $k \in \mathbb{N}$
\[
\|a^k\| \leq |H|^{1/2}D^{1/2}(2Mk + 1)^{3/2}\|a^k\|_2.
\]

Hence
\[
\begin{align*}
    r(a) &= \limsup_{k \to \infty} \|a^k\|^{1/k} \\
      &\leq \limsup_{k \to \infty} |H|^{1/2k}D^{1/2k}(2Mk + 1)^{3/2k}\|a^k\|_2^{1/k} \\
      &= \limsup_{k \to \infty} \|a^k\|_2^{1/k} \\
      &= r_2(a).
\end{align*}
\]

Hence $r(a) = r_2(a)$ and Claim 5.6b is proved.

Now it follows from Claims 5.6a and 5.6b that $vuF$ is semifree and has the $\ell^2$-spectral radius property, and hence the Lemma is proved.

Proof of Theorem 1.6. If $H$ is trivial then $sr(C^*_\lambda(\Gamma)) = 1$ as mentioned after the statement of the theorem, so assume $H$ is nontrivial. If $G_1$ and $G_2$ are both finite, then $G_1$ and $G_2$ have property (RD), hence by the result of Jolissaint that we give as Example 3.2.iv, it follows that $\Gamma$ has (RD), hence has the $\ell^2$-spectral radius property. Moreover, by Proposition 5.1 and Example 4.4.iv, it follows that $\Gamma$ has the free semigroup property. Therefore, by Theorem 1.4, $sr(C^*_\lambda(\Gamma)) = 1$.

Now suppose $G_2$ is infinite. Since double cosets $HgH$ are finite, there must be $y, z \in G_2$ as required in Lemma 5.6, which then implies that $sr(C^*_\lambda(\Gamma)) = 1$. 

\qed
§6. ON THE REAL RANK OF REDUCED C*-ALGEBRAS OF GROUPS.

Our first observation is that, for a group $\Gamma$ which is abelian, the following are equivalent:

(i) $\Gamma$ is locally finite,
(ii) $\Gamma$ is a torsion group,
(iii) $\text{RR}(C^*_\lambda(\Gamma)) = 0$.

Indeed, the equivalence (i)$\iff$(ii) is straightforward. The implication (i)$\implies$(iii) follows from the equality $\text{RR}(A) = 0$ for any finite dimensional C*-algebra. For (iii)$\implies$(i), one may argue by contradiction as follows: if $\Gamma$ is not locally finite, then $\Gamma$ has a subgroup isomorphic to $\mathbb{Z}$, so that the Pontryagin dual $\hat{\Gamma}$ surjects onto $\hat{\mathbb{Z}} \cong \mathbb{R}/\mathbb{Z}$, and consequently $\dim(\hat{\Gamma}) \geq 1$, which is the negation of (iii).

If $A$ is an infinite dimensional C*-algebra having a faithful state, $\phi$, and if $\text{RR}(A) = 0$, then it is well known that $A$ must have nonzero projections, $p$, such that $\phi(p)$ is arbitrarily small. The contrapositive assertion provides a way of showing that a given C*-algebra has nonzero real rank. An extreme case of when this analysis applies is when $A$ is an infinite dimensional C*-algebra whose only projections are 0 and 1; then we must have $\text{RR}(A) > 0$.

The Kaplansky–Kadison conjecture is that $C^*_r(\Gamma)$ has only projections 0 and 1 whenever $\Gamma$ is a torsion free group. If $\Gamma$ is a torsion free group for which the Baum–Connes conjecture holds, then $\Gamma$ also satisfies the Kaplansky–Kadison conjecture; (see [BC88], [Va89] and [Ju] for surveys of these and other ideas). The Baum–Connes conjecture has been proved for a number of groups. Thus we have the following.

**Theorem 6.1** ([Ka84], [JK91], [BCV95], [HK]). Let $\Gamma$ be a torsion free group that is either

(i) a discrete subgroup in a connected Lie group whose semi-simple part is locally isomorphic to a product of compact groups, of Lorentz groups $\text{SO}(n, 1)$, and of groups $\text{SU}(n, 1)$ (where $n \geq 2$);

or

(ii) amenable.

Then $C^*_\lambda(\Gamma)$ has no idempotent distinct from 0 and 1, and in particular

$$\text{RR}(C^*_\lambda(\Gamma)) > 0.$$ 

**Citations.** In (i), the case for discrete subgroups of products of compact and Lorentz groups is [Ka84] and the additional $\text{SU}(n, 1)$ is [JK91]; however, this follows also from the more
recent work of Higson and Kasparov [HK], for which our source of information is P. Julg’s work [Ju]. The case (ii) of amenable groups follows from [HK] and [BCV95].

\[ \square \]

Suppose that $\Gamma$ is a torsion free, cocompact lattice of a Lie group that is either the connected component of $\text{SO}(n, 1)$ or the quotient of $\text{SU}(n, 1)$ by its center. Then $\Gamma$ has nonzero real rank by virtue of Theorem 6.1(i), and $\Gamma$ is hyperbolic. Thus, using also Theorem 1.5, we have

\[ \text{sr}(C^*_\Lambda(\Gamma)) = 1 \quad \text{and} \quad \text{RR}(C^*_\Lambda(\Gamma)) = 1. \]

E. Germain, extensively generalizing results of J. Cuntz [Cu83], has in [Ge96], [Ge97a], and [Ge97b] shown that if $(A, \tau_i) = \bigotimes_{i=1}^{n} (A_i, \tau_i)$ is the reduced free product of nuclear $\text{C}^*$-algebras, $A_i$, with respect to faithful traces, $\tau_i$, then the range of $\tau$ on $K_0(A)$ is equal to the subgroup of $\mathbb{R}$ generated by the union of the ranges of the $(\tau_i)_*$ on $K_0(A_i)$. Taken in the case of reduced $\text{C}^*$-algebras of groups and their canonical traces, and using Theorem 6.1(ii), this implies the following theorem.

**Theorem 6.2.** Let $I$ be a countable set having at least two elements and for every $i \in I$ let $G_i$ be either a finite group or a torsion free amenable group or a direct product of a finite group and a torsion free amenable group. Let $\Gamma = \ast_{i \in I} G_i$ be the free product of groups. If there is $n \in \mathbb{N}$ such that no $G_i$ has a finite subgroup of order greater than $n$, then $C^*_\Lambda(\Gamma)$ has no projections whose trace is less than $1/(n!)$. Consequently,

\[ \text{RR}(C^*_\Lambda(\Gamma)) > 0. \]

Combined with the result [DHR97], that such free products have stable rank one, we get that for $\Gamma$ from Theorem 6.2,

\[ \text{RR}(C^*_\Lambda(\Gamma)) = 1. \]

Moreover, a partial converse of Theorem 6.2 is given by the following theorem:

**Theorem 6.3 ([DR]).** Let $\Gamma = \ast_{i=1}^{\infty} G_i$ be the free product of infinitely many nontrivial groups, $G_i$. Suppose that for every $n$ there is a finite subgroup of $\Gamma$ having order at least $n$. Then

\[ \text{RR}(C^*_\Lambda(\Gamma)) = 0. \]

**Open Problem 6.4.** Let $\Gamma$ be an infinite group and suppose that $C^*_\Lambda(\Gamma)$ is simple. Is it true that

\[ \text{RR}(C^*_\Lambda(\Gamma)) = 0 \]

if and only if $\Gamma$ has finite subgroups of arbitrarily high order?


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