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Maximal Hamiltonian tori for polygon spaces

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1 Introduction

Let $M$ be a symplectic manifold and let $\mathcal{S}(M)$ be the group of symplectomorphisms of $M$. A sub-torus of $\mathcal{S}(M)$ is called a symplectic torus; these tori are partially ordered by inclusions. In this paper, we study the maximal symplectic tori of polygon spaces with a particular emphasis on bending tori (see the definitions below). Since polygon spaces are simply connected, symplectic tori act on $M$ in a Hamiltonian fashion so we refer to them as Hamiltonian tori.

Let $E$ be a finite set together with a function $\lambda : E \rightarrow \mathbb{R}_+$. Define the space $\widetilde{\text{Pol}} (E, \lambda)$ by

$$\widetilde{\text{Pol}} (E, \lambda) := \left\{ \rho : E \rightarrow \mathbb{R}^3 \mid \sum_{e \in E} \rho(e) = 0 \text{ and } |\rho(e)| = \lambda(e) \forall e \in E \right\}.$$

The polygon space $\text{Pol} (E, \lambda)$ is the quotient $\text{Pol} (E, \lambda) := \widetilde{\text{Pol}} (E, \lambda) / SO_3$. By choosing a bijection between $E$ and $\{1, \ldots, m\}$, the space $\text{Pol} (E, \lambda)$ is regarded as the space of configurations in $\mathbb{R}^3$ of a polygon with $m$ edges of length $\lambda_1, \ldots, \lambda_m$, modulo rotation, whence the name “polygon space”. Also, we call an element of $E$ an edge and $\lambda$ the length function.

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A length function $\lambda$ is called \textit{generic} if there is no map $\varepsilon : E \to \{\pm 1\}$ so that $\sum_{e \in E} \varepsilon(e)\lambda(e) = 0$. This guarantees that the polygon cannot collapse to a line. In this paper, we always assume that $\lambda$ is generic and that $\text{Pol}(E, \lambda)$ is not empty. In this case, $\text{Pol}(E, \lambda)$ is a closed smooth symplectic manifold of dimension $2(|E| - 3) \geq 0$. The polygon spaces are better known as the moduli spaces of (weighted) ordered points on $\mathbb{P}^1$, and also arise via other symplectic reductions (see [Kl], [KM], [HK1] and the proof of Proposition 2.4 below).

A subset $I$ of $E$ is called \textit{lopsided} if there exists $e_0 \in I$ such that $\lambda(e_0) > \sum_{e \in I \setminus \{e_0\}} \lambda(e)$. The empty set is not lopsided, while a singleton $\{e\}$ is always lopsided since the length function takes strictly positive values. The total set $E$ is not lopsided since $\text{Pol}(E, \lambda)$ is assumed to be non-empty.

For $I \subset E$ define $\rho_I : \text{Pol}(E, \lambda) \to \mathbb{R}^3$ by $\rho_I := \sum_{e \in I} \rho(e)$. The continuous function and $f_I : \text{Pol}(E, \lambda) \to \mathbb{R}$ by $f_I(\rho) := |\sum_{i \in I} \rho_i|$ descends to a function on $\text{Pol}(E, \lambda)$, still called $f_I$. When $I$ is lopsided, this function does not vanish and is therefore smooth. Its Hamiltonian flow $\Phi_I^t$ is called the \textit{bending flow} associated to $I$. Bending flows have been introduced in [Kl] and [KM]. They are periodic (see [Kl, §2.1] or [KM, Corollary 3.9]): $\Phi_I^t$ rotates at constant speed the set of vectors $\{\rho(e) \mid e \in I\}$ around the axis $\rho_I$.

A \textit{bending torus} is a Hamiltonian torus in $\mathcal{S}(\text{Pol}(E, \lambda))$ generated by bending flows. Since the dimension of $\text{Pol}(E, \lambda)$ is $2(|E| - 3)$, the dimension of any Hamiltonian torus is at most $|E| - 3$.

In this paper, we study the poset of bending tori and compare it with that of Hamiltonian ones. For instance, the following result is proved in Section 3 (see Corollary 3.2):

\textbf{Theorem A} Let $N(\lambda)$ be the minimal number of lopsided subsets which are necessary for a partition of $E$. Then the maximal dimension of a bending torus for $\text{Pol}(E, \lambda)$ is $|E| - \max\{3, N(\lambda)\}$.

We also give a more general statement that allows us to characterize maximal bending tori. In some cases, these coincide with maximal Hamiltonian tori:\n
\textbf{Theorem B} Let $T$ be a bending torus of $\text{Pol}(E, \lambda)$ of dimension $\geq |E| - 5$. Then $T$ is a maximal Hamiltonian torus if and only if it is a maximal bending torus.

In Section 5, we give several examples where maximal Hamiltonian tori are not all of the same dimension. Using the work of Y. Karshon [Ka], we
show the existence of Hamiltonian tori which are not conjugate to a bending torus (Proposition 5.5). Finally, the relationship with maximal tori in the contactomorphism group of pre-quantum circle bundles, due to E. Lerman [Le], is mentioned in 5.6.

2 Preliminaries - Bending sets

Lemma 2.1 Let $\mathcal{I}$ be a family of lopsided subsets of $E$. The following conditions are equivalent:

a) The bending flows $\{\Phi_t^I \mid I \in \mathcal{I}\}$ generate a bending torus.

b) For each pair $A, B \subseteq \mathcal{I}$, either $A \cap B = \emptyset$ or one is contained into the other.

Proof: By [Kl, §2.1] or [KM, Corollary 3.9], the bending flows are periodic. Therefore, a) is equivalent to the fact that $\{f_A, f_B\} = 0$ for all $A, B \in \mathcal{I}$, where $\{\cdot, \cdot\}$ denotes the Poisson bracket. Proposition 2.1.2 of [Kl] shows that $\{f_A^2, f_B^2\} = 0$ if and only if the pair $A, B$ satisfies Condition b).

Since $f_A$ and $f_B$ never vanish, the formula

$$\{f_A^2, f_B^2\} = 4 f_A f_B \{f_A, f_B\}$$

implies that $\{f_A^2, f_B^2\} = 0$ if and only if $\{f_A, f_B\} = 0$. □

A set $\mathcal{I}$ of lopsided subsets of $E$ is called a bending set if it contains every singleton $\{e\}$ and satisfies the following “absorption condition”: for each pair $A, B \subseteq \mathcal{I}$, either $A \cap B = \emptyset$ or one is contained in the other.

Bending sets are technically convenient to parametrize bending tori. Indeed, let $\mathcal{I}$ be a bending set. By 2.1), the bending flows $\{\Phi_t^I \mid I \in \mathcal{I}\}$ generate a bending torus $T_\mathcal{I}$. Conversely, if $T$ is a bending torus, there is at least one set $\mathcal{I}$ of lopsided subsets satisfying the absorption condition such that $T = T_\mathcal{I}$, and one can add singletons to $\mathcal{I}$ to make it a bending set.

The elements of $\mathcal{I}$ are partially ordered by inclusions, so one can associate to $\mathcal{I}$ the family $\mathcal{M}_\mathcal{I}$ of its maximal elements. A direct consequence of the definition is that $\mathcal{M}_\mathcal{I}$ is a partition of $E$.

A bending set $\mathcal{I}$ is called full if, for each $I \in \mathcal{I}$ which is not a singleton, there exist $I', I'' \in \mathcal{I}$ so that $I$ is the disjoint union of $I'$ and $I''$. It is easy to check that this condition is equivalent to either of the following.

a) Given $I$ and $I'$ in $\mathcal{I}$ such that $I' \subset I$, the union $\mathcal{I} \cup \{I'\}$ is not a bending set. This justifies the term “full”: one can no longer add elements to $\mathcal{I}$ and keep the latter a bending set.

b) For all $I \in \mathcal{I}$ the set $\{I' \in \mathcal{I} : I' \subseteq I\}$ contains $2|I| - 1$ elements.
**Remark**  Let $\mathcal{I}$ be a bending set. The reader might find it helpful to consider the graph of this poset. It is a union of disjoint trees, each of which contains a unique maximal element. The bending set $\mathcal{I}$ is full iff these trees are binary: each vertex has one edge leaving it (except the maximal ones which have none) and 2 edges pointing into it (except the singletons which have none).

**Lemma 2.2** Let $\mathcal{I}$ be a bending set. Then there exists a (non-unique) bending set $\hat{\mathcal{I}}$ such that the following conditions hold

1) $\mathcal{I} \subset \hat{\mathcal{I}}$ (therefore $T_\mathcal{I} \subset T_{\hat{\mathcal{I}}}$).
2) $\hat{\mathcal{I}}$ is full.
3) $\mathcal{M}_{\hat{\mathcal{I}}} = \mathcal{M}_\mathcal{I}$.

**Proof:** If $\mathcal{I}$ is full we are done. Otherwise, we proceed by induction on the number of “non-full” elements of $\mathcal{I}$: those $I \in \mathcal{I}$ which are not singletons and are not the disjoint union of 2 elements of $\mathcal{I}$. Let $I \in \mathcal{I}$ be a minimal “non-full” element.

Let $I_1, \ldots, I_r$ be the maximal proper subsets of $I$ which are elements of $\mathcal{I}$. One of them, say $I_1$, contains the longest edge of $I$. For $i = 2, \ldots, r - 1$, define $R_i := I_1 \cup \cdots \cup I_i$ and let $\tilde{\mathcal{I}} := \mathcal{I} \cup \{R_2\} \cup \cdots \cup \{R_{r-1}\}$. One has $I = R_{r-1} \cup I_r$, $R_{r-1} = R_{r-2} \cup I_{r-1}$ etc. As $I$ was minimal, it is no longer non-full in $\tilde{\mathcal{I}}$. This gives the inductive step.

We shall now compute the dimension of a bending tori. We need some knowledge about the critical points of the maps $f_I$ and its symplectic reduction. The following lemma comes from [Ha, Theorem 3.2].

**Lemma 2.3** Let $I$ be a lopsided subset of $E$. An element $\rho \in \text{Pol}(E, \lambda)$ is a critical point for $f_I$ if and only if either the set $\{\rho(e) | e \in I\}$ or the set $\{\rho(e) | e \notin I\}$ lies in a line.

**Proposition 2.4** Let $A \subset E$. Define $\tilde{A} := A \cup \{A\}$ and $\lambda^{A,t} : \tilde{A} \to \mathbb{R}$ by $\lambda^{A,t}(e) := \lambda(e)$ for $e \in A$ and $\lambda^{A,t}(A) := t$. Then, if $A$ is lopsided, the symplectic reduction of $\text{Pol}(E, \lambda)$ at $t$, for the action of the bending circle $T_A$, is symplectomorphic to the product of the two polygon spaces

$$\text{Pol}(E, \lambda) \sslash T_A \cong \text{Pol}(\tilde{A}, \lambda^{A,t}) \times \text{Pol}(E - \tilde{A}, \lambda^{E-A,t}).$$
Remark 2.5  a) Proposition 2.4 holds true even if $t$ is not a regular value.
If it is, the two right hand polygon spaces of the formula are generic by Lemma 2.3.

b) The following is clear from the proof below: if $T_I$ is a bending torus
and $A \in I$, then the action of $T_I$ descends to the reduced space, giving rise
to a product of two bending tori: one for the bending set \{ $I \in I \mid I \subset A$\}
and the other for \{ $I \in I \mid I \not\subset A$\}

c) In this paper, Proposition 2.4 is used only for $|A| = 2$. In this case,
the reduction of Pol $(E, \lambda)$ at $t$ is symplectomorphic to a polygon space with
$|E| - 1$ edges, since Pol $(\tilde{A}, \lambda^{A,t})$ is a point. However, the hypothesis $|A| = 2$
does not simplify the proof.

Proof of Proposition 2.4 : First recall the precise definition for the
symplectic structure on Pol $(E, \lambda)$ (for details, see [HK1, § 1]). For $s \in \mathbb{R}$,
let $\mathcal{O}(s)$ the coadjoint orbit of $SO(3)$ with symplectic volume $2s$. With
the usual identification of $so(3)^*$ with $\mathbb{R}^3$, $\mathcal{O}(s)$ is the 2-sphere centered in
0 of radius $r$. For $A \subset E$, let $\mu_A : \prod_{e \in E} \mathcal{O}(\lambda(e)) \to \mathbb{R}^3$ be the partial
sum $\mu_A((z_e)) := \sum_{e \in A} z_e$. This is the moment map for the diagonal action
of $SO(3)$ on the component indexed by $e \in A$. The space Pol $(E, \lambda) = \mu_E^{-1}(0)/SO(3)$ is then the symplectic reduction

$$Pol(E, \lambda) = \prod_{e \in E} \mathcal{O}(\lambda(e)) \parallel SO(3)$$

for the diagonal action of $SO(3)$. This determines the symplectic structure
on Pol $(E, \lambda)$.

The codimension 2-embedding

$$V_t := \mu_A^{-1}(\mathcal{O}(t)) \cap \mu_E^{-1}(0) \hookrightarrow \mu_A^{-1}(\mathcal{O}(t)) \times \mu_{E-A}^{-1}(\mathcal{O}(t))$$

(1)
gives rise to a diffeomorphism

$$\frac{[V_t/SO(3)]}{T_A} \cong \frac{\mu_A^{-1}(\mathcal{O}(t))/SO(3) \times \mu_{E-A}^{-1}(\mathcal{O}(t))/SO(3)}{\mu_A^{-1}(\mathcal{O}(t)) \times \mu_{E-A}^{-1}(\mathcal{O}(t))}$$

(2)

As the embedding (1) is the restriction of the obvious symplectomorphism

$$\prod_{e \in E} \mathcal{O}(\lambda(e)) \cong \prod_{e \in A} \mathcal{O}(\lambda(e)) \times \prod_{e \in E-A} \mathcal{O}(\lambda(e)).$$

(3)
and as all group actions preserve the symplectic forms, the diffeomorphism \( \mathfrak{H} \) is a symplectomorphism. \( \blacksquare \)

**Proposition 2.6** Let \( I \) be a bending set for \( \text{Pol} (E, \lambda) \). Then

\[
\dim T_I \leq |E| - \max\{3, |\mathcal{M}_I|\}
\]

with equality if and only if \( I \) is full.

**Proof:** By Lemma 2.2, it is enough to prove the formula when \( I \) is full. We proceed by induction on the number of elements of \( I \) which are not singletons. If there are none, then \( \dim T_I = 0 = |E| - |E| \) and the formula holds true (recall that \( |E| \geq 3 \) since we suppose that \( \text{Pol} (E, \lambda) \neq \emptyset \)).

Otherwise, as \( I \) is full, there is \( A \in I \) with \( |A| = 2 \).

If \( |E| = 3 \), the formula holds true (the 0-torus, being a quotient of \( R^0 \), is of dimension 0). We may then assume that \( |E| \geq 4 \).

The map \( f_A : \text{Pol} (E, \lambda) \rightarrow R \) is a moment map for the bending circle \( T_A \). As \( |E| \geq 4 \), it is not constant. Let \( s \) be a regular value of \( f_A \) (\( s > 0 \) since \( A \) is lopsided). By Proposition 2.4, the symplectic reduction of \( \text{Pol} (E, \lambda) \) at \( s \) is a generic polygon space with \( |E| - 1 \) edges. By Part b) of Remark 2.5, the bending set \( I \) coinduces a bending set \( \bar{I} \) for \( \lambda \) which is full. The number of non-singletons elements of \( \bar{I} \) is one less than that of \( I \). By induction hypothesis, one has

\[
\dim T_{\bar{I}} = |E| - 1 - \max\{3, |\mathcal{M}_{\bar{I}}|\}.
\]

As \( \dim T_I = \dim T_{\bar{I}} + 1 = \mathcal{M}_{\bar{I}} = \mathcal{M}_I \), one gets the required expression for \( \dim T_I \). \( \blacksquare \)

### 3 Maximal bending tori

In this section, we study the poset of bending tori. Let \( K \) and \( L \) be two partitions of \( E \). We say that \( L \) is coarser than \( K \) if each element of \( L \) is a union of elements of \( K \).

**Theorem 3.1** Let \( I \) be a bending set for \( \text{Pol} (E, \lambda) \). Let \( N(\lambda, I) \) be the minimal number of lopsided subsets which are necessary for a partition of \( E \) which is coarser than \( \mathcal{M}_I \). Then, the maximal dimension \( n(\lambda, I) \) of a bending torus for \( \text{Pol} (E, \lambda) \) containing \( T_I \) is

\[
n(\lambda, I) = |E| - \max\{3, N(\lambda, I)\}.
\]
Proof: Let $T$ be a bending torus containing $T_I$. By Section 2, $T = T_J$ for a bending set $J$. By Lemma 2.1, the partition $\mathcal{M}_J$ is coarser than $\mathcal{M}_I$. By Lemma 2.3, one has
\[
\dim T_J \leq |E| - \max\{3, |\mathcal{M}_J|\} \leq |E| - \max\{3, N(\lambda, I)\}
\]
and therefore
\[
n(\lambda, I) \leq |E| - \max\{3, N(\lambda, I)\}.
\]

Conversely, let $J_0$ be a partition of $E$ into lopsided subsets, coarser than $\mathcal{M}_I$, with $N(\lambda, I)$ elements. Let $J := J_0 \cup I$. One can easily check that $J$ is a bending set. Let $\hat{J}$ be a full bending set associated to $J$ as in Lemma 2.2. One has $\mathcal{M}_{\hat{J}} = J_0$ and, by Proposition 2.4, one has,
\[
n(\lambda, I) \geq \dim T_{\hat{J}} = |E| - \max\{3, N(\lambda, J)\}.
\]

As a corollary, we obtain Theorem A of the introduction:

**Theorem 3.2 (Theorem A)** Let $N(\lambda)$ be the minimal number of lopsided subsets which are necessary for a partition of $E$. Then the maximal dimension of a bending torus for $\text{Pol}(E, \lambda)$ is $|E| - \max\{3, N(\lambda)\}$.

**Proof:** Set $I$ be the sets of singletons of $E$ in the statement of Theorem 3.1

We now give a characterization of the maximal bending tori which will be used later. We can restrict our attention to those $T_I$, for $I$ a full bending set, whose dimension is less than $|E| - 3$ (the maximal possible dimension of a Hamiltonian torus of $\text{Pol}(E, \lambda)$).

**Proposition 3.3** Let $I$ be a full bending set so that $\dim T_I < |E| - 3$. Then, $T_I$ is a maximal bending torus iff
\[
\bigcap_{J \in \mathcal{M}_J} \text{Image}(f_J) \neq \emptyset
\]

**Proof:** Observe that $T_I$ is a maximal bending torus if and only if for each pair $I, J \in \mathcal{M}_I$, one has $\text{Image}(f_I) \cap \text{Image}(f_J) \neq \emptyset$ ($I \cup J$ is not lopsided). The condition of Proposition 3.3 is a priori stronger than that but in fact equivalent, thanks to the following lemma.

**Lemma 3.4** Let $A_0, \ldots, A_n$ be intervals of the real line. If $A_i \cap A_j \neq \emptyset$ for all $i, j$, then $A_1 \cap \cdots \cap A_n \neq \emptyset$.  

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Proof: By induction on $n$, starting with $n = 2$. The condition $A_i \cap A_j \neq \emptyset$ for all $i, j$ implies that $A := A_1 \cup \cdots \cup A_n$ is connected and hence is an interval. The set $\mathcal{A} := \{A_0, \ldots, A_n\}$ is an acyclic covering of $A$ and therefore its nerve $\mathcal{N}(\mathcal{A})$ can be used to compute the cohomology of $A$: $H^*(A) = H^*(\mathcal{N}(\mathcal{A}))$. By induction hypothesis, the simplicial set $\mathcal{N}(\mathcal{A})$ contains the $n-1$ skeleton of the simplex $\Delta^n$. As $H^{n-1}(A) = 0$, $\mathcal{N}(\mathcal{A})$ must contain $\Delta^n$ which is to say $A_1 \cap \cdots \cap A_n \neq \emptyset$.

4 Maximal Hamiltonian tori

We start with an important special case which illustrate the technique: the almost regular pentagon. A function $\lambda : \{1, \ldots, 5\} \to \mathbb{R}_+$ is called the length function of an almost regular pentagon if $\lambda(i) = 1$ for $i = 1, \ldots, 4$ and $1 < \lambda(5) < 2$. In this case, $\dim \text{Pol}(E, \lambda) = 4$.

Proposition 4.1 Let $\lambda : \{1, \ldots, 5\} \to \mathbb{R}_+$ be a length function of an almost regular pentagon. Then, the maximal bending tori of $\text{Pol}(E, \lambda)$, which are 1-dimensional, are maximal Hamiltonian tori.

Proof: The maximal lopsided subset of $E$ are of the form $\{k, 5\}$. Therefore, all maximal bending tori are of dimension 1. Since they are all of the same form, it is enough to prove Proposition 4.1 for one of them, say $T_{4,5}$ with $\mathcal{I} := \{\{1\}, \{2\}, \{3\}, \{4, 5\}\}$. This gives a Hamiltonian circle action with moment map $f := f_{\{4,5\}} = |\rho(4) + \rho(5)|$. By Lemma 2.3 this map has three critical values:

a) The two extremals $z = \lambda(5) - 1$ and $z = \lambda(5) + 1$ are of course critical values. In both cases, the critical set is a 2-sphere, the configuration spaces of the quadrilateral with side length $(1, 1, 1, z)$.

b) the value 1 for which the critical set consists of three points, namely the configurations $\rho : \{1, \ldots, 5\} \to \mathbb{R}^3$ given by one of the line of equations below

$$
-\rho(1) = \rho(2) = \rho(3) = -\rho(4) - \rho(5),
\rho(1) = -\rho(2) = \rho(3) - \rho(4) - \rho(5) \text{ or }
\rho(1) = \rho(2) = -\rho(3) - \rho(4) - \rho(5).
$$

The proof then follows from the lemma below.

Lemma 4.2 Let $\mu : M \to \mathbb{R}^{m-1}$ be the moment map for a Hamiltonian action of of $T^{m-1}$ on a compact symplectic manifold $M^{2n}$. Denote by $\text{Crit} \mu \subset M$ the set of critical points of $\mu$. Suppose that there is a point
δ in the interior of the moment polytope μ(M) such that μ−1(δ) ∩ Crit μ has at least 3 connected components. Then the action does not extend to an effective Hamiltonian action of a m-torus.

Proof: Suppose that T extends to a Hamiltonian action of \( T \times S^1 \) with moment map \( \Phi : Pol(E, \lambda) \to \mathbb{R}^n \). Then the moment map \( f \) is the composition of \( \Phi \) with the projection \( \mathbb{R}^n \to \mathbb{R} \) onto the last coordinate. Additionally, this action, being effective, would make \( Pol(\lambda) \) a symplectic toric manifold. Thus, \( \Phi(\rho) \) are distinct points on the boundary of the moment polytope \( \phi(Pol(E, \lambda)) \) (see [De]), which all project to 1. As at most two points of this boundary can project onto one point of \( \mathbb{R} \), we get a contradiction. \( \square \)

The rest of this section is devoted to the proof of our second main result:

**Theorem 4.3 (Theorem B)** Let \( T \) be a bending torus of \( Pol(E, \lambda) \) of dimension \( \geq |E| - 5 \). Then \( T \) is a maximal Hamiltonian torus if and only if it is a maximal bending torus.

We only need to prove Theorem B in the cases \( \dim T = |E| - 4 \) and \( |E| - 5 \), since it is obvious for \( \dim T = |E| - 3 \).

**Proof for \( \dim T = |E| - 4 \):** Let \( I \) be a bending set so that \( T_I \) is a maximal bending torus of dimension \( |E| - 4 \). We suppose that there is a Hamiltonian circle \( S^1 \) commuting with \( T_I \); we shall prove that the resulting action of \( \hat{T} := T_I \times S^1 \) is not effective.

Let \( f_I : Pol(E, \lambda) \to \mathbb{R}^2 \) be the product map \( f_I := \prod_{A \in I} f_A \). This is a moment map for the action of \( T_I \). Its image \( \Delta \) is a convex polytope of dimension \( |E| - 4 \). Let \( \mu \) be the composition of \( f_I \) with the projection to the affine space spanned by \( \Delta \) (the “essential” moment map).

By Proposition 2.6, \( I \) is full and has 4 maximal elements: \( \mathcal{M}_I = \{I, J, K, L\} \). By Proposition 3.3, there exists a point \( c \) in the intersection of the images of \( f_I, f_J, f_K \), and \( f_L \). The proof divides into 3 cases:

**Case a):** Suppose that \( c \) is in the interior of each image. Then \( \tilde{c} := (c, c, c, c) \) belongs to the interior of the image of the product map \( f := f_I \times f_J \times f_K \times f_L : Pol(E, \lambda) \to \mathbb{R}^4 \). This product map is the composition of \( \mu \) with the projection to \( \mathbb{R}^{\mathcal{M}_I} \). Hence, there exists \( \delta \) in the interior of \( \Delta \) which projects to \( \tilde{c} \).
For any $\rho \in \widetilde{\text{Pol}}(E, \lambda)$ such that $\mu(\rho) = \delta$, there exists $R_I, R_J, R_K, R_L \in SO(3)$ such that
\[ R_I(\rho_I) = R_J(\rho_J) = -R_K(\rho_K) = -R_L(\rho_L). \]
Then the configuration $\rho'$ defined by
\[ \rho'(e) := R_I(\rho(e)) \text{ if } e \in I, \rho'(e) := R_J(\rho(e)) \text{ if } e \in J, \text{ etc.} \]
also satisfies $\mu(\rho') = \delta$ and moreover $\rho'_I = \rho'_J = -\rho'_K = -\rho'_L$. This implies that $\rho'$ is a critical point for the function $h := f_I + f_J - f_K - f_L$ and hence for $\mu$. Indeed, the Hamiltonian flow of $\dot{h}$ would be a global rotation around the axis $\rho_I$, and therefore induces the identity on $\text{Pol}(E, \lambda)$.

Similarly, one constructs critical configurations in $\mu^{-1}(\delta)$ with $\rho_I = -\rho_J = \rho_K = -\rho_L$ and $\rho_I = -\rho_J = -\rho_K = \rho_L$. By lemma 1.2, this completes the first case.

Case b): the argument of Case a) works as well if $c$ is in the interior of the image $f_A$ for each $A \in M_I$ which is not a singleton (by genericity of $\lambda$, there exists at least one such element).

Case c): in the general case, there may be some set $A \in M_I$, such that $c$ is in the boundary of the image of $f_A$. Let $M' \subset M_I$ be the set of such $A$'s and let $M'$ be the partition of $E$ generated by $M'$ (formed by the elements of $M'$ and the singletons). Call $I'$ the largest sub-poset of $I$ so that $M_{I'} = M'$; this is a full bending set.

In this case, $\tilde{P} := f^{-1}(c)$ is a symplectic submanifold of $\text{Pol}(E, \lambda)$ on which $T_{I'}$ acts trivially. As $\tilde{P}$ coincides with the result of successive symplectic reductions at $c$ for the various $f_A$ with $A \in M'$, it is, by Proposition 2.4, symplectomorphic to the polygon space $\text{Pol}(M', \tilde{\lambda})$, where
\[ \tilde{\lambda}(\{e\}) = \lambda(e) \quad \text{and} \quad \tilde{\lambda}(A) = c \text{ if } A \in M'. \]
The bending torus $T_{I'}$ acts on $\tilde{P}$, giving rise to a bending torus $T_{\tilde{I}}$ isomorphic to $T_{I'/T_{I'}}$. Observe that $\tilde{I}$ has 4 maximal elements and that we are in Case b). Therefore, $T_{\tilde{I}}$ is a maximal Hamiltonian torus and the induced action of $\tilde{T}$ on $\tilde{P}$ has a kernel of dimension strictly larger than that of $T_{I'}$. Therefore, as
\[ \dim \text{Pol}(E, \lambda) - \dim \tilde{P} = 2\left( \sum_{A \in M'} |A| - |M'| \right) = 2 \dim T_{I'}, \]
there is a circle in $\tilde{T}$ acting trivially on a tubular neighborhood of $\tilde{P}$. Hence, by the generic orbit type theorem [Al, § 2.2], the action of $\tilde{T}$ on Pol $(E, \lambda)$ is not effective. \[ \square \]
Proof for \( \dim T = |E| - 5 \): Let \( \mathcal{I} \) be a bending set so that \( T_{\mathcal{I}} \) is a maximal bending torus of dimension \( |E| - 5 \). We suppose that there is a Hamiltonian circle \( S^1 \) commuting with \( T_{\mathcal{I}} \) and we shall prove that the resulting action of \( \hat{T} := T_{\mathcal{I}} \times S^1 \) is not effective.

Let \( \mu : \text{Pol}(E, \lambda) \to \mathbb{R}^{|E| - 5} \) be the essential moment map, defined as in the proof for \( \dim T = |E| - 4 \), and let \( \Delta \) be the image of \( \mu \). Let \( \hat{\mu} : \text{Pol}(E, \lambda) \to \Delta \times \mathbb{R} \) be a moment map for the action of \( \hat{T} \) with first component equal to \( \mu \) and let \( \hat{\Delta} \) be the image of \( \hat{\mu} \).

By Proposition 2.6, \( M_{\mathcal{I}} \) has 5 elements. By Proposition 3.3, there exists a point \( c \) in the intersection of the images of \( f_A \) for \( A \in M_{\mathcal{I}} \). The proof divides into several cases:

Case 1) : Suppose that \( |E| = 5 \). Then \( T_{\mathcal{I}} \) is of dimension 0 and we have to know that a maximal Hamiltonian torus for a regular pentagon space is also of dimension 0. This is the contents of \([HK2, \text{Theorem 3.2}]\).

Case 2) : Suppose that each \( A \in M_{\mathcal{I}} \) contains exactly 2 elements (hence \( |E| = 10 \)) and \( c \) is in the interior of the image of \( f_A \). This implies that \( \vec{c} := (c, c, c, c, c) \) is a regular value of \( \mu \). The reduction \( Q \) of \( \text{Pol}(E, \lambda) \) at \( \vec{c} \) is then symplectomorph to a regular pentagon space (apply Proposition 2.4 five times). The induced Hamiltonian action of \( \hat{T} \) on \( Q \) is then trivial by Case 1). This implies that the image of the differential \( D\hat{\mu} \) at any point of \( \mu^{-1}(\vec{c}) \) is parallel to \( \Delta \times \{0\} \). By convexity, we deduce that \( \hat{\Delta} \) and \( \Delta \) have the same dimension and therefore the action of \( \hat{T} \) is not effective.

Case 3) : The argument of Case 2) works as well if each \( A \in M_{\mathcal{I}} \) has \( \leq 2 \) elements and \( c \) is in the interior of the image of \( f_A \) when \( |A| = 2 \). Also, if there are sets \( A \in M_{\mathcal{I}} \) with \( |A| = 2 \) and \( c \) is in the boundary of the image of \( f_A \), one proceeds as in Case c) of the proof for \( \dim T_{\mathcal{I}} = |E| - 4 \) to deduce that the action of \( \hat{T} \) is not effective. Thus, we are able to prove our result when all the elements of \( M_{\mathcal{I}} \) are either singletons or doubletons.

General case) : For \( A \in M_{\mathcal{I}} \), let \( k_A := \max\{0, |A| - 2\} \) and \( k := \sum_{A \in M_{\mathcal{I}}} k_A \). The proof goes by induction on \( k \), the case \( k = 0 \) being established in Case 3). If \( k > 0 \), let \( A \in M_{\mathcal{I}} \) such that \( |A| \geq 3 \). If \( c \) lies in the boundary of the image of \( f_A \), one proceeds as in Case c) of the proof for \( \dim T_{\mathcal{I}} = |E| - 4 \) to deduce that the action of \( \hat{T} \) is not effective (using the induction hypothesis). Otherwise, as \( \mathcal{I} \) is full, there exists \( B \in \mathcal{I} \) such that \( |B| = 2 \), \( B \subset A \) and \( f_B(f_A^{-1}(c)) \) is an interval of positive length. It contains an open interval \( J \) of regular values of \( f_B \). For \( t \in J \), the reduction of \( \text{Pol}(E, \lambda) \) for the action of the Hamiltonian circle with moment map \( f_B \) is, by Proposition
symplectomorphic to an \((|E| − 1)\)-gon space \(\bar{P}\). The bending torus \(T_\bar{\Sigma}\) descends to a bending torus \(T_\Sigma\) for \(\bar{P}\). One has \(\mathcal{M}_\bar{\Sigma} = \mathcal{M}_\Sigma\) and \(k = k - 1\). By induction hypothesis, \(T_\bar{\Sigma}\) is a maximal Hamiltonian torus. This implies that each point of \(f_B^{-1}(t)\) has a stabilizer of positive dimension for the action of \(\bar{T}\). This holds true for all \(t \in J\), therefore for an open set of \(\text{Pol}(E, \lambda)\). By the generic orbit type theorem [Au, § 2.2], this implies that the action of \(\hat{T}\) on \(\text{Pol}(E, \lambda)\) is not effective.  5 Examples Notations : When \(E = \{1, \ldots, n\}\), we describe \(\text{Pol}(E, \lambda)\) by writing the values of \(\lambda\). For instance, \(\text{Pol}(1, 1, 1, 2)\) stands for \(\text{Pol}(\{1, 2, 3, 4\}, \lambda)\) with \(\lambda(1) = \lambda(2) = \lambda(3) = 1\) and \(\lambda(4) = 2\). A bending set is described by listing its elements which are not singletons and labeling the edges by their length.

5.1 The “two long edge” case : Suppose that the set of edges \(E\) contains two elements \(a, b\) such that

\[
\lambda(a) + \lambda(b) > \sum_{e \in E - \{a, b\}} \lambda(e).
\]

Then \(E\) is the disjoint union of \(E_a\) and \(E_b\) so that \(E_a\) is lopsided with longest edge \(a\) and \(E_b\) is lopsided with longest edge \(b\). One then has \(N(\lambda) = 2\) and, by Theorem 3.1, \(\text{Pol}(E, \lambda)\) admits a bending torus of dimension \(|E| - 3\). In particular, \(\text{Pol}(E, \lambda)\) is a toric manifold.

5.2 Almost regular pentagon : The almost regular pentagon \(\text{Pol}(1, 1, 1, 1, a)\) with \(1 < a < 2\) (or \(0 < a < 1\)) is a very important special case, already used in Proposition 4.1. Notice \(\text{Pol}(E, \lambda)\) is diffeomorphic to \(\mathbb{C}P^2 \sharp 4 \mathbb{C}P^2\) (see [HK2], Example 10.4).

We used the result of [HK2] that the regular pentagon space admits no non-trivial circle action. This is not known for regular polygon spaces with more edges. Nor it is known whether an almost regular pentagon space is diffeomorphic to a toric manifold.

5.3 Hamiltonian tori of different dimensions : Consider a generic pentagon space of the form \(P_{a,b} := \text{Pol}(1, 1, 1, a, b)\) with \(a \neq 1 \neq b\) and \(0 < a - b < 1 < a + b\). The bending circle \(\{a, b\}\) is a maximal Hamiltonian torus by
Proposition 3.3 and 4.3. However, \( \operatorname{Pol}(1, 1, 1, a, b) \) is a toric manifold by the bending tori \( T_I \) of the form \( I := \{(1, a), \{1, b\}\} \). In this example, one sees that maximal bending tori, as well as maximal Hamiltonian tori, are not all of the same dimension.

The moment polytope for \( T_I \) shows that \( P_{a,b} \) is diffeomorphic to \( \mathbb{CP}^2 \# 4 \mathbb{CP}^2 \) if \( a + b < 3 \) and to \( \mathbb{CP}^2 \# 3 \mathbb{CP}^2 \) if \( a + b > 3 \) (the case \( a + b = 3 \) is not generic). It is known that the other pentagon spaces are 4-manifolds with second Betti number \(< 3. For them, any Hamiltonian circle action extends to a toric action by \([s, \text{ Th. 1}].

An example with maximal Hamiltonian tori of 3 different dimensions is provided by the heptagon spaces \( \operatorname{Pol}(1, 1, 2, 2, 3, 3, 3) \) (it is generic since lengths are integral and the perimeter is odd). The 3 bending sets with maximal (non-singleton) elements of the form

\[
\{\{2, 1\}, \{2, 1\}\} , \{\{2, 1\}, \{3, 1\}, \{3, 2\}\} , \{\{3, 1, 1\}, \{3, 2\}, \{3, 2\}\}
\]

determine maximal Hamiltonian tori of dimension respectively 2, 3 and 4. Observe that the bending circle \( \{3, 2\} \) is contained in two maximal tori of different dimension.

Examples in higher dimension can be constructed by adding “little edges” to the previous one, for instance the \((7 + m)\)-gon space

\[ \operatorname{Pol}(1, 1, 2, 2, 3, 3, 3, 1/2, 1/4, \ldots , 1/2^m). \]

It admit full bending sets with maximal (non-singleton) elements of the form

- \( \{\{2, 1\}, \{2, 1\}, \{3, 1/2, 1/4, \ldots , 1/2^m\}\} \)
- \( \{\{2, 1\}, \{3, 1\}, \{3, 2\}, \{3, 1/2, 1/4, \ldots , 1/2^m\}\} \)
- \( \{\{3, 1, 1\}, \{3, 2\}, \{3, 2\}, \{3, 1/2, 1/4, \ldots , 1/2^m\}\} \)

which determine maximal Hamiltonian tori of dimension respectively \( m + 2 \), \( m + 3 \) and \( m + 4 \).

5.4 Let \( T_1 \) and \( T_2 \) be two Hamiltonian tori of dimension \( n \) for a symplectic manifold \( M^{2n} \). Choose isomorphisms \( \operatorname{Lie}(T_1)^* \approx \mathbb{R}^n \approx \operatorname{Lie}(T_2)^* \). the moment polytopes \( \Delta_1 \) and \( \Delta_2 \) of the two actions are in \( \mathbb{R}^n \). By Delzant’s theorem, \( T_1 \) is conjugate to \( T_2 \) in the group \( S(M) \) of symplectomorphism.
of $M$ if and only if the moment polytopes $\Delta(T_i)$ satisfy $\Delta(T_2) = \psi(\Delta(T_1))$
where $\psi$ is a composition of translations and transformations in $GL(\mathbb{Z}^n)$.

Consider the pentagon space $P := \text{Pol}(1, a, c, c, c)$, with $c > a + 1 > 2$. The two bending tori $T_1 = \{\{c, 1\}, \{c, a\}\}$ and $T_2 = \{\{c, 1\}, \{c, a, 1\}\}$ have moment polytopes

$\Delta(T_1)$  
\[2 \quad 2a\]

$\Delta(T_2)$  
\[2a - 2 \quad 2a + 2\]

Therefore, $T_1$ and $T_2$ are not conjugate in the group $S(P)$. One can check that any other bending torus is conjugate to either $T_1$ or $T_2$.

On the other hand, the polytope $\Delta(T_1)$ shows that $P$ is symplectomorphic to $(S^2 \times S^2, \omega_1 + a\omega_2)$, where $\omega_1$ and $\omega_2$ are the pull back of the standard area form on $S^2$ via the two projection maps. By [Ka, Th. 2], the number of conjugacy classes of maximal Hamiltonian tori is equal to $[a]$, the smallest integer greater than or equal to $a$. This proves the following

**Proposition 5.5** If $c > a + 1 > 3$, then $\text{Pol}(1, a, c, c, c)$ admits Hamiltonian tori which are not conjugate to a bending torus.

5.6 Let $(M, \omega)$ be a simply connected symplectic manifold such that $[\omega] \in H^2(M; \mathbb{R})$ is integral. Then there exists a principal circle bundle $S^1 \to Q \to M$ with Euler class $[\omega]$ and $Q$ carries a natural contact distribution by a theorem of Boothby and Wang [BW, Th. 3]. In [Le, Th. 1], E. Lerman recently proved that maximal Hamiltonian tori in $M$ (of dimension $k$) give rise to maximal tori (of dimension $k + 1$) in the group of diffeomorphism of $Q$ preserving the contact distribution.

By [HK1, Prop. 6.5], the symplectic form on $\text{Pol}(E, \lambda)$ is integral when, for example, $\lambda$ takes integral values. Then, our examples in 5.3 give rise to contact manifolds with maximal tori of different dimensions in their group of contactomorphisms (see [Le, Example 2]).
References


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