Studying the multivariable Alexander polynomial by means of Seifert surfaces

CIMASONI, David

Abstract

We show how Seifert surfaces, so useful for the understanding of the Alexander polynomial $\Delta_L(t)$, can be generalized in order to study the multivariable Alexander polynomial $\Delta_L(t_1,\ldots,t_\mu)$. In particular, we give an elementary and geometric proof of the Torres formula.


arxiv : math/0406150v1
STUDYING THE MULTIVARIABLE ALEXANDER POLYNOMIAL BY MEANS OF SEIFERT SURFACES

DAVID CIMASONI

Abstract. We show how Seifert surfaces, so useful for the understanding of the Alexander polynomial $\Delta_L(t)$, can be generalized in order to study the multivariable Alexander polynomial $\Delta_L(t_1, \ldots, t_\mu)$. In particular, we give an elementary and geometric proof of the Torres formula.

1. Introduction

The technique of Seifert surfaces, discovered by Herbert Seifert [12] in 1935, enabled him to make great progress in the study of the Alexander polynomial of a knot. In particular, he succeeded in characterizing among all Laurent polynomials $\Delta(t)$ those that can be realized as the Alexander polynomial of a knot. The introduction by Ralph Fox of the multivariable Alexander polynomial $\Delta_L(t_1, \ldots, t_\mu)$ of a $\mu$-component oriented link $L$ naturally gave rise to the corresponding question for this new invariant (see [6, Problem 2]). Guillermo Torres made use of the free differential calculus – developed at that time by Fox – to give several conditions for a polynomial $\Delta$ in $\mathbb{Z}[t_1^{\pm 1}, \ldots, t_\mu^{\pm 1}]$ to be the Alexander polynomial of a $\mu$-component link [13, 5]. Since then, very little progress has been made: it is known that the Torres conditions are not sufficient in general [7, 11], but a complete algebraic characterization remains out of reach.

In this paper, we present an original approach to this problem. We show how the technique of Seifert surfaces can be generalized to obtain a new geometric interpretation of $\Delta_L(t^{m_1}, \ldots, t^{m_\mu})$ for any integers $m_1, \ldots, m_\mu$ (see Proposition 2.1 and Corollary 3.4). If an equality holds for $\Delta_L(t^{m_1}, \ldots, t^{m_\mu})$ for any integers $m_1, \ldots, m_\mu$, then it also holds for $\Delta_L(t_1, \ldots, t_\mu)$ (Lemma 2.2), therefore, it is possible to prove properties of $\Delta_L$ with this method. As an example, we give an elementary and geometric proof of the celebrated Torres formula, valid for any link in a homology 3-sphere. We also present several properties of $\Delta_L$ which turn out to be equivalent to the Torres conditions (Proposition 4.2).

2. Preliminaries

Let us consider an oriented ordered link $L = L_1 \cup \ldots \cup L_\mu$ in a homology 3-sphere $\Sigma$, and let $X$ be the exterior of $L$. If $\hat{X}$ denotes the universal abelian covering of $X$ and $\hat{X}^0$ the inverse image by $\hat{p}$ of a base point $X^0$ of $X$,
the homology \( H_1(\bar{X}, \bar{X}^0) \) is endowed with a natural structure of a module over the ring \( \Lambda_\mu = \mathbb{Z}[t_1^{\pm 1}, \ldots, t_\mu^{\pm 1}] \). Given an \( m \times n \) presentation matrix of \( H_1(\bar{X}, \bar{X}^0) \) – that is, the matrix \( P \) corresponding to a presentation with \( n \) generators and \( m \) relations – the \((n-i) \times (n-i)\) minor determinants of \( P \) span an ideal of \( \Lambda_\mu \) denoted by \( E_iH_1(\bar{X}, \bar{X}^0) \). The greatest common divisor of these minor determinants is denoted by \( \Delta_\mu H_1(\bar{X}, \bar{X}^0) \): this invariant is well defined up to multiplication by units of \( \Lambda_\mu \), that is, by \( \pm t_1^{\nu_1} \cdots t_\mu^{\nu_\mu} \) with \( \nu_i \in \mathbb{Z} \). In the sequel, we will write \( \Delta \vdash \Delta' \) if two elements \( \Delta, \Delta' \) of a ring \( R \) satisfy \( \Delta = \varepsilon \Delta' \) for some unit \( \varepsilon \) of \( R \). The Laurent polynomial \( \Delta_\mu H_1(\bar{X}, \bar{X}^0) \) is called the Alexander polynomial of the link \( L_{\bar{X}}^\mu \). It is denoted by \( \Delta_\mu(t_1, \ldots, t_\mu) \).

Our method will be to prove statements on this polynomial in an indirect way, by studying all the infinite cyclic coverings of \( X \). Since these coverings are classified by \( \text{Hom}(H_1(X), \mathbb{Z}) \cong H^1(X; \mathbb{Z}) \cong H_1(L) = \bigoplus_{i=1}^n \mathbb{Z}L_i \), this leads to the following definition [3]. A multilink is an oriented link \( L = L_1 \cup \ldots \cup L_\mu \) in a homology sphere \( \Sigma \) together with an integer \( m_i \) associated with each component \( L_i \), with the convention that a component \( L_i \) with multiplicity \( m_i \) is the same as \( -L_i \) (\( L_i \) with reversed orientation) with multiplicity \( -m_i \). Throughout this paper, we will write \( m \) for the ordered set of integers \( m_1, \ldots, m_\mu \), \( d \) for their greatest common divisor, and \( L(m) \) for the multilink. Finally, we will also denote by \( m \) the morphism \( H_1(X) \to \mathbb{Z} \) given by \( m(\gamma) = \sum_{i=1}^\mu m_i \ell_k(L_i, \gamma) \). Let \( \bar{X} \) be the regular \( \mathbb{Z} \)-covering determined by \( m \). If \( \bar{X}^0 = \bar{X} \) \( \xrightarrow{\mathcal{D}} \) \( X \), the homology \( H_1(\bar{X}, \bar{X}^0) \) can be thought of as a module over the ring \( \mathbb{Z}[t^{\pm 1}] \). The Laurent polynomial \( \Delta_{L(m)}(t) = \Delta_1H_1(\bar{X}, \bar{X}^0) \) is called the Alexander polynomial of the multilink \( L(m) \). Note that if \( m \neq 0 \), the exact sequence of the pair \((\bar{X}, \bar{X}^0)\) implies at once that \( \xi_1H_1(\bar{X}, \bar{X}^0) = \xi_0H_1(X) \). Therefore, \( \Delta_{L(m)}(t) \) is also equal to \( \Delta_0H_1(X) \).

Here is the dictionary between the polynomials \( \Delta_L \) and \( \Delta_{L(m)} \):

**Proposition 2.1 (Eisenbud-Neumann [3]).**

\[
\Delta_{L(m)}(t) = \begin{cases} \Delta_L(t^{m_1}) & \text{if } \mu = 1; \\ (t^d - 1) \Delta_L(t^{m_1}, \ldots, t^{m_\mu}) & \text{if } \mu \geq 2. \end{cases}
\]

**Proof.** To check this equality, we need the well-known fact that \( \xi_1H_1(X, \bar{X}^0) = (\Delta_\ast) \cdot I \), where \( I \) is the augmentation ideal \((t_1 - 1, \ldots, t_\mu - 1)\) and \( \Delta_\ast \) some polynomial in \( \Lambda_\mu \). This can be proved by purely homological algebraic methods using the fact that the group \( \pi_1(X) \) has defect \( \geq 1 \) (see [3, Theorem 6.1]). By considering a finite presentation of \( H_1(\bar{X}, \bar{X}^0) \) given by an equivariant cellular decomposition of \( \bar{X} \), it is easy to show that \( H_1(X, \bar{X}^0) \otimes_{\Lambda_\mu} \mathbb{Z}[t^{\pm 1}] = H_1(X, \bar{X}^0) \), where \( \mathbb{Z}[t^{\pm 1}] \) is endowed with the structure of \( \Lambda_\mu \)-algebra given by \( t_i \mapsto t^{m_i} \) for \( i = 1, \ldots, \mu \). Hence,

\[
\xi_1H_1(X, \bar{X}^0) = \xi_1(H_1(X, \bar{X}^0) \otimes_{\Lambda_\mu} \mathbb{Z}[t^{\pm 1}]) = (\Delta_\ast(t^{m_1}, \ldots, t^{m_\mu})) \cdot (t^{m_1} - 1, \ldots, t^{m_\mu} - 1) = (\Delta_\ast(t^{m_1}, \ldots, t^{m_\mu})) \cdot (t^d - 1).
\]
Since $\Delta_L = (t_1 - 1)\Delta_*$ if $\mu = 1$ and $\Delta_L = \Delta_*$ if $\mu \geq 2$, the proposition is proved.

In order to show that properties of $\Delta_L(m)$ translate directly into properties of $\Delta_L$, we also need the following lemma.

**Lemma 2.2.** Consider two polynomials $\Delta$ and $\Delta'$ in $\Lambda_\mu$ such that

$$\Delta(t^{m_1}, \ldots, t^{m_\mu}) = \Delta'(t^{m_1}, \ldots, t^{m_\mu})$$

for all $(m_1, \ldots, m_\mu)$ in $\mathbb{Z}^\mu$ except possibly a finite number of them. Then, $\Delta \equiv \Delta'$ in $\Lambda_\mu$.

**Proof.** Without loss of generality, it may be assumed that $\Delta = \sum a_{i_1 \ldots i_\mu} t_1^{i_1} \cdots t_\mu^{i_\mu}$ and $\Delta' = \sum b_{j_1 \ldots j_\mu} t_1^{j_1} \cdots t_\mu^{j_\mu}$ with $a_{0 \ldots 0} > 0$, $b_{0 \ldots 0} > 0$, and only non-negative indices $i_k, j_k \geq 0$. By hypothesis, there are maps $\mathbb{Z}^\mu \rightarrow \{\pm 1\}$ and $\mathbb{Z}^\mu \rightarrow \mathbb{Z}$ such that the equality

$$\sum a_{i_1 \ldots i_\mu} t^{\sum k m_k i_k} = \varepsilon(m_1, \ldots, m_\mu) t^{\nu(m_1, \ldots, m_\mu)}$$

holds for all but a finite number of $(m_1, \ldots, m_\mu)$ in $\mathbb{Z}^\mu$. Let us choose an integer $N$ greater than $\max_k \deg t_k \Delta$ and $\max_k \deg t_k \Delta'$, and set $m_1 = 1$, $m_2 = N$, $\ldots$, $m_\mu = N^{\mu-1}$. By choosing $N$ sufficiently large, it may be assumed that the equality above holds for this ordered set of integers. Since all these integers are positive as well as the coefficients $a_{0 \ldots 0}$ and $b_{0 \ldots 0}$, it follows that $\varepsilon(1, N, \ldots, N^{\mu-1}) = +1$ and $\nu(1, N, \ldots, N^{\mu-1}) = 0$. This gives

$$\sum a_{i_1 \ldots i_\mu} t_1^{i_1 + Ni_2 + \ldots + N^{\mu-1} i_\mu} = \sum b_{j_1 \ldots j_\mu} t_1^{j_1 + Nj_2 + \ldots + N^{\mu-1} j_\mu}.$$

But the equality $i_1 + Ni_2 + \ldots + N^{\mu-1} i_\mu = j_1 + Nj_2 + \ldots + N^{\mu-1} j_\mu$ with $0 \leq i_k, j_k < N$ for all $k$ implies that $(i_1, \ldots, i_\mu) = (j_1, \ldots, j_\mu)$. Hence, $a_{i_1 \ldots i_\mu} = b_{i_1 \ldots i_\mu}$ for all multi-indices $(i_1, \ldots, i_\mu)$, which proves the result. □

### 3. Generalized Seifert surfaces

One of the advantages of multilinks is that they can be studied via generalized Seifert surfaces [3]. A **Seifert surface** for a multilink $L(m)$ is an open embedded oriented surface $F \subset \{\Sigma \setminus L\}$ such that, if $F_0$ denotes $F \cap (\Sigma \setminus \text{int} N(L))$, the closure $\partial F$ of $F$ intersects a closed tubular neighborhood $N(L_i)$ of $L_i$ as follows for each $i$:

- If $m_i \neq 0$, $\partial F \cap N(L_i)$ consists of $|m_i|$ sheets meeting along $L_i$; $F$ is oriented such that $\partial F_0 = m_i L_i$ in $H_1(N(L_i))$.
- If $m_i = 0$, $\partial F \cap N(L_i)$ consists of discs transverse to $L_i$; $F$ is oriented such that the intersection number of $L_i$ with each of these discs is the same (either always +1 or always −1).
This is illustrated in Figure 1. Note that $F \subset \Sigma \setminus L$ and $F_0 \subset \Sigma \setminus \text{int} N(L)$ determine each other up to isotopy; to simplify the notation, we will consider both of them as Seifert surfaces, and denote both by $F$. From now on, we will write $\mathcal{F}$ for the union of $F \subset \Sigma \setminus L$ and $L$.

**Lemma 3.1** (Eisenbud-Neumann [3]). Let $F$ be a Seifert surface for a multilink $L^m$. Then, for $i = 1, \ldots, \mu$, the intersection $F \cap \partial N(L_i)$ gives a $d_i$ component link which is the $(d_i p_i, d_i q_i)$-cable about $L_i$, where $p_i$ and $q_i$ are coprime, $d_i p_i = m_i$ and $d_i q_i = -\sum_{j \neq i} m_j \ell_k(L_i, L_j)$.

**Proof.** Let us denote by $(P_i, M_i)$ a basis of $H_1(\partial N(L_i))$ given by a standard parallel and meridian. Since $F$ is a Seifert surface for $L^m$, $F \cap \partial N(L_i) = m_i P_i + n_i M_i$ in $H_1(\partial N(L_i))$ for some integer $n_i$. Furthermore, $\partial F = \sum_{j \neq i} m_j L_j + m_i P_i + n_i M_i$ in $H_1(\Sigma \setminus \text{int} N(L_i))$. By Alexander duality, this module is isomorphic to $H^1(N(L_i) ; \mathbb{Z}) = \mathbb{Z}$, and the isomorphism is given by the linking number with $L_i$. It follows $0 = \ell_k(L_i, \partial F) = \sum_{j \neq i} m_j \ell_k(L_i, L_j) + n_i$, which gives the result.

In the usual case of an oriented link, a Seifert surface needs to be connected in order to be useful. In the general case of a multilink, it has to be “as connected as possible”. More precisely, a Seifert surface for $L(m)$ is a **good Seifert surface** if it has $d = \gcd(m)$ connected components.

**Lemma 3.2.** Given a multilink $L(m)$, there exists a good Seifert surface for $L(m)$.

**Proof.** One easily shows that there exists a Seifert surface for $L(m)$ (see [3 Lemma 3.1]). If $d > 1$, a good Seifert surface for $L(m)$ is given by $d$ parallel copies of a connected Seifert surface for $L(m)$. Therefore, it may be assumed that $d = 1$. Let $F$ be any Seifert surface for $L(m)$ without closed component,
and let us denote by \( i_+ \) (resp. \( i_- \)) the epimorphism \( H_0(F) \to H_0(\Sigma \setminus \overline{F}) \) induced by the push in the positive (resp. negative) normal direction off \( F \). If \( i_+ \) and \( i_- \) are not isomorphisms, it is possible to reduce the number of connected components of \( F \) by handle attachment(s) have been performed, yielding \( F = F_1 \cup \ldots \cup F_n \) with isomorphisms \( i_+, i_- : H_0(F) \to H_0(\Sigma \setminus \overline{F}) \). The automorphism of \( H_0(F) \) given by \( h = (i_-)^{-1} \circ i_+ \) cyclically permutes the connected components of \( F \). (Indeed, consider a component \( F_i \) of \( F \); since \( X = (\Sigma \setminus \overline{F}) \cup F \) is path connected and \( i_+, i_- \) are isomorphisms, there exists an integer \( m \) such that \( F_i = h^m(F_i) \).) It easily follows that \( \partial F_i = \partial F_j \) in \( H_1(\mathcal{N}(L)) \) for \( i, j = 1, \ldots, n \). Therefore, the equality \( \sum_{i=1}^n m_i L_i = \partial F = \sum_{i=1}^n \partial F_i = n \partial F_1 \) holds in \( H_1(\mathcal{N}(L)) = \bigoplus_{i=1}^n \mathbb{Z} L_i \). Hence, \( n \) divides \( m_i \) for \( i = 1, \ldots, \mu \). Since \( \gcd(m_1, \ldots, m_\mu) = 1 \), \( F \) is connected. \( \square \)

Let us now turn to the natural generalization to multilinks of the Seifert form. Given \( F \) a good Seifert surface for \( L(\mathbf{m}) \), the Seifert forms associated to \( F \) are the bilinear forms

\[
\alpha_+, \alpha_- : H_1(F) \times H_1(\overline{F}) \to \mathbb{Z}
\]

given by \( \alpha_+(x, y) = \ell k(i_+ x, y) \) and \( \alpha_-(x, y) = \ell k(i_- x, y) \), where \( i_+ \) (resp. \( i_- \)) is the morphism \( H_1(F) \to H_1(\Sigma \setminus \overline{F}) \) induced by the push in the positive (resp. negative) normal direction off \( F \). (Note that we use the same notation for the morphisms \( H_0(i_\pm) \) and \( H_1(i_\pm) \); it will always be clear from the context which dimension is concerned.) Let us denote by \( A_+ \) and \( A_- \) matrices of these forms, called Seifert matrices. Here is the generalization of Seifert’s famous theorem.

**Theorem 3.3.** Let \( F \) be a good Seifert surface for \( L(\mathbf{m}) \), and let \( A_+, A_- \) be associated Seifert matrices. Then, \( A_+ - t A_- \) is a presentation matrix of the module \( H_1(\overline{X}) \).

**Proof.** Given \( F \) a good Seifert surface for \( L(\mathbf{m}) \), let us denote \( \Sigma \setminus \overline{F} \) by \( Y \). By the proof of Lemma 3.2, it is possible to number the connected components \( F = F_1 \cup \ldots \cup F_d \) and \( Y = Y_1 \cup \ldots \cup Y_d \) such that \( i_+ F_k = Y_k \) and \( i_- F_k = Y_{k-1} \) (with the indices modulo \( d \)). Let us set \( N = F \times (-1; 1) \) an open bicollar of \( F \), \( N_+ = F \times (0; 1) \), \( N_- = F \times (-1; 0) \) and \( \{ Y^i \}_{i \in \mathbb{Z}} \) (resp. \( \{ N^i \}_{i \in \mathbb{Z}} \)) copies of \( Y \) (resp. \( N \)). Define

\[
E = \bigcup_{i \in \mathbb{Z}} Y^i \cup \bigcup_{i \in \mathbb{Z}} N^i / \sim,
\]

where \( Y^i \supseteq N_+ \sim N_+ \subseteq N^i \) and \( Y^i \supseteq N_- \sim N_- \subseteq N^{i+1} \). The obvious projection \( E \to X \) is the infinite cyclic covering \( \overline{X} \to X \) determined by \( \mathbf{m} \). Indeed, a loop \( \gamma \) in \( X \) lifts to a loop in \( E \) if and only if the intersection number of \( \gamma \) with \( F \) is zero, that is, if \( 0 = \gamma \cdot F = \ell k(L(\mathbf{m}), \gamma) = m(\gamma) \).

Consider the Mayer-Vietoris exact sequence of \( \mathbb{Z}[t^{\pm 1}] \)-modules associated to the decomposition \( \overline{X} = (\bigcup_i Y^i) \cup (\bigcup_i N^i) \); it gives

\[
(H_1(F) \oplus H_1(F)) \otimes \mathbb{Z}[t^{\pm 1}] \xrightarrow{\phi_1} (H_1(Y) \oplus H_1(F)) \otimes \mathbb{Z}[t^{\pm 1}] \xrightarrow{\psi} H_1(\overline{X}) \to
\]
If the proof of Lemma 3.2, a Seifert surface $F$ is good, the homomorphisms $i_+(H_0(F)) \rightarrow H_0(\Sigma \setminus \bar{F})$ are injective, and so is $\phi_0$. Therefore, $\psi$ is surjective and there is an exact sequence

$$(H_0(F) \oplus H_0(F)) \otimes \mathbb{Z}[t^{\pm 1}] \xrightarrow{\phi_0} (H_0(F) \oplus H_0(F)) \otimes \mathbb{Z}[t^{\pm 1}],$$

where the homomorphism $\phi_0$ is given by $(\alpha, \beta) \mapsto (i_+ \alpha + t i_- \beta, \alpha + \beta)$. Since $F$ is good, the homomorphisms $i_+(H_0(F)) \rightarrow H_0(\Sigma \setminus \bar{F})$ are injective, and so is $\phi_0$. Therefore, $\psi$ is surjective and there is an exact sequence

$$(H_1(F) \oplus H_1(F)) \otimes \mathbb{Z}[t^{\pm 1}] \xrightarrow{\phi_1} (H_1(F) \oplus H_1(F)) \otimes \mathbb{Z}[t^{\pm 1}] \rightarrow H_1(\bar{X}) \rightarrow 0,$$

with $\phi_1(\alpha, \beta) = (i_+ \alpha + t i_- \beta, \alpha + \beta)$. This can be transformed into

$$H_1(F) \otimes \mathbb{Z}[t^{\pm 1}] \xrightarrow{\tilde{\phi}} H_1(F) \otimes \mathbb{Z}[t^{\pm 1}] \rightarrow H_1(\bar{X}) \rightarrow 0,$$

where $\tilde{\phi}(\alpha) = i_+ \alpha - t i_- \alpha$. Let us fix basis $\mathcal{B}$ for $H_1(F)$, $\mathcal{B}$ for $H_1(F)$, and consider the basis $\mathcal{B^*}$ for $H_1(F)$ which is dual to $\mathcal{B}$ under Alexander duality. The matrix of $i_+$ (resp. $i_-$) with respect to $\mathcal{B}$ and $\mathcal{B^*}$ is given by $A^+_T$ (resp. $A^-_T$), where $A_+$ and $A_-$ are the Seifert matrices with respect to the basis $\mathcal{B}$ and $\mathcal{B}$. Therefore, a matrix of $\tilde{\phi}$ is given by $A^+_T - t A^-_T$. This concludes the proof.

**Corollary 3.4.** Let $L(m)$ be a multilink with $m \neq 0$. If $m_i = \sum_{j \neq i} m_j \ell k(L_i, L_j) = 0$ for some index $i$, then $\Delta_{L(m)}(t) = 0$. If there is no such index, the matrices $A_+$ and $A_-$ are square, and $\Delta_{L(m)}(t) \doteq \det(A_+ - t A_-)$.

**Proof.** By the proof of Lemma 3.2 a Seifert surface $F$ is good if and only if $\text{rk} H_0(F) = \text{rk} H_0(\Sigma \setminus \bar{F})$ which is equal to $\text{rk} H_2(F)$ by Alexander duality. It is easy to show that $\text{rk} H_0(F) = r$, the number of indices $i$ with $m_i = \sum_{j \neq i} m_j \ell k(L_i, L_j) = 0$. Since $\chi(F) = \chi(\bar{F})$, it follows that $\text{rk} H_2(F) = \text{rk} H_1(F) + r$. So if $r = 0$, $A_+ - t A_-$ is a square presentation matrix of $H_1(\bar{X})$ and if $r > 0$, it has more generators than relations. It follows that $\Delta_0 H_1(\bar{X}) = \det(A_+ - t A_-)$ if $r = 0$, and $\Delta_0 H_1(\bar{X}) = 0$ if $r > 0$.

### 4. The Torres conditions

Let us now illustrate how Corollary 3.4 along with Proposition 2.1 and Lemma 3.2 can be used to study the multivariable Alexander polynomial. As an example, we present an elementary proof of the Torres formula [13], quite simpler than the original proof. (On the other hand, it should be mentioned that more perspicuous proofs have since been given, for example in [9].)

Throughout this section, we will denote by $\ell k(L_i, L_j)$ the linking number $\ell k(L_i, L_j)$.

**Lemma 4.1.** Let $L(m) = L(m_1, \ldots, m_{1-1}, 0)$ be a multilink, and let $L'(m') = L'(m_1, \ldots, m_{1-1})$ be the multilink obtained from $L(m)$ by removing the last component $L_\mu$. Then,

$$\Delta_{L(m)}(t) = (t^{\sum_{\ell \mu} \ell k_{ij}} - 1) \Delta_{L'(m')}(t).$$

Proof. If $m_i = \sum_{j \neq i} m_j \ell_{ij} = 0$ for some index $i$, the lemma holds by Corollary 3.4. It may therefore be assumed that there is no such index. Let $F$ be a good Seifert surface for $L(m)$; then, a good Seifert surface for $L'(m')$ is given by $F' = F \cup (F \cap N(L_m))$. By Lemma 3.1, $F \cap N(L_m)$ consists of $d_m = \sum_{i=1}^{\mu} m_i \ell_{im}$ discs (recall Figure 1). Furthermore, $F = F' \cup L_m$. Therefore, we have the natural isomorphisms

$$H_1(F) = H_1(F') \oplus \bigoplus_{j=1}^{d_m} \mathbb{Z} T_j$$
and

$$H_1(F) = H_1(F') \oplus \bigoplus_{j=1}^{d_m} \mathbb{Z} \gamma_j,$$

where the cycles $T_j$ correspond to the boundaries of the discs, and the $\gamma_j$ are the transverse cycles depicted in Figure 2. The associated Seifert matrices $A_+$ and $A'_-$ are related by

$$A_+ = \begin{pmatrix} A'_+ & 0 \\ * & 1 \\ \vdots & \vdots \\ 0 & 1 \end{pmatrix}$$
and

$$A_- = \begin{pmatrix} A'_- & 0 \\ * & 1 \\ \vdots & \vdots \\ 0 & 1 \end{pmatrix}.$$

Corollary 3.4 then gives

$$\Delta_{L(m)} = \det(A_+ - tA_-) = \begin{vmatrix} A'_+ - t A'_- & 0 \\ 1 & -t \\ * & \vdots \\ -t & 1 \end{vmatrix} = (t^{d_m} - 1) \det(A'_+ - t A'_-) = (t^{d_m} - 1) \Delta_{L(m')}(t),$$

and the lemma is proved. \qed
The demonstration of the Torres formula is now a mere translation of Lemma 2.1 via Proposition 2.1.

**Torres formula.** Let \( L = L_1 \cup \ldots \cup L_\mu \) be an oriented link with \( \mu \geq 2 \) components, and let \( L' \) be the sublink \( L_1 \cup \ldots \cup L_{\mu-1} \). Then,

\[
\Delta_L(t_1, \ldots, t_{\mu-1}, t_1) = \begin{cases} 
\frac{t_{12}^{\ell_2-1}}{t_{1}^{\ell_1-1}} \Delta_{L'}(t_1) & \text{if } \mu = 2; \\
(t_1^{\ell_{\mu}} \cdots t_{\mu-1}^{\ell_{\mu-1}} - 1) \Delta_{L'}(t_1, \ldots, t_{\mu-1}) & \text{if } \mu > 2.
\end{cases}
\]

**Proof.** Let us denote by \( \Delta' \) the right-hand side of this formula, and let \( m_1, \ldots, m_{\mu-1} \) be arbitrary integers with \( d = \gcd(m_1, \ldots, m_{\mu-1}) > 0 \). We have the equalities

\[
\Delta'(t^{m_1}, \ldots, t^{m_{\mu-1}}) = \begin{cases} 
\frac{t_{12}^{\ell_2-1}}{t_{1}^{\ell_1-1}} \Delta_{L'}(t^{m_1}) & \text{if } \mu = 2; \\
(t_{1}^{\ell_{\mu}} \cdots t_{\mu-1}^{\ell_{\mu-1}} - 1) \Delta_{L'}(t^{m_1}, \ldots, t^{m_{\mu-1}}) & \text{if } \mu > 2,
\end{cases}
\]

(Proposition 2.1) \[ \frac{1}{t^{d-1}} \Delta_{L}(t) \Delta_{L'}(t^{m_1}, \ldots, t^{m_{\mu-1}})
\]

(Proposition 2.1) \[ \frac{1}{t^{d-1}} \Delta_{L}(t)
\]

(Proposition 2.1) \[ \Delta_{L}(t^{m_1}, \ldots, t^{m_{\mu-1}}, 1)
\]

and the proof is settled by Lemma 2.2.

Using the same method, it is not hard to show the following result.

**Fox-Torres relation.** Let \( L = L_1 \cup \ldots \cup L_\mu \) be an oriented link with \( \mu \geq 2 \) components. Then,

\[
\Delta_L(t_1^{-1}, \ldots, t_\mu^{-1}) = (-1)^\mu t_1^{\nu_1-1} \cdots t_\mu^{\nu_\mu-1} \Delta_L(t_1, \ldots, t_\mu)
\]

with integers \( \nu_i \) such that \( \nu_i \equiv \sum_j \ell_{ij} \pmod{2} \) if \( \Delta_L \neq 0 \).

These results provide necessary conditions for a polynomial \( \Delta \) in \( \Lambda_\mu \) to be the Alexander polynomial of a \( \mu \)-component link with fixed \( \ell_k(L_i, L_j) = \ell_{ij} \). They are known as the Torres conditions (see [10] for a precise statement). Since these conditions are not sufficient [7] [11], the problem is now to find stronger conditions. By means of a close study of the homology \( H_1(F) \) and \( H_1(\bar{F}) \), it is possible to find necessary conditions for a polynomial \( \Delta \) in \( \mathbb{Z}[t^{\pm 1}] \) to be the Alexander polynomial of a multilink. Via Proposition 2.1 this translates into the following result (see [2] for a proof).

**Proposition 4.2.** Let \( L \) be an oriented link with \( \mu \geq 2 \) components. Then, its Alexander polynomial \( \Delta_L \) satisfies the following conditions. For all integers \( \mu = (m_1, \ldots, m_\mu) \) with \( d = \gcd(m_1, \ldots, m_\mu) \) and \( d_i = \gcd(m_i, \sum_j m_j \ell_{ij}) \), there exists some polynomial \( \nabla_{L(\mu)}(t) \in \mathbb{Z}[t^{\pm d}] \) such that:

- \[ \prod_{i=1}^\mu (t^{d_i} - 1) \nabla_{L(\mu)}(t) \equiv (t^{d} - 1)^2 \Delta_L(t^{m_1}, \ldots, t^{m_\mu}); \]
\[ \nabla_L(m)(t^{-1}) = \nabla_L(m)(t); \]

- \(|\nabla_L(m)(1)| = \frac{a^2 D}{d_1 \cdots d_{\mu-1}} \), where \( D \) is any \((\mu - 1) \times (\mu - 1)\) minor determinant of the matrix

\[
\begin{pmatrix}
- \sum_j m_1 m_j \ell_{1j} & m_1 m_2 \ell_{12} & \cdots & m_1 m_\mu \ell_{1\mu} \\
m_1 m_2 \ell_{12} & - \sum_j m_2 m_j \ell_{2j} & \cdots & m_2 m_\mu \ell_{2\mu} \\
\vdots & \vdots & \ddots & \vdots \\
m_1 m_\mu \ell_{1\mu} & m_2 m_\mu \ell_{2\mu} & \cdots & - \sum_j m_\mu m_j \ell_{\mu j}
\end{pmatrix};
\]

- If \( m_i = 0 \) for some index \( i \), then \( \nabla_L(m) = \nabla_L(m') \), where \( L' \) denotes the sublink \( L \setminus L_i \) and \( m' = (m_1, \ldots, \hat{m}_i, \ldots, m_\mu) \).

This result easily implies the Torres conditions. It can also be thought of

as a generalization of a theorem of Hosokawa [8], which corresponds to the case

\( m_1 = \ldots = m_\mu = 1 \). At first sight, it might therefore seem more general than

the Torres conditions. Unfortunately, this is not the case: it can be shown

that every polynomial \( \Delta \) which satisfies the Torres conditions also satisfies the

conditions of Proposition 4.2 (see [2]).

By means of a somewhat closer study of the Seifert matrices \( A_{\pm} \), it should be

possible to find new properties of \( \Delta_L(m) \). They would translate into properties

of \( \Delta_L \), and provide new conditions, stronger than the ones of Torres.

Acknowledgments

I wish to express my thanks to Jerry Levine, to my advisor Claude Weber,

and to Mathieu Baillif.

References


