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On the existence of polynomial first integrals of quadratic homogeneous systems of ordinary differential equations

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Abstract

We consider systems of ordinary differential equations with quadratic homogeneous right hand side. We give a new simple proof of a result already obtained in [8,10] which gives the necessary conditions for the existence of polynomial first integrals. The necessary conditions for the existence of a polynomial symmetry field are given. It is proved that an arbitrary homogeneous first integral of a given degree is a linear combination of a fixed set of polynomials.

1 Introduction

In the present paper we study the system of ordinary differential equations with quadratic homogeneous right hand side

\[ \dot{x}_i = f_i(x_1, \ldots, x_n), \quad f_i = \sum_{j,k=1}^{n} a_{ijk} x_j x_k, \quad a_{ijk} \in \mathbb{C}, \quad i = 1, \ldots, n. \]  

(1.1)

Systems of such a form arise in many problems of classical mechanics: Euler–Poincaré equations on Lie algebras, the Lotka–Volterra systems, etc.

The main concern of this paper is to find the values of the parameters \(a_{ijk}\) for which equations (1.1) have first integrals.

In the paper [2] the necessary conditions are found for the existence of polynomial first integrals of the system

\[ \dot{x}_i = V_i(x_1, \ldots, x_n), \quad i = 1, \ldots, n, \]  

(1.2)

where \(V_i \in \mathbb{C}[x_1, \ldots, x_n]\) are homogeneous polynomials of weighted degree \(s \in \mathbb{N}\). In the case \(s = 2\) we obtain equations (1.1).

The method given in [2] is based on ideas of Darboux [1,6,7] who used a special type of particular solutions of the system (1.2)

\[ x_i(t) = d_i \phi(t), \quad i = 1, \ldots, n, \]

where \(\phi(t)\) satisfies the differential equation \(\dot{\phi} = \lambda \phi^s\), \(\lambda\) is an arbitrary number and \(d = (d_1, \ldots, d_n)^T \neq 0\) is a solution of the following algebraic system

\[ V_i(d) = \lambda d_i, \quad i = 1, \ldots, n. \]

In this paper we generalize this method.

It was shown in [8,10] that the weighted degree of a polynomial first integral of the system (1.1) is a certain integer linear combination of Kovalevskaya exponents (see [9]). In Section 2 we give a new simple proof of this result. In Section 3 a similar theorem for polynomial symmetry fields is proved. As an example, we consider the well known Halphen equations. Section 4 contains our main result. We present so called base functions and prove that every homogeneous polynomial first integral of a fixed degree is a certain linear combination of the corresponding base functions. In Section 5 we give an application of previous results to planar homogeneous quadratic systems where necessary and sufficient conditions for the existence of polynomial first integrals in terms of Kovalevskaya exponents are found.

2 The existence of polynomial first integral. Necessary conditions

Following the paper [2], we consider solution \(C = (c_1, \ldots, c_n)^T \neq (0, \ldots, 0)^T\) of algebraic equations

\[ f_i(c_1, \ldots, c_n) + c_i = 0, \quad i = 1, \ldots, n. \]  

(2.1)
Define the *Kovalevskaya matrix* $K$ [3]

$$K_{ij} = \frac{\partial f_i}{\partial x_j}(C) + \delta_{ij}, \quad i, j = 1, \ldots, n.$$  

where $\delta_{ij}$ is the Kronecker symbol. Let us assume that $K$ can be transformed to diagonal form

$$K = \text{diag}(\rho_1, \ldots, \rho_n).$$

The eigenvalues $\rho_1, \ldots, \rho_n$ are called *Kovalevskaya exponents*.

**Lemma 1.** ([3]) Vector $C$ is an eigenvector of the matrix $K$ with eigenvalue $\rho_1 = -1$.

Consider the following linear differential operators

$$D_+ = \sum_{i=1}^{n} f_i \frac{\partial}{\partial x_i}, \quad D_0 = \sum_{i,j=1}^{n} K_{ij} x_j \frac{\partial}{\partial x_i},$$

$$U = \sum_{i=1}^{n} x_i \frac{\partial}{\partial x_i}, \quad D_- = \sum_{i=1}^{n} c_i \frac{\partial}{\partial x_i},$$

which satisfy relations

$$[D_- , D_+] = D_0 - U, \quad [D_0, D_-] = D_-,$$  

where $[A, B] = AB - BA$. [8,10]

**Theorem 2.** Suppose that the system (1.1) possesses a homogeneous polynomial first integral $F_M$ of degree $M$, and $\rho_1 = -1, \rho_2, \ldots, \rho_n$ are Kovalevskaya exponents. Then there exists a set of non-negative integers $k_2, \ldots, k_n$ such that

$$k_2 \rho_2 + \cdots + k_n \rho_n = M, \quad k_2 + \cdots + k_n \leq M.$$  

**(2.4)**

**Proof.** By definition of a first integral $D_+ F_M = 0$. Considering identities

$$D_+^l (D_+ F_M) = 0, \quad \text{for} \quad l \in \mathbb{N},$$

we obtain the following set of polynomials

$$F_M, F_{M-1}, \ldots, F_{\rho+1}, F_{\rho},$$

defined by the recursive relations

$$D_- F_{i+1} = (M - i) F_i, \quad i = \rho, \ldots, M - 1,$$

where the number $1 \leq \rho \leq M$ is determined by the condition $D_- F_{\rho} = 0$. Using (2.3), (2.5) we deduce the following chain of equations

$$D_0 F_M = M F_M - D_+ F_{M-1},$$

$$D_0 F_{M-1} = M F_{M-1} - D_+ F_{M-2},$$

$$\ldots$$

$$D_0 F_{\rho+1} = M F_{\rho+1} - D_+ F_{\rho},$$

$$D_0 F_{\rho} = M F_{\rho}.$$  

**(2.6)**

Let $J_1, \ldots, J_n$ – linearly independent eigenvectors of the Kovalevskaya matrix $K$ corresponding to the eigenvalues $\rho_1 = -1, \rho_2, \ldots, \rho_n$. According to Lemma 1 we can always put $J_1 = C$.

Consider the linear change of variables

$$x_i = \sum_{j=1}^{n} L_{ij} y_j, \quad i = 1, \ldots, n,$$  

**(2.7)**

where $L = (L_{ij})$ is a nonsingular matrix defined by

$$L = (C, J_2, \ldots, J_n),$$

where $C$ is a constant.
then obviously

$$L^{-1} KL = \text{diag}(-1, \rho_2, \ldots, \rho_n).$$

With help of (2.7) and Lemma 1 one finds the following expressions for the operators \(D_0, D_-\) in the new variables

$$D_0 = \sum_{i=1}^n \rho_i y_i \frac{\partial}{\partial y_i}, \quad D_- = -\frac{\partial}{\partial y_1},$$

and the equation (2.6) becomes

$$\left( \rho_2 y_2 \frac{\partial}{\partial y_2} + \cdots + \rho_n y_n \frac{\partial}{\partial y_n} \right) F_\rho = MF_\rho. \quad (2.8)$$

We can write the polynomial \(F_\rho\) as follows

$$F_\rho = \sum_{|k|=\rho} A_{k_2\ldots k_n} y_2^{k_2} \cdots y_n^{k_n}, \quad |k| = k_2 + \cdots + k_n, \quad k_i \in \mathbb{Z}_+. \quad (2.9)$$

Substituting (2.9) into (2.8), one obtains the following linear system

$$(k_2 \rho_2 + \cdots + k_n \rho_n)A_{k_2\ldots k_n} = MA_{k_2\ldots k_n}, \quad \text{for} \quad |k| = \rho. \quad (2.10)$$

Taking into account that \(F_\rho\) is not zero identically, we conclude that there exists at least one nonzero set \(k_2, \ldots, k_n, |k| \leq M\) such that

$$k_2 \rho_2 + \cdots + k_n \rho_n = M. \quad (2.11)$$

This relation implies (2.4). Q.E.D.

**Remark.** Theorem 2 does not impose any restrictions on \(\text{grad}(F_M)\) calculated at the point C. Thus, it generalizes the theorem of Yoshida ([3], p.572), who used essentially the condition \(\text{grad}(F_M) \neq 0\).

**Corollary 3.** The Halphen equations

$$\begin{align*}
\dot{x}_1 &= x_3 x_2 - x_1 x_3 - x_1 x_2, \\
\dot{x}_2 &= x_1 x_3 - x_2 x_1 - x_2 x_3, \\
\dot{x}_3 &= x_2 x_1 - x_3 x_2 - x_3 x_1, 
\end{align*} \quad (2.12)$$

admit no polynomial first integrals.

Indeed, the system (2.12) has Kovalevskaya exponents \(\rho_1 = \rho_2 = \rho_3 = -1\). It is easy to verify that conditions (2.4) are not fulfilled for any positive integer \(M\). Moreover, as proved in [2], the system (2.12) has no rational first integrals.

## 3 Existence of polynomial symmetry fields. Necessary conditions

The first integrals are the simplest tensor invariants of the system (1.1). In [4] Kozlov considered tensor invariants of weight-homogeneous differential equations which include the system (1.1). In particular, he found necessary conditions for the existence of symmetry fields. Below we propose a generalization of his result.

Recall that the linear operator \(W = \sum_{i=1}^n w_i(x_1, \ldots, x_n) \frac{\partial}{\partial x_i}\), is called the *symmetry field* of (1.1), if \([W, D_+] = 0\), where \(D_+\) is defined by (2.2). If \(w_1, \ldots, w_n\) are homogeneous functions of degree \(M + 1\) then the degree of \(W\) is \(M\) ([4]).

**Theorem 4.** Suppose that the system (1.1) possesses a polynomial symmetry field of degree \(M\) and \(\rho_1 = -1, \rho_2, \ldots, \rho_n\) are Kovalevskaya exponents. Then there exist non-negative integers \(k_2, \ldots, k_n, |k| \leq M + 1\) such that at least one of the following equalities holds

$$k_2 \rho_2 + \cdots + k_n \rho_n = M + \rho_i, \quad i = 1, \ldots, n. \quad (3.1)$$
After the change of variables (2.7) the system (1.1) takes the form

\begin{equation}
\frac{dy_i}{dt} = (p_i - 1)y_i + \varphi_i(y_2, \ldots, y_n), \quad i = 2, \ldots, n,
\end{equation}

where \( \varphi_i \) are quadratic homogeneous polynomials in the variables \( y_2, \ldots, y_n \).

According to (2.2) define operators \( D_+, D_0, D_- \).

A homogeneous polynomial \( P_M(y_1, \ldots, y_n) \) of degree \( M \) satisfies the condition

\begin{equation}
D_-(D_+P_M) = 0,
\end{equation}

is called a base function of the system (4.1). In other words, the function \( D_+P_M \) does not depend on \( y_1 \). It is clear that base functions of degree \( M \) form a linear space \( L_M \) over field \( \mathbb{C} \).

**Lemma 6.** If the system (4.1) has a homogeneous polynomial first integral \( F_M \) of degree \( M \), then \( F_M \in L_M \).

Indeed, by definition, we have \( D_+F_M = 0 \), hence, in view of (4.2), \( F_M \in L_M \).

Let \( J(M) = \{ z \in \mathbb{Z}^{n-1}_+ \mid z_2\rho_2 + \cdots + z_n\rho_n = M, |z| \leq M \} \) be the set of integer-valued vectors \( z = (z_2, \ldots, z_n)^T \) for which the condition (2.11) is fulfilled. Put \( m = |J(M)| \) and suppose \( J(M) \neq \emptyset \).

**Theorem 7.** The dimension \( d \) of \( L_M \) satisfies the condition \( 1 \leq d \leq m \).

**Proof.** Let us assume the set \( J(M) \) contains vectors \( z^{(1)}, \ldots, z^{(m)} \) which are ordered by the norm \( |z| = z_2 + \cdots + z_n \)

\[ |z^{(1)}| \leq \cdots \leq |z^{(m)}|. \]

Define the vector \( \rho = (\rho_2, \ldots, \rho_n)^T \) and put \( (\rho, z) = \rho_2z_2 + \cdots + \rho_nz_n, |z^{(i)}| = n_i, i = 1, \ldots, m. \)

Following the proof of Theorem 2, for each \( i = 1, \ldots, m \) consider the system of linear partial differential equations

\begin{equation}
\begin{align*}
D_0P_{i,n_1} &= MP_{i,n_1}, \\
D_0P_{i,n_1+1} &= MP_{i,n_1+1} - D_+P_{i,n_1}, \\
&\vdots \\
D_0P_{i,M-1} &= MP_{i,M-1} - D_+P_{i,M-2}, \\
D_0P_{i,M} &= MP_{i,M} - D_+P_{i,M-1}, \\
D_-P_{i,l+1} &= (M-l)P_{i,l}, \quad l = n_i, \ldots, M - 1,
\end{align*}
\end{equation}

which defines polynomials \( P_{i,n_1}, \ldots, P_{i,M} \) recurrently.

It follows from \( (z^{(i)}, \rho) = M \) that the first equation in (4.3) has the particular solution \( P_{i,n_1} = y_2^{z_2^{(i)}} \cdots y_n^{z_n^{(i)}}. \)

Equations (4.3), (4.4) define certain base function \( P_{i,M} \). Indeed, according to (4.4), we have

\begin{equation}
P_{i,M-1} = D_-P_{i,M}.
\end{equation}

Substituting (4.5) into the last equation in (4.3), and using relations (2.3) we find

\[ D_0P_{i,M} = MP_{i,M} - D_+D_-P_{i,M} = MP_{i,M} - (D_-D_+ - D_0 + U)P_{i,M}. \]

Hence \( D_-(D_+P_{i,M}) = 0. \)
Now consider the problem on the existence of a solution of (4.3), (4.4) in form of homogeneous polynomials \( P_{i,n}, \ldots, P_{i,M} \). Fix certain \( i = 1, \ldots, m \) and put \( a_i = M - n_i \). Using the relations (4.4) we can write

\[
P_{i,n_i} = I_{i,n_i},
\]

\[
P_{i,n_i+p} = \sum_{j=0}^{p} \binom{p-j}{a_{i,j}} y_1^{p-j} I_{i,n_i+j}, \quad p = 1, \ldots, a_i,
\]

(4.6)

where \( I_{i,k}(y_2, \ldots, y_n) \) are certain homogeneous polynomials of degrees \( k = n_i, \ldots, M \). Notice that \( I_{i,k} \) does not depend on \( y_1 \).

Differential operators \( D_+, D_0 \) can be represented in the form

\[
D_+ = (-y_1^2 + \varphi_1) \frac{\partial}{\partial y_1} + y_1 (A_0 - \tilde{U}) + A_+,
\]

\[
D_0 = -y_1 \frac{\partial}{\partial y_1} + A_0,
\]

(4.7)

where

\[
A_+ = \sum_{k=2}^{n} \varphi_k \frac{\partial}{\partial y_k}, \quad A_0 = \sum_{k=2}^{n} \rho_k y_k \frac{\partial}{\partial y_k}, \quad \tilde{U} = \sum_{k=2}^{n} y_k \frac{\partial}{\partial y_k}.
\]

(4.8)

Using (4.3), (4.6), (4.7) one deduces the following equations for determination of \( I \)

\[
A_0 I_{i,n_i} = M I_{i,n_i},
\]

\[
A_0 I_{i,n_i+1} = M I_{i,n_i+1} - A_+ I_{i,n_i},
\]

\[
A_0 I_{i,n_i+2} = M I_{i,n_i+2} - a_i \varphi_1 I_{i,n_i} - A_+ I_{i,n_i+1},
\]

\[
A_0 I_{i,n_i+3} = M I_{i,n_i+3} - (a_i - 1) \varphi_1 I_{i,n_i+1} - A_+ I_{i,n_i+2},
\]

\[
\cdots
\]

\[
A_0 I_{i,M} = M I_{i,M} - 2 \varphi_1 I_{i,M-2} - A_+ I_{i,M-1}.
\]

(4.9)

We can write each equation of (4.9) as follows

\[
A_0 X_l = M X_l + Y_l,
\]

(4.10)

where \( X_l, Y_l \) are homogeneous polynomials of weighted degree \( l = n_i, \ldots, M \). Let us assume

\[
X_l = \sum_{|i|=l} c_{i_2 \ldots i_n} y_2^{i_2} \cdots y_n^{i_n}, \quad Y_l = \sum_{|i|=l} d_{i_2 \ldots i_n} y_2^{i_2} \cdots y_n^{i_n}, \quad |i| = i_2 + \cdots + i_n,
\]

(4.11)

where \( c_{i_2 \ldots i_n}, d_{i_2 \ldots i_n} \) are constant parameters. Then substituting (4.11) into (4.10), we obtain the following linear system with respect to \( c_{i_2 \ldots i_n} \)

\[
(i_2 \rho_2 + \cdots + i_n \rho_n - M) c_{i_2 \ldots i_n} = d_{i_2 \ldots i_n},
\]

(4.12)

for \( i_2, \ldots, i_n = 0, 1, \ldots, |i| = l \).

Suppose there exists a set \( k_2, \ldots, k_n \) for which the following conditions are fulfilled

\[
(k_2, \ldots, k_n)^T \in J(M), \quad d_{k_2 \ldots k_n} \neq 0, \quad |k| = l,
\]

(4.13)

then the solution \( I_{i,n_i}, \ldots, I_{i,M} \) does not exist. In this case we put \( P_{i,M} = 0 \).

If the conditions (4.13) are not satisfied, we obtain the base function

\[
P_{i,M} = \sum_{j=0}^{a_i} y_1^{a_i-j} I_{i,n_i+j},
\]

(4.14)

It is easy to show that polynomials \( \{P_{i,M}\}_{i=1}^{m} \) are linearly independent over the field \( \mathbb{C} \).

Taking into account that \( n_1 \leq \cdots \leq n_m \) and using (4.13), we see that in case \( i = m \) we always can determine the base function \( P_{i,M} \). Therefore, under the assumption \( J(M) \neq \emptyset \), the space \( L_M \) always contains a nonzero function. Q.E.D.

**Corollary 8.** If at least one resonance condition of the form

\[
(z, \rho) = M, \quad |z| \leq M, \quad z \in \mathbb{Z}_+^{n-1},
\]

is fulfilled, then there exists a base function of degree \( M \).
5 Polynomial first integrals in the case of quadratic plane vector field.

The first classification of integral curves of two-dimensional quadratic homogeneous systems can be found in the paper of Lyagina [5] and later was completed by numerous authors.

In this section we apply the previous results to this problem to illustrate the method of basis functions. Consider the system

\[
\begin{align*}
\dot{x}_1 &= a_1 x_1^2 + b_1 x_1 x_2 + d_2 x_2^2, \\
\dot{x}_2 &= a_2 x_2^2 + b_2 x_1 x_2 + d_1 x_1^2,
\end{align*}
\]

(5.1)

where \(a_i, b_i, d_i\) are constant parameters.

Let \(c^{(1)} = (c_1^{(1)}, c_2^{(1)})^T, c^{(2)} = (c_1^{(2)}, c_2^{(2)})^T\) be any two linearly independent solutions of the algebraic system (2.1). The exceptional cases when the system (2.1) has only one or admit no solutions are excluded for the discussion below.

Assume that Kovalevskaya exponents corresponding to \(c^{(1)}, c^{(2)}\) are

\[
\Re_{1} = (-1, \rho_1)^T, \quad \Re_{2} = (-1, \rho_2)^T.
\]

(5.2)

Lemma 9. The system (5.1) has a homogeneous polynomial first integral of degree \(M\) if and only if there exists integer \(k = 1, \ldots, M - 1\) such that \(\rho_1 = M/k\) and \(\rho_2\) is one of the following numbers

\[
\frac{M}{M - k} \frac{M - 1}{M - k - 1} \ldots \frac{M}{2}.\]

Proof. Consider the following change of coordinates

\[
\begin{pmatrix}
  x_1 \\
  x_2
\end{pmatrix} =
\begin{pmatrix}
  c_1^{(1)} & c_1^{(2)} \\
  c_2^{(1)} & c_2^{(2)}
\end{pmatrix}
\begin{pmatrix}
  p_1 \\
  p_2
\end{pmatrix}
\]

(5.3)

which exists because of linear independence of vectors \(c^{(1)}, c^{(2)}\). In coordinates \((p_1, p_2)\) the system (5.1) takes a more simple form

\[
\begin{align*}
\dot{p}_1 &= -p_1 + (\rho_2 - 1)p_1 p_2, \\
\dot{p}_2 &= -p_2 + (\rho_1 - 1)p_1 p_2,
\end{align*}
\]

(5.4)

It is easy to show that under the change (5.3), the vectors \(c^{(1)}, c^{(2)}\) turn into \(\tilde{c}^{(1)} = (1, 0)^T, \tilde{c}^{(2)} = (0, 1)^T\) respectively. Obviously, the system (5.4) has the same Kovalevskaya exponents (5.2). The matrix \(K\), calculated for \(\tilde{c}^{(1)}\) is

\[
K = \begin{pmatrix}
  -1 & \rho_2 - 1 \\
  0 & \rho_1
\end{pmatrix}
\]

Under the assumption \(\rho_1 \neq -1\), we can reduce \(K\) to a diagonal form using the following change of coordinates

\[
\begin{pmatrix}
  p_1 \\
  p_2
\end{pmatrix} = L \begin{pmatrix}
  y_1 \\
  y_2
\end{pmatrix}
\]

with the constant matrix \(L\)

\[
L = \begin{pmatrix}
  1 & \rho_2 - 1 \\
  0 & \rho_1 - 1
\end{pmatrix}
\]

The case \(\rho_1 = -1\) will be considered below.

Finally, equations (5.4) take the form (4.1)

\[
\begin{align*}
\dot{y}_1 &= -y_1^2 + \varphi_1, \\
\dot{y}_2 &= (\rho_1 - 1)y_1 y_2 + \varphi_2,
\end{align*}
\]

(5.5)

where

\[
\varphi_1 = ay_2^2, \quad \varphi_2 = by_2^2, \\
a = (\rho_2 - 1)(\rho_1 + \rho_2), \quad b = (\rho_1 - 1)(\rho_2 - 1) - \rho_1 - 1.
\]
For the operators (4.8) we get

\[ A_+ = \varphi_2 \frac{\partial}{\partial y_2}, \quad A_0 = \rho_1 y_2 \frac{\partial}{\partial y_2}, \quad \tilde{U} = y_2 \frac{\partial}{\partial y_2}. \]

Let \( F_M \) be a polynomial first integral of (5.5) of degree \( M \).

According to Theorem 2, there exists an integer \( k = 1, \ldots, M - 1 \) such that

\[ k \rho_1 = M. \tag{5.6} \]

We exclude the case \( k = M (\rho_1 = 1) \), since if \( \rho_1 = \pm 1 \), then the system (5.1) has no polynomial first integrals. This can be shown directly using equations (5.4), (5.5).

Next, calculate the base function \( P_M \) corresponding to the resonance condition (5.6). Consider the equations (4.9). It is obvious, that polynomials \( I_1, k, \ldots, I_1, M \) can be represented in the following form

\[ I_{1, k+i} = \alpha_i y_2^{k+i}, \quad i = 0, \ldots, M - k, \tag{5.7} \]

where \( \alpha_0, \ldots, \alpha_{M-k} \) are constant parameters.

Substituting (5.7) into (4.9) we obtain

\[ \begin{align*}
\alpha_0 &= 1, \\
\alpha_1 &= \frac{bk}{\rho_1(k+1) - M}, \\
\alpha_i &= \frac{\alpha_i(M-k-i+2)\alpha_{i-1} + b(k+i-1)\alpha_{i-2}}{\rho_1(k+i) - M}, \quad i = 2, \ldots, M - k.
\end{align*} \tag{5.8} \]

According to (4.14), we get the following expression for the base function \( P_M \)

\[ P_M = \sum_{j=0}^{M-k} \alpha_j y_1^{M-k-j} y_2^{k+j}. \tag{5.9} \]

By definition of the base function it is clear that

\[ D_+ P_M = \delta y_2^{M+1}, \tag{5.10} \]

where

\[ \delta = a \alpha_{M-1} + b M \alpha_M. \tag{5.11} \]

Thus, the linear space \( L_M \) contains only one polynomial \( P_M \). Hence, taking into account Lemma 6, \( F_M = \text{const} \cdot P_M \).

Using (5.10), we conclude that \( P_M \) is a first integral if and only if \( \delta = 0 \). In view of (5.6), (5.8), (5.11) and the above condition, we arrive at Lemma 9.

**Theorem 10.** The system (5.1) possesses a homogeneous polynomial first integral of degree \( M \) if and only if the following conditions are fulfilled

a) \( \rho_i, i = 1, 2 \) – are positive rational numbers,

b) \( \rho_1^{-1} + \rho_2^{-1} \leq 1, \)

c) \( \frac{M}{\rho_1} \in \mathbb{N}. \)

This is an obvious consequence of Lemma 9.

As an example consider the following system

\[ \begin{align*}
\dot{x}_1 &= x_1^2 - 9x_2^2, \\
\dot{x}_2 &= -3x_1^2 - 8x_1x_2 + 3x_2^2.
\end{align*} \tag{5.12} \]
The vectors $c^{(1)}$, $c^{(2)}$ have the form
\[ c^{(1)} = (1/8, -1/8)^T, \quad c^{(2)} = (1/8, 1/8)^T. \]
Calculating the corresponding Kovalevskaya exponents (5.2) one obtains
\[ \Re_1 = (-1, 3)^T, \quad \Re_2 = (-1, 3/2)^T. \]
We have $\rho_1 = 3$, $\rho_2 = 3/2$, $\rho_1^{-1} + \rho_2^{-1} = 1$. So, the conditions a), b) of Theorem 10 are fulfilled. By the condition c) one gets $M = 3$, $l \in \mathbb{N}$. Thus, the equations (5.12) possess a cubic first integral $F_3$. Using formulas (5.8), (5.9), we obtain
\[ F_3 = x_1^3 + x_1^2 x_2 - x_1 x_2^2 - x_2^3. \]

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