Kneading operators, sharp determinants, and weighted Lefschetz zeta functions in higher dimensions

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Kneading Operators, Sharp Determinants and Weighted Lefschetz Zeta Functions in Higher Dimension

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Abstract

We study a transfer operator \( \mathcal{M}(k) \) associated to a family \( \{\psi_i\} \) of \( C^k \) transversal local diffeomorphisms of \( \mathbb{R}^n \), with \( C^k \) compactly supported weights \( g_\psi \), and let it act on \( k \)-forms in \( \mathbb{R}^n \). Using the definitions of sharp trace \( \text{Tr}^\# \) and flat trace \( \text{Tr}^\flat \), the following formula holds between formal power series: \( \text{Det}^\#(1-z\mathcal{M}) = \prod_{k=0}^{n} \text{Det}^\flat(1-z\mathcal{M}^{(k)}) (-1)^k \). Following ideas of Kitaev, we define the kneading operators \( D_k(z) \), which are kernel operators. Our main result is the equality in odd dimension

\[
\text{Det}^\#(1-z\mathcal{M}) = \prod_{k=0}^{n-1} \text{Det}^\flat(1+D_k(z)) (-1)^{k+1}
\]

as formal power series (the determinant \( \text{Det}^\flat \) is defined by the trace \( \text{Tr}^\flat \) which is the usual trace of kernel operators).

As a consequence, we obtain that the weighted Lefschetz zeta-function

\[
\zeta^\#(z) = 1/\text{Det}^\#(1-z\mathcal{M})
\]

has a positive radius of convergence. This (partially) generalises results obtained by Baladi, Kitaev, Ruelle and Semmes in dimension 1, complex and real.

Introduction

Kneading theory in dimension one has been a powerful and successful approach in the study of dynamical systems. In their famous paper [MT88], Milnor and Thurston introduced kneading coordinates and used them in order to get a lot of results about piecewise monotone and continuous interval maps: semi-conjugacy with a piecewise linear map, relationship with the lap number, and a surprising “magical formula” showing the equality (modulo a polynomial error factor \( p(t) \)) between the reduced zeta function \( \zeta^R(t) \) of a continuous piecewise monotone map \( f \), and the inverse of the determinant \( \Delta(t) \) of the kneading matrix ([MT88]):

\[
\zeta^R(t) \cdot \Delta(t) = p(t)
\]

(1)

The kneading matrix entries are built using the orbits of the turning points of \( f \) (i.e., the endpoints of the maximal intervals of monotonicity) and are power series, holomorphic in the unit disc. The reduced zeta function is a power series built with the periodic points of \( f \). The
meromorphicity of the reduced zeta function in the unit disc, which is not at all trivial, follows immediately from (1).

This theory has been pushed further by Baladi and Ruelle who introduced piecewise constant weights [BR94], and by Baladi who generalised the result to weights of bounded variation [Bal95].

Nevertheless, kneading theory was not really useful (at first sight) for the study of the statistical properties of a given map $f$. Recall that if $f : X \to X$ is a map for which any point $x$ in $X$ has a finite number of preimages (this condition could be weakened), we can define the Ruelle transfer operator $L$ associated to $f$ and acting on some Banach space of functions by

$$L\phi(x) = \sum_{f(y) = x} g(x) \cdot \phi \circ f(x)$$

where $g$ is a suitable weight (for instance, $g$ equals 1 over the Jacobian of $f$). The statistical properties of $f$ such as decay of correlations for an ergodic invariant measure are often strongly related to the spectrum of the transfer operator $L$. For instance, one can often show that the existence of a spectral gap for $L$ (that is, $L$ admits an isolated simple eigenvalue of maximal modulus) implies exponential decay of correlations (see for instance [Bal00]). Thus, it would be interesting to have a tool to investigate the spectrum of $L$. For instance, one would like to have some kind of ‘determinant’, i.e. a map $\text{det}(1 - z L)$, holomorphic in $z$ on some domain of $\mathbb{C}$, whose zeroes in this domain are exactly the inverse of the eigenvalues of $L$ with multiplicity. For a transfer operator, it happens that such a ‘determinant’ sometimes exists and takes the form of (the inverse of) a zeta function (see for instance [Rue75], [Rue90] and [Rug96]), leading us to the question: can we use the kneading theory, or at least get inspired by it, to understand something about the spectral properties of the transfer operator $L$ for an interval map?

Unfortunately, Milnor and Thurston’s work did not directly reveal any direct relationship between the reduced zeta function and the (eigenvalues of) the transfer operator $L$ associated to $f$. The step was achieved when Baladi and Ruelle [BR96] realised that the idea of regularisation was lying behind the kneading theory. More precisely, they considered a finite collection of homeomorphisms $\psi_\omega : U_\omega \to \psi_\omega(U_\omega)$ (representing the inverse branches of a piecewise monotone map), $\omega \in \Omega$ being a finite set of indices, $U_\omega$ being open intervals; and compactly supported, continuous and of bounded variation weights $g_\omega$. The transfer operator acting on maps of bounded
variation can then be written as

\[ \mathcal{M} \phi(x) = \sum_{\omega} g_{\omega}(x) \cdot \phi \circ \psi_{\omega}(x). \]

They defined the *sharp trace* of \( \mathcal{M} \) as follows:

\[ \text{Tr}^\# \mathcal{M} := \sum_{\omega} \sum_{\psi_{\omega}(x) = x} g_{\omega}(x) \cdot L(x, \psi_{\omega}) \quad (2) \]

where the *Lefschetz sign* \( L(x, \psi_{\omega}) \in \{0, \pm 1\} \) is the sign of \( 1 - \psi_{\omega}'(x) \) if \( \psi_{\omega} \) is differentiable (see further).

Using the formula \( \det(A) = \exp \text{tr} \log(A) \) (see Section 6 below), they defined the *sharp determinant* \( \text{Det}^\#(1 - z\mathcal{M}) \) of \( \mathcal{M} \).

Then, they introduced *kneading operators* \( D(z) \) (\( z \) being a complex variable), which could be seen as limits of kneading matrices. \( D(z) \) is a kernel operator with kernel in \( L^2 \), thus it is possible to apply Hilbert space theory (see [Sim79] or [GGK00]) to conclude that \( D(z) \) is in fact (almost) trace class, which means that we can define a determinant \( \text{Det}(1 + D(z)) \), and this determinant is a ‘real’ one in the sense that \( \text{Det}(1 + D(z)) = 0 \) if and only if \(-1\) is an eigenvalue of \( D(z) \). (In fact, only \( (D(z))^2 \) is trace class, and one has to deal with regularised determinants \( \text{det}_2(1 + D(z)) \).) The main result in [BR96] is the following equality (which plays the role of (1) in this setting):

\[ \text{Det}^\#(1 - z\mathcal{M}) \cdot \text{Det}(1 + D(z)) = 1 \quad (3) \]

Then, using (3), the relationship between the zeroes of \( \text{Det}(1 - z\mathcal{M}) \) and the eigenvalues of \( \mathcal{M} \) follows relatively easily from the definition of \( D(z) \) (see [BR96], and [Rue96] for a complement).

Some results were obtained by Baladi, Kitaev, Ruelle and Semmes [BKRS97] in the 1-dimensional (complex) holomorphic case. They defined kneading operators and were able to prove (3) as well as some trace class property of \( D(z) \). In fact, \( D(z) \) has an unbounded kernel, and they used a family \( D_t(z) \), where \( D_0(z) = D(z) \), and for each \( 0 < t \leq 1 \), \( D_t(z) \) is a regularisation of \( D(z) \), and they showed that the trace-norm of \( (D_t(z))^2 \) is uniformly bounded for \( 0 < t \leq 1 \). Unfortunately, their paper does not contain any spectral interpretation.

In an unpublished note [Kit95], Kitaev proposed a definition of the kneading operators and sketched a proof of an analog of (3) in dimension \( n \). It requires another definition of trace, the *flat trace* \( \text{Tr}^f \), whose name is inspired from the work of Atiyah and Bott.

In the present paper, we follow Kitaev’s ideas and give a complete proof of a formula generalising (3) in odd dimension (this is Theorem...
As a consequence, we obtain (using the theory of regularised determinants) that in all dimensions (even or odd), \( \text{Det}^\#(1 - z\mathcal{M}) \) is holomorphic near 0, and has a meromorphic extension to a (possibly larger) disk. (this is Theorem 8.4). A spectral interpretation will be given in a forthcoming paper.

The paper is organised as follows:
- In Section 1, we fix the general notations.
- In Section 2, the definitions of (transversal) transfer operators, sharp trace \( \text{Tr}^\# \) and flat trace \( \text{Tr}^1 \) of a transversal transfer operator are given.
- In Section 3, we introduce the operators \( S \) and \( N \) with which we will later build the kneading operators \( D_k(z) = N_k(1 - z\mathcal{M}^{(1)})^{-1}S_k \).
- In Section 4, we define the vector spaces of operators \( K^S_k, K^S_L, K^S_R \) and \( K^d_L \), and the star trace of these operators. We show that the star trace and the flat trace coincide on transversal transfer operators
- In Section 5, we show that in our vector spaces, we can perform some commutations of operators without changing their star traces. A sign appears in even dimension that destroys the proof of Theorem 6.1 in that case.
- In Section 6, we define the kneading operators, state and prove Theorem 6.1, which is the main theorem of this paper. We use the commutations given in Section 5.
- In Section 7, we give some complementary results that allow us to extend a little bit Theorem 6.1.
- Section 8 is devoted to the proof of the holomorphic nature of \( \text{Det}^\#(1 - z\mathcal{M}) \) near 0. Using Kaloshin's results [Kal00], we show some examples of diffeomorphisms for which it is not obvious that the series for \( \text{Det}^\#(1 - z\mathcal{M}) \) converges.

Sections 3 and 6 are based on Kitaev's sketch of a proof given in [Kit95]. The proof of Theorem 6.1, as we give it here, is basically his. The reader who does not want to look at all details is encouraged to read Sections 2, 3 and 6 first.

The paper requires basic knowledge on differential forms. For the definitions of differential forms, currents, pullback and exterior derivative, see for instance [Spi65].

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1- Preliminaries

Let us begin by recalling some notations.

Let \( n \in \mathbb{N} \). A \( k \)-tuple of ordered indices \( (0 \leq k \leq n) \) will be denoted by \( I(k) \). Sometimes, \( \langle k \rangle \) will be omitted. \( I'(n-k) \) is the ordered complement in \( \{1, \ldots, n\} \) of \( I(k) \). If \( I(k) \) and \( J(\ell) \) are two multi-indices, \( (I \cup J)(k+\ell) \) will be their ordered union, \( (1, \ldots, n) \) will be denoted by \( N \). If \( I(k) \) is a multi-index, we define \( \gamma(I) \) such that

\[
(-1)^{\gamma(I)} \text{ is the sign of the permutation } (I(k), I'(n-k)) \to (1, \ldots, n)
\]

(4)

Let \( A \) be a \((n \times n)\)-matrix. We will use ‘det’ and ‘tr’ for determinant and trace. We denote by

\[
\left( A \right)_{I(k)}^{J(\ell)}
\]

the sub-matrix of \( A \) with rows \( I(k) \) and columns \( J(\ell) \).

We denote by \( \bigwedge^k A \) the \( k \)-th exterior power of \( A \) (acting on \( \bigwedge^k (\mathbb{R}^n)^* \), the alternate \( k \)-linear maps from \( \mathbb{R}^n \) to \( \mathbb{R} \)). We recall the formula:

\[
\text{tr}\left( \bigwedge^k A \right) = \sum_{I(k)} \det\left( A \right)_{I(k)}^{I(k)}
\]

(5)

If \( I(k) = (i_1, \ldots, i_k) \), we will use the notation \( dx_{I(k)} \) (or, shortly, \( dx_I \)) for \( dx_{i_1} \wedge \cdots \wedge dx_{i_k} \). We will denote a differential form in \( \mathbb{R}^n \) by \( \phi(x, dx) \).

Recall that a \( k \)-form is of the form \( \sum_I \phi_I(x) dx_I \). A form on \( \mathbb{R}^n \times \mathbb{R}^n \) depending on two variables will be denoted by \( \phi(x, y, dx, dy) \). A current is a form with coefficients that are not functions but distributions.

Let \( \phi(x, dx) = \phi(x) \cdot dx_N \) be a \( L^1(dx) \) \( n \)-form in \( \mathbb{R}^n \), its integral is just the integral of \( \phi(x) \) with respect to the \( n \)-dimensional Lebesgue measure:

\[
\int_x \phi(x, dx) = \int_{\mathbb{R}^n} \phi(x) dx_1 \cdots dx_n
\]

We will need to integrate some forms that depend on two variables, so let us fix the following convention:

If \( \phi(x, y, dx, dy) = \sum_{I(k)} \phi_I(x, y) dy_1 \wedge dx_N \) is a \( n \)-form in \( x \) and a \( k \)-form in \( y \), the integral of \( \phi \) over \( x \) will be a \( k \)-form in \( y \), we define

\[
\int_x \phi(x, y, dx, dy) = \sum_{I(k)} \int_x \phi_I(x, y) dy_1 \wedge dx_N
\]

\[
= \sum_{I(k)} \left( \int_{\mathbb{R}^n} \phi_I(x, y) dx_1 \cdots dx_n \right) \wedge dy_1
\]
Hence,
\[
\int_x \int_y \phi(x, y)dx_N \wedge dy_N = \int_y \int_x \phi(x, y)dy_N \wedge dx_N.
\]
Let \(\phi(x, y, dx, dy)\) be a \(n\)-form in each variable. Therefore:
\[
\int_x \int_y \phi(x, y, dx, dy) = (-1)^n \int_y \int_x \phi(x, y, dx, dy).
\]
Let \(d_k\) denote the operators acting on \(k\)-forms which send a form to its exterior derivative. We will often drop the index. The derivatives are to be taken in the sense of distributions, such that \(d\phi(x, dx)\) is defined (as a current) whenever \(\phi\) is in \(L^1_{\text{loc}}\). Let \(f : \mathbb{R}^n \supset U \to f(U) \subset \mathbb{R}^n\) be a differentiable function. We denote the pullback by \(f\) of \(\phi(x, dx)\) by \(f^*\phi(x, dx)\). We recall that \(d\) has the two following properties:
\[
\begin{align*}
    dd\phi(x, dx) &= 0 \\
    f^*d\phi(x, dx) &= df^*\phi(x, dx)
\end{align*}
\]
or shortly: \(d^2 = 0, f^*d = df^*\). (cf [Spi65])
If \(\phi(x, y, dx, dy)\) depends on two variables, \(dx^k \phi\) — respectively \(f_{x}^*\phi\) — will denote the exterior derivative — respectively pullback — of \(\phi\) with respect to \(x\). We also have the commutations: \(f_{x}^*d^x = d^x f_{x}^*\), \(d^y f_{x}^* = f_{x}^*d^y\), and so on.
From now on, all derivatives have to be taken in the sense of distributions.

Let us end these preliminaries by an easy lemma, which we will constantly use throughout the paper (especially point a)). If \(\varphi\) is a form in \(\mathbb{R}^n \times \mathbb{R}^n\), we adopt the following notation:
\[
\text{RD}(\varphi(x, y, dx, dy)) := \varphi(x, x, dx, dx)
\]
(RD is for Restriction on the Diagonal.) That is, one replaces \(y\) by \(x\) and each \(dy_i\) by \(dx_i\). We have:

**Lemma 1.1:** Let \(\Psi\) be a diffeomorphism from an open set \(U \subset \mathbb{R}^n\) to its image and \(\varphi(x, y, dx, dy)\) a \(k\)-form in \(x\) and a \(\ell\)-form in \(y\). Then,
\[
\begin{align*}
    \text{a) } \text{RD}(\Psi_{x}^* \varphi(x, y, dx, dy)) &= \Psi_{x}^* \left( \text{RD}(\Psi^{-1}_{y}^* \varphi(x, y, dx, dy)) \right) \\
    \text{b) } \text{Moreover, if each coefficient function of } \varphi \text{ is of the form } \phi(x - y), \text{ with } \phi : \mathbb{R}^n \to \mathbb{R}, \text{ we have:}
\end{align*}
\]
\[
\begin{align*}
    \text{RD}(\psi_{x}^*d^x \varphi(x, y, dx, dy)) &= d^x \text{RD}(\psi_{x}^* \varphi(x, y, dx, dy)) - \text{RD}(\psi_{x}^*d^y \varphi(x, y, dx, dy))
\end{align*}
\]
Proof of Lemma 1.1:

The proof is by recurrence on $k + \ell$, which is the total order of $\varphi(x, y, dx, dy)$. One checks easily that a) and b) hold for $k = \ell = 0$ and $k + \ell = 1$. If $k + \ell > 1$, we can write $\varphi$ as a finite sum of $\nu \wedge \eta$, where $\nu, \eta$ are forms of total order (order in $x$ plus order in $y$) strictly less than $k + \ell$.

Since $\text{RD}(\nu \wedge \eta) = \text{RD}(\nu) \wedge \text{RD}(\eta)$, we have that

$$\text{RD}(\Psi^*_x \varphi) = \text{RD}(\Psi^*_x (\nu \wedge \eta)) = \text{RD}(\Psi^*_x \nu) \wedge \text{RD}(\Psi^*_x \eta) = \Psi^*_x \text{RD}((\Psi^{-1})^*_y \nu) \wedge \Psi^*_x \text{RD}((\Psi^{-1})^*_y \eta) = \Psi^*_x \text{RD}((\Psi^{-1})^*_y \varphi)$$

which proves a). To prove b), we can assume that $\nu$ has total degree 1. Since $d^x (\nu \wedge \eta) = d^x \nu \wedge \eta - \nu \wedge d^x \eta$ and $\text{RD}$ is linear, we have

$$\text{RD}(\Psi^*_x d^x \varphi) = \text{RD}(\Psi^*_x d^x \nu) \wedge \text{RD}(\Psi^*_x \eta) - \text{RD}(\Psi^*_x \nu) \wedge \text{RD}(\Psi^*_x d^x \eta)$$

By recurrence,

$$\text{RD}(\Psi^*_x d^x \varphi) = (d^x \text{RD}(\Psi^*_x \nu) - \text{RD}(\Psi^*_x d^y \nu)) \wedge \text{RD}(\Psi^*_x \eta)$$

$$\quad - \text{RD}(\Psi^*_x \nu) \wedge (d^x \text{RD}(\Psi^*_x \eta) - \text{RD}(\Psi^*_x d^y \eta))$$

$$= d^x \text{RD}(\Psi^*_x \nu) \wedge \text{RD}(\Psi^*_x \eta) - \text{RD}(\Psi^*_x (d^y \nu \wedge \eta - \nu \wedge d^y \eta))$$

$$= d^x \text{RD}(\Psi^*_x \varphi) - \text{RD}(\Psi^*_x d^y \varphi)$$

$\square$

2. Transfer operators; Flat and Sharp Trace

From now on, $n$ denotes the dimension and is fixed.

Definition 2.1: We denote by $A_k$ the set of $k$-currents of order $\leq 2$ in $\mathbb{R}^n$, and by $A^C_k \subset A_k$ the subset of compactly supported $k$-currents.

(A current is of order $\ell$ if its coefficient distributions acts on $C^\ell$ functions with compact support.) When it is not misleading, we will use a functional notation $\phi(x, dx)$ for any member of $A_k$ (despite the fact that a distribution is not defined at a point $x \in \mathbb{R}^n$).

A family $\{\psi_\omega, g_\omega\}_{\omega \in \Omega}$ is called adapted if:

- $\Omega$ is a finite set,
- $\psi_\omega$ is a $C^3$ diffeomorphism from an open set $U_\omega \subset \mathbb{R}^n$ to its image,
- $g_\omega$ is a $C^3$ function $\mathbb{R}^n \to \mathbb{C}$ with compact support $\subset U_\omega$.

Let $\epsilon_\omega = \pm 1$ depending on whether $\psi_\omega$ preserves or reverses the orientation.
We say that an operator acting on $A_k$ is a \textit{transfer operator} if it admits a representation:

$$M^{(k)}\phi(x,dx) = \sum_{\omega} g_{\omega}(x) \cdot \psi_{\omega}^{*}\phi(x,dx) \quad (8)$$

where $\{\psi_{\omega}, g_{\omega}\}_{\omega \in \Omega}$ is an adapted family. Notice that the representation (8) is in general not unique (one can use partition of the unity).

Since the $g_{\omega}$s are $C^3$ and the $\psi_{\omega}$s are $C^3$ diffeomorphisms, it is obvious that a transfer operator maps $A_k$ to $A^C_k$. (Here, $C^2$ would be enough for the $g_{\omega}$s, but we shall later need $d_g g_{\omega}$ to be $C^2$.)

For a fixed adapted family, we denote by $M$ the operator acting on $\bigcup_k A_k$ whose restriction to $A_k$ is $M^{(k)}$.

Suppose that $\forall \omega \in \Omega$, $\psi_{\omega}$ has finitely many fixed points.

\textbf{Definition 2.2:} We define the \textit{sharp trace} of the representant (8) of $M$ by:

$$Tr^# M := \sum_{\omega} \sum_{x \in Fix_{\psi_{\omega}}} g_{\omega}(x) \cdot L(x, \psi_{\omega}) \quad (9)$$

Where $L(x, \psi_{\omega}) = s\text{gn}(\det(1 - d_x \psi_{\omega}))$ is the Lefschetz number of the fixed point $x$, which can take the values 0, ±1.

Note that the composition of transfer operators is a transfer operator. Let $\ell \in \mathbb{N}$.

\textbf{Definition 2.3:} We say that the family $\{\psi_{\omega}\}_{\omega \in \Omega}$ is \textit{transversal of degree $\ell$} if (7) holds for each $\psi_{\omega}$, and if for all $k < \ell$ and every set of indices $\omega_1, \ldots, \omega_k$, whenever $\psi_{\omega_1} \circ \cdots \circ \psi_{\omega_k}(x) = x$, then 1 is not an eigenvalue of $d_x (\psi_{\omega_1} \circ \cdots \circ \psi_{\omega_k})$. A family is transversal if it is transversal of degree $\ell$ for all $\ell$.

Recall that a map belonging to a transversal family has finitely many fixed points on any compact set (see for instance [GP74]).

\textbf{Definition 2.4:} If transversality of degree 1 holds, we can define the (formal) \textit{flat trace} of (the representant of) $M^{(k)}$ by:

$$Tr^f M^{(k)} := \sum_{\omega} \sum_{x \in Fix_{\psi_{\omega}}} g_{\omega}(x) \cdot \text{tr} \text{tr} \bigwedge_{d_x \psi_{\omega}}^k \frac{1}{\det(1 - d_x \psi_{\omega})} \quad (10)$$

By using the formula $\det(1 - A) = \sum_{k=0}^n (-1)^k \text{tr} \bigwedge^k A$, it is easy to
check that
\[ \text{Tr}^\# \mathcal{M} = \sum_{k=0}^{n} (-1)^k \text{Tr}^1 \mathcal{M}^{(k)} \] (11)

**Definition 2.5:** A transversal transfer operator \( \mathcal{M}^{(k)} \) is a transfer operator which has a representation (8) with a transversal family.

The flat trace of \((\mathcal{M}^{(k)})\)' is thus defined for all \(\ell\).

Note: To avoid confusion, we will use capital letters for the operator traces 'Tr#' and 'Tr\[\]' and small letters for the matrix traces 'tr'; and for the determinants as well.

The next result shows that the flat and the sharp traces are well defined and are independent of the representation of \(\mathcal{M}\).

**Lemma 2.1:** \(\text{Tr}^1 \mathcal{M}^{(k)}\) depends only on \(\mathcal{M}^{(k)}\) as an operator on the subset of \(A_k\) containing \(C^\infty k\)-forms. Thus, \(\text{Tr}^\# \mathcal{M}\) is independent of the transversal representation we choose.

**Proof of Lemma 2.1:**

The statement about \(\text{Tr}^\# \mathcal{M}\) takes into account the fact that a transversal transfer operator could have a non-transversal representant, for which the sharp trace is still defined. However, our proof does not show that the sharp trace of a transversal representant and the one of a non-transversal representant coincide.

Let \(\phi_m : \mathbb{R}^n \to \mathbb{R}\) be a sequence of \(C^\infty\) maps with compact support which approximate \(\delta(x)\), the Dirac mass at 0 in the sense that for each continuous \(f : \mathbb{R}^n \to \mathbb{R}\),

\[ \int_{\mathbb{R}^n} f(x)\phi_m(x)dx \to f(0) \text{ as } m \to \infty \]

Recall that \(N := (1, \cdots , n)\).

The following expressions depend only on \(\mathcal{M}^{(k)}\) as an operator acting on \(C^\infty k\)-forms:

\[ \sum_{l(k)} \int_y \int_x (-1)^l \mathcal{M}^{(k)}_x (\phi_m (x - y)dx_1) \wedge \delta(x - y)dx_1, \wedge dy_N \]

\[ = \sum_{l(k)} \int_y \int_x (-1)^l \sum_\omega g_\omega (x) \cdot \left( (\psi_\omega)_x^* (\phi_m (x - y)dx_1) \right) \wedge \delta(x - y)dx_1, \wedge dy_N \]
\begin{align*}
&= \sum_{l(k)} \sum_{\omega} \int (\!(-1)^{\gamma(l)}g_{\omega}(x) \cdot \phi_m (\psi_\omega(x) - x) \cdot \psi_\omega^*(dx_1) \wedge dx_N \\
&= \sum_{l(k)} \sum_{\omega} \int g_{\omega}(x) \cdot \phi_m (\psi_\omega(x) - x) \cdot \det (d_x \psi_\omega)^l \cdot dx_N \\
&\to_{m \to \infty} \sum_{\omega} \int g_{\omega}(x) \cdot \text{tr} \bigwedge^k (d_x \psi_\omega) \cdot \delta(\psi_\omega(x) - x) \cdot dx_N \\
&= \sum_{\omega} \sum_{x \in \text{Fix}_\omega} \frac{g_{\omega}(x) \cdot \text{tr} \bigwedge^k d_x \psi_\omega}{|\det (1 - d_x \psi_\omega)|} = \text{Tr}^{(k)} \mathcal{M}^{(k)}
\end{align*}

We used definition (4), formula (5), the identity
\[ (-1)^{\gamma(l)} \psi^*(dx_1) \wedge dx_N = \det (d_x \psi)^l dx_N \]
and a change of variable. \[ \square \]

The term ‘trace’ is justified by the next proposition.

**Proposition 2.2:** If \( \mathcal{M}_1 \) and \( \mathcal{M}_2 \) are two transversal transfer operators, then
\[ \text{Tr}^{(k)} \mathcal{M}_1^{(k)} \mathcal{M}_2^{(k)} = \text{Tr}^{(k)} \mathcal{M}_2^{(k)} \mathcal{M}_1^{(k)} \ (k = 0, \cdots, n) \]

The proof follows directly from Lemma 7.4 (see below). There is a more direct proof that we won’t give here. See [BR96] for a more general result in dimension one.

### 3- The operators \( S \) and \( N \)

In this section, we define the two auxiliary operators which will allow us to build the kneading operators.

First, we will need operators \( S_k : A_{k+1}^{(c)} \to A_k \) such that
\[ d_{k-1}S_{k-1} + S_k d_k = 1 \quad (1 \text{ being the identity on } A_0^{(c)}) \]

Let \( \rho(z, dz) \) be the Dirac \( n \)-current \( \delta(z)dz_N \). Let \( \sigma(z, dz) \) be a \( n-1 \)-form such that
\[ d\sigma(z, dz) = \rho(z, dz) \]

(as currents acting on \( \mathcal{C}^2 \) functions). We take
\[ \sigma(z, dz) = \frac{2}{2\pi} \left( \frac{z}{(z)^2} \right) \cdot \frac{1}{|z|^n} \sum_{j=1}^n (1)^{j+1} z_j \cdot d_z z_1 \wedge \cdots \wedge d_z z_j \wedge \cdots \wedge d_z z_n \]
Where \( \Gamma \) is the Euler function, \( | \cdot | \) is the Euclidian norm on \( \mathbb{R}^n \) and \( \partial_i \) means that we omit \( dz_i \). Notice that \( \sigma \) is a \( L^p \) form for \( p < \frac{n}{n-1} \).

**Proof of Formula (13):**

We denote by \( C(n) \) the constant \( \frac{\Gamma(n)}{2\pi^n} \). We have

\[
d\sigma(z, dz) = C(n) \sum_{i=1}^{n} \partial_i \frac{z_i}{|z|^n} \cdot dz_N
\]

Notice that \( \frac{z_i}{|z|^n} = -\frac{1}{n-2} \partial_i \frac{1}{|z|^{n-2}} \) in dimension \( n \geq 3 \), and \( \frac{z_i}{|z|^2} = \partial_i \log(|z|) \) in dimension 2. We use then the following classical theorem:

**Theorem:** Let \( \Delta = \frac{\partial^2}{\partial z_1^2} + \cdots + \frac{\partial^2}{\partial z_n^2} \), and

\[
E(x) = \left\lfloor \frac{|x|}{2\pi} \right\rfloor \quad n = 1
\]

\[
E(x) = \frac{1}{2\pi} \log(|x|) \quad n = 2
\]

\[
E(x) = -\frac{C(n)}{n-2} \cdot \frac{1}{|x|^{n-2}} \quad n \geq 3.
\]

Then, \( \Delta E(x) = \delta \) (in the sense that for any \( C^2 \) function \( f \),
\[
\int \Delta E(x) \cdot f(x) dx = f(0).
\]

See, for instance, [Sch66]. This proves (13).

Now, let us consider \( \rho(x - y, dx - dy) \). This is a formal sum of forms in \( x \) and \( y \). More precisely, one can check that

\[
\rho(x - y, dx - dy) = \sum_{k=0}^{n} (-1)^{n(k+1)} \rho_k(x, y, dx, dy)
\]

where \( \rho_k(x, y, dx, dy) \) is given by

\[
\rho_k(x, y, dx, dy) = \delta(x - y) \cdot \sum_{I(k)} (-1)^{\gamma(I)} dx_1 \wedge dy_1,
\]

(see (4) for the definition of \( \gamma(I) \)). It is a \( k \)-current in \( x \) and a \( n-k \)-current in \( y \). Notice that we have the following interesting property:

\[
\int_y \rho_k(x, y, dx, dy) \wedge \phi(y, dy) = \phi(x, dx).
\]

(Recall that the Dirac delta is the identity for the convolution operation in distributions, see [Sch66].)
We define $\sigma_k(x, y, dx, dy)$ implicitly by the following conditions:

$$
\sigma_k(x, y, dx, dy) \text{ is a } k \text{-form in } x \text{ and a } n - k - 1 \text{-form in } y
$$

$$
\sigma(x - y, dx - dy) = \sum_{k=0}^{n-1} (-1)^k \sigma_k(x, y, dx, dy)
$$

(16)

Now, let $\phi(x, dx)$ be a $k$-form in $x$, and let $u = x - y$, with $x, y \in \mathbb{R}^n$. We have then $du = dx - dy$, and one can easily check that $d^y \phi(u, du) = d^y \phi(x - y, dx - dy) + d^y \phi(x - y, dx - dy)$. Thus, applying $d^y$ to both sides of (16) and grouping the terms yields

$$
d^y_k \sigma_{k-1} + (-1)^n d^y_{n-k-1} \sigma_k = \rho_k
$$

(17)

(we omitted $(x, y, dx, dy)$ as currents acting on $\mathcal{C}^2$ functions. Notice that $d^y \sigma_k$ and $d^y \rho_k$ are currents of order one (their coefficient distributions are derivatives of $L^p$ functions, for $p < \frac{n}{n-1}$).

We define $S_k : A^C_{k+1} \rightarrow A_k$ ($k = 0, \cdots, n-1$) as

$$
S_k \phi(x, dx) := \int_y \sigma_k(x, y, dx, dy) \wedge \phi(y, dy)
$$

(18)

The fact that $S_k$ sends $A^C_{k+1}$ to $A_k$ follows from a property of the convolution: if $\varphi$ is a $\mathcal{C}^2$ compactly supported function, $\chi$ a $L^1$ function and $F$ a compactly supported distribution of order two, then the convolution $F \ast \chi = \chi \ast F$ applied to $\varphi$ is equal to $F$ applied to $\chi \ast \varphi$ (see, for instance, [Sch66]). Since $S_k$ maps $\mathcal{C}^2$ forms compactly supported to $\mathcal{C}^2$ forms by convolution with $\sigma_k(x, y, dx, dy)$, $S_k$ maps $A^C_{k+1}$ to $A_k$.

**Lemma 3.1**: $d^y_{k-1} S_k - 1 + S_k d^y_k = 1$ on $A^C_k$.

*Proof of Lemma 3.1* :

Let $\phi(x, dx)$ be a $\mathcal{C}^2$ $k$-form.

$$
(d^y_{k-1} S_k - 1 + S_k d^y_k) \phi(x, dx)
$$

$$
= \int_y d^y_{k-1} \sigma_{k-1}(x, y, dx, dy) \wedge \phi(y, dy) + \int_y \sigma_k(x, y, dx, dy) \wedge d^y_k \phi(y, dy)
$$

$$
= \int_y (d^y_{k-1} \sigma_{k-1}(x, y, dx, dy) - (-1)^n d^y_{n-k-1} \sigma_k) \wedge \phi(y, dy)
$$

$$
= \phi(x, dx)
$$

Hence, $d^y_{k-1} S_k - 1 + S_k d^y_k$ is the identity on $\mathcal{C}^2$ $k$-forms compactly supported. By the same argument as before, it is then the identity on $A^C_{k+1}$. \qed

We will now define the second ingredient. It is ‘almost’ a transfer operator, but acts from $A_k$ to $A^C_{k+1}$: we put

$$
\mathcal{N}_k = d^y_k \mathcal{M}^{(k)} - \mathcal{M}^{(k+1)} d_k.
$$

(19)
for $k = 0, \cdots, n-1$. As for $\mathcal{M}^{(k)}$ and $\mathcal{M}$, we denote by $\mathcal{N}$ the operator acting on $\bigcup_k A_k$ whose restriction to $A_k$ is $\mathcal{N}_k$. In fact, $\mathcal{N}_k$ acts on $A_k$ in the following way:

$$\mathcal{N}_k \phi(x, dx) = \sum_\omega dg_\omega(x) \cdot \psi_\omega^* \phi(x, dx)$$  \hspace{1cm} (20)

as the Leibniz rule shows. We are allowed to use the Leibniz rule because $g_\omega$ and $\psi_\omega$ are $C^3$ (and therefore $C^2$). Of course, by definition, $\mathcal{N}$ depends only on $\mathcal{M}$ and not on the representation.

4- Kernel operators and generalised kernel operators

The Star trace

From now on, we will assume for convenience that each transversal family we are working with is contained in a fixed transversal ‘super’-family, such that the transversality property always holds when we compose the diffeomorphisms. This is useful if, for instance, we want to work with different transfer operators.

**Definition 4.1:** We say that a linear operator $\mathcal{K}$ acting from $k$-forms to $\ell$-forms ($k, \ell = 0, \cdots, n$) is a kernel operator if it admits a representation:

$$\mathcal{K} \phi(x, dx) = \int_y K(x, y, dx, dy) \wedge \phi(y, dy)$$  \hspace{1cm} (21)

where $K(x, y, dx, dy)$, the kernel, is a $n-k$ form in $y$ and a $\ell$-form in $x$.

For the moment, we ask the coefficients to be $L^1$ (class of) functions, and say for convenience that $\mathcal{K}$ acts on bounded $k$-forms.

If $k = \ell$, it is natural, following Fredholm theory, to formally define the *star trace* of $\mathcal{K}$ as:

**Definition 4.2:**

$$\text{Tr}^* \mathcal{K} := \int_x K(x, x, dx, dx) = \int_x \text{RD}(K(x, y, dx, dy))$$  \hspace{1cm} (22)

This trace is not well-defined in general. Indeed, without any assumptions on the kernel, we can change the value of the coefficients of $K(x, y, dx, dy)$ arbitrary in the diagonal without changing $\mathcal{K}$, since the diagonal is negligible in $\mathbb{R}^n \times \mathbb{R}^n$ for the Lebesgue measures. Meanwhile, if $\mathcal{K}$ admits a representation (21) with continuous kernel, it is
easy to see that (22) — where \( K(x, y, dx, dy) \) is the continuous representative of the kernel — depends only on \( K \) acting on continuous forms.

The following lemmas will however allow us to compute the traces of operators obtained from \( S \) (which have non continuous kernels) and belonging to a suitable vector space. In particular, the kneading operator will belong to this vector space, and thus its star trace will be well defined (see Section 6).

**Definition 4.3:** We say that an operator \( K \) acting on \( k \)-forms is of type \( M \) if it is a transversal transfer operator. We say that \( K \) is of type \( N \) if it acts from \( k \)-forms to \( k+1 \)-forms as follows:

\[
K \phi(x, dx) = \sum_{\omega \in \Omega} \eta_\omega(x, dx) \wedge \Psi_\omega^* \phi(x, dx)
\]

where the family \( \{\Psi_\omega\}_{\omega \in \Omega} \) is transversal, and \( \eta_\omega \) is \( C^2 \).

Let \( k, \ell \in \{0, \ldots, n\} \). We will call \( K^S_{k, \ell} \) the vector space of operators \( A^C_k \to A_\ell \) which are linear combinations of products of operators of type \( M, N \) and of \( S \), with at least one \( S \), at least one operator of type \( M \) or \( N \), and with the restriction that two \( S \)'s can not be adjacent.

We say that an operator \( K \in K^S_{k,\ell} \) is in \( K^S_{n,\ell} \) if \( K = K_1 K_2 \cdots K_j \) with \( K_j \neq S \). If \( K_j \neq S \), \( K \) is in \( K^R_{n,\ell} \).

Note that the product of a type \( N \) operator with a type \( M \) operator is of type \( N \).

The next lemma is very similar to Lemma 1.1 in [BKRS97]. The proof uses strongly the transversality of our diffeomorphisms.

**Lemma 4.1:** Operators in \( K^S_{k,\ell} \) are kernel operators, and their kernels can be written as linear combination of terms

\[
\tilde{K}(x, y, dx, dy) = h(x) \cdot \tilde{h}(y) \cdot K(x, y, dx, dy),
\]

where

\[\begin{align*}
1) & \quad \begin{aligned}
& \text{•} h, \tilde{h} \text{ are } C^\infty \text{ functions} \\
& \text{•} \tilde{K}(x, y, dx, dy) \text{ is a } C^\infty \text{ form, except in the set of points} \\
& \text{determined by } \Psi(x) = y \text{ (} \Psi \text{ belonging to a transversal family)} \\
& \text{where it is maybe singular.} \\
& \text{•} h(x) - \text{ resp. } \tilde{h}(y) - \text{ is compactly supported if } K \in K^R_{k,\ell} \\
& \text{ - resp. } K^S_{k,\ell} \\
2) & \quad \begin{aligned}
& \tilde{K}(x, y, dx, dy) \in L^p(dx \times dy) \forall p < \frac{n}{n-1} \\
& \text{If } K \in K^S_{k,\ell} \cup K^R_{k,\ell}, \text{ then } \tilde{K}(x, x, dx, dx) \in L^p(dx) \forall p < \frac{n}{n-1}.
\end{aligned}
\end{align*}\]
Proof of Lemma 4.1

Let us do it by induction on the number of factors, starting our decomposition from the right. Let $\mathcal{H}$ be of type $M$ (8), and $\mathcal{J}$ be of type $N$ (23).

Obviously, 1) holds for $\mathcal{J}S_k$ whose kernel is

$$\sum_{\omega} \eta_{\omega}(x, dx) \wedge (\psi_{\omega})^*_k(x, y, dx, dy).$$

Indeed, if we expand the formula, we see the coefficient functions are linear combinations of terms of the form

$$C \cdot (\eta_{\omega})(x) \cdot \det (d_x \psi_{\omega})^J \cdot \frac{(\psi_{\omega})_j(x) - y}{|\psi_{\omega}(x) - y|^n}.$$

$S_k\mathcal{H}$ is also easy:

$$S_k\mathcal{H}\phi(x, dx) = \sum_{\omega} \int_y \sigma_k(x, y, dx, dy) \wedge g_{\omega}(y) \cdot (\psi_{\omega})^*_y \phi(y, dy)$$

$$= \sum_{\omega} \epsilon_{\omega} \int_y g_{\omega}(y) \cdot \psi_{\omega}^{-1}(y)(\psi_{\omega}^{-1})^*_y \sigma_k(x, y, dx, dy) \wedge \phi(y, dy)$$

Therefore the coefficient functions of the kernel will be a sum of

$$C \cdot g_{\omega}(y) \cdot \psi_{\omega}^{-1}(y) \cdot \frac{x_j - (\psi_{\omega}^{-1})_j(y)}{|x - \psi_{\omega}^{-1}(y)|^n}.$$

Idem for $S_k\mathcal{J}$. By induction, let $\mathcal{K} \in K_{\ell, k}^S$ be of the required form. Obviously from what precedes, $\mathcal{J}\mathcal{K}$ satisfies 1). Let us study $S_{k-1}\mathcal{K}$.

It acts as follows:

$$S_{k-1}\mathcal{K}\phi(x, dx)$$

$$= \int_z \sigma_{k-1}(x, z, dx, dz) \wedge \left( \int_y \tilde{K}(z, y, dz, dy) \wedge \phi(y, dy) \right)$$

$$= (-1)^n \int_y \left( \int_z \sigma_{k-1}(x, z, dx, dz) \wedge \tilde{K}(z, y, dz, dy) \wedge \phi(y, dy) \right)$$

$$H(x, y, dx, dy)$$

It is clear that $H(x, y, dx, dy)$ is $C^3$ where $\tilde{K}(x, y, dx, dy)$ is.

Let us now prove 2). As seen from the point of view of coefficient functions, the action of $S$ is nothing but a convolution with a $L^p$ function, $p < \frac{n}{n-1}$ (since the coefficient functions of $\sigma_k(x, y, dx, dy)$ are $C \cdot \frac{x_j - y_j}{|x - y|^n}$).

Hence, $S$ sends $L^q$ forms to $L^q$ forms $\forall p > 0$. (Recall that if $f \in L^p$,}
$g \in L^q$, then $f * g \in L^r$ with $\frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1$. See for instance [Sch66], page 151.) Obviously, if $\phi(x, dx) \in L^q(dx)$, then $M\phi(x, dx)$ and $N\phi(x, dx)$ are in $L^q(dx)$. Hence, each operator in $K^{SR}_{k,\ell}$ sends $L^q$-forms to $L^q$-forms. (In fact, $g$ even increases.)

The kernel of any operator in $K^{SR}_{k,\ell}$ is obtained by successive convolutions and multiplications by $C^2$ compactly supported forms, and is therefore in $L^p(dx \times dy)$ for all $p < \frac{n}{n-1}$, since $\sigma_k \in L^p$ for all $p < \frac{n}{n-1}$.

The second part of 2) remains to be proven. Let $K(x, y, dx, dy)$ be the kernel of an operator $K$ in $K^{SR} \cup K^{SL}$, such that there is $s+1$ $S$s in the decomposition of $K$. To prove that $K(x, x, dx, dy) \in L^p(dx)\forall p < \frac{n}{n-1}$, let us precise the notations : $u^{(1)}, \ldots, u^{(s)}$ will belong to $\mathbb{R}^n$, and we will denote by $\Psi_i$ the $i$th coordinate of the map $\Psi : \mathbb{R}^n \to \mathbb{R}^n$. The family $(\Psi^{(i)})_{i=1,\ldots,s}$ will denote a transversal family (in fact, $\Psi^{(i)}$ is a composition of $\psi_{i,s})$.

Let $K_I(x, y)$ be a coefficient function of $K(x, y, dx, dy)$. It is then a linear combination of $F(x, y)$, which we can write as

$$\int_{\mathbb{R}^n} \cdots \int_{\mathbb{R}^n} G(x, u^{(1)}, \ldots, u^{(s)}, y) \cdot H(x, u^{(1)}, \ldots, u^{(s)}, y) du^{(1)} \ldots du^{(s)}$$

where $G$ is $C^3$ and compactly supported in each $u^{(i)}$ and in $x$ or $y$, and $H$ is of the form:

$$H(x, u^{(1)}, \ldots, u^{(s)}, y) := \prod_{i=0}^{s-1} \frac{\Psi^{(i)}(u^{(i)}) - u^{(i+1)}}{\Psi^{(i)}(u^{(i)}) - u^{(i+1)}} \cdot \prod_{i=0}^{s-1} \frac{\Psi^{(i)}(u^{(i)}) - u^{(i+1)} \cdot \psi_{ij}}{\Psi^{(i)}(u^{(i)}) - u^{(i+1)} \cdot \psi_{ij}}$$

Thus, it suffices to show that $H(u^{(0)}, u^{(1)}, \ldots, u^{(s)}, u^{(0)})$ is in $L^p_{L^1}(du^{(0)} \times \cdots \times du^{(s)})$ for all $p < \frac{n}{n-1}$ to conclude that $K_I(x, x)$ is in $L^p(dx)$.

(If $K \in K^{SR} \cap K^{SL}$, put $u^{(0)} = x = y$. If $K \in K^{SR}$, put $z = x = y$, then use the change of variable $u^{(0)} = \Psi(z)$. Now, the integrand of $F(x, x)$ is just $H$ multiplied by a function which is $C^2$ and compactly supported in each variable.)

Note first that the ‘worse singularity’ of $H$ occurs at points $\hat{u} := (\hat{u}^{(0)}, \ldots, \hat{u}^{(s)})$ where $\Psi^{(i)}(\hat{u}^{(i)}) - \hat{u}^{(i+1)} = 0$ (for $i = 0, \ldots, s-1$), and $\Psi^{(i)}(\hat{u}^{(i)}) - \hat{u}^{(i+1)} = 0$. Hence, $\Psi^{(i)} \circ \cdots \circ \Psi^{(0)}(\hat{u}^{(0)}) = \hat{u}^{(0)}$. By transversality, on every compact, there is only a finite number of such $\hat{u}$, which are isolated. Let us fix one of those $\hat{u}$.

We perform the following change of variable near $\hat{u}$:

$$\eta^{(i)} = \Psi^{(i)}(u^{(i)}) - u^{(i+1)} \quad \text{for } i = 0, \ldots, s-1$$

$$\eta^{(s)} = \Psi^{(s)}(u^{(s)}) - u^{(0)}$$

(24)
The fact that the Jacobian $J\text{ac}$ of this change of variable is non-zero is checked later.

Hence,
\[
\int \cdots \int_{C_{p1}} |H(u^{(0)}, \ldots, u^{(s)})|^p \, du^{(0)} \cdots du^{(s)}
\]
\[
= \int \cdots \int_{C_{p1}} \left| \frac{1}{\text{Jac}} \cdot \left| \prod_{i=0}^{s} |\eta^{(i)}|^{n_i} \right|^p \right| \, d\eta^{(0)} \cdots d\eta^{(s)}
\]
\[
\leq M \cdot \prod_{i=0}^{s} \int_{C_{p1}} \left( \frac{|\eta^{(i)}|}{|\eta^{(i)}|^p} \right)^p \, d\eta^{(i)}
\]

($M$ being the sup of $\frac{1}{\text{Jac}}$ on the compact set) and we are done since $\frac{1}{\text{Jac}} \in L^p_{OC} (dx)$ for all $p < \frac{1}{\text{Jac}}$.

Let us now check that the Jacobian of our change of variable (24) is not zero near the point $\hat{u}$. The matrix is of the form:

\[
J = \begin{pmatrix}
M_0 & -id & 0 & \cdots & 0 \\
0 & M_1 & -id & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & \cdots & 0 & M_{s-1} & -id \\
-id & 0 & \cdots & 0 & M_s
\end{pmatrix}
\]

where $M_i = d_{\hat{u}(i)} \Psi^{(i)} (i = 0, \ldots, s)$

Now, suppose that $\text{Jac} = \det(J) = 0$. Then, there is an eigenvector $0 \neq w = (w^{(0)}, \ldots, w^{(s)})^T$ (with $w^{(i)} \in \mathbb{R}^n$) such that $J \cdot w = 0$.

Looking at the blocks, this formula reads

\[
\begin{pmatrix}
d_{\hat{u}(0)} \Psi^{(0)} \cdot w^{(0)} \\
d_{\hat{u}(1)} \Psi^{(1)} \cdot w^{(1)} \\
\vdots \\
d_{\hat{u}(s-1)} \Psi^{(s-1)} \cdot w^{(s-1)} \\
d_{\hat{u}(s)} \Psi^{(s)} \cdot w^{(s)}
\end{pmatrix}
= \begin{pmatrix}
w^{(1)} \\
w^{(2)} \\
\vdots \\
w^{(s)} \\
w^{(0)}
\end{pmatrix}
\]

Hence, if $w \neq 0$, each $w^{(i)} \neq 0$. Let $\hat{\Psi} = \Psi^{(s)} \circ \cdots \circ \Psi^{(0)}$. We have that

\[
w^{(0)} = \left( d_{\hat{u}(s)} \Psi^{(s)} \cdot d_{\hat{u}(s-1)} \Psi^{(s-1)} \cdots d_{\hat{u}(0)} \Psi^{(0)} \right) \cdot w^{(0)}
= (d_{\Psi^{(s)}(s)} \circ \cdots \circ \Psi^{(0)}(s) \circ \cdots \circ \Psi^{(0)}(0)) \Psi^{(s)}(s-1) \cdots \Psi^{(s)}(0) \Psi^{(0)}(s-1) \cdots \Psi^{(0)}(0) \cdot w^{(0)}
= \left( d_{\hat{u}(0)} \Psi^{(s)} \circ \cdots \circ \Psi^{(0)} \Psi^{(0)} \right) \cdot w^{(0)}
= d_{\hat{u}(0)} \hat{\Psi}
\]

Since $w^{(0)} \neq 0$, 1 is an eigenvalue of $d_{\hat{u}(0)} \hat{\Psi}$. But $\hat{\Psi}(\hat{u}^{(0)}) = \hat{u}^{(0)}$, and $\hat{\Psi}$ is a composition of $\psi_\omega$’s, which is a contradiction since each composition in the family $\{\psi_\omega\}$ is transversal.
By continuity, \( \text{Jac} = \det(J) \neq 0 \) in a neighborhood of \( \hat{u} \), where \( H \) is then integrable.

We proved that \( H \) is in \( L^p_{\text{loc}} \) on the neighborhood of the points \( \hat{u} := (\hat{u}^{(0)}, \ldots, \hat{u}^{(s)}) \) where \( \Psi^{(i)}(\hat{u}^{(i)}) - \hat{u}^{(i+1)} = 0 \) (for \( i = 0, \ldots, s-1 \)), and \( \Psi^{(s)}(\hat{u}^{(s)}) - \hat{u}^{(0)} = 0 \). We will now take care of the other points.

Let us denote by \( T_i(x, y) \) the function \( \frac{x_i - y_i}{|x-y|^2} \), and by \( R_i(x, y) \) the function \( T_i \circ (\Psi^{(i)} \times \text{id}) \). We then have

\[
H(u^{(0)}, \ldots, u^{(s)}) = R_0(u^{(0)}, u^{(1)}) \cdot R_1(u^{(s)}, u^{(0)}) \cdot \prod_{i=1}^{s-1} R_i(u^{(i)}, u^{(i+1)})
\]

We fix \( \epsilon > 0 \). Let us suppose that there is a \( k \) such that \( |\Psi^{(k)}(u^{(k)}) - u^{(k+1)}| > \epsilon \) on some open set \( U \) with compact closure. Up to performing a (circular) change of indices, we can suppose that \( k = s \). On \( U \), \( |R_k(u^{(s)}, u^{(0)})|^p \) is then bounded, let us say by \( B \). Hence we have

\[
\int \cdots \int_U |H(u^{(0)}, \ldots, u^{(s)})|^p du^{(0)} \cdots du^{(s)} \leq B \cdot \int \cdots \int_U R_0(u^{(0)}, u^{(1)}) \cdots R_{s-1}(u^{(s-1)}, u^{(s)})|^p du^{(0)} \cdots du^{(s)}
\]

Note that \( u^{(0)} \) and \( u^{(s)} \) appear only once in this integral. Hence, \( \int |R_{s-1}(u^{(s-1)}, u^{(s)})|^p du^{(s)} \) is bounded for almost all \( u^{(s-1)} \) if the integral is taken on a (subset of a) compact set, since it is the convolution of \( T_{s-1} \) with a compactly supported \( C^2 \) function \( \phi \) which takes the value 1 on \( U \), composed with the diffeo \( \Psi^{(s-1)} \times \text{id} \).

By repeating that argument \( s \) times, it follows that \( H(u^{(0)}, \ldots, u^{(s)})|^p \) is integrable on \( U \).

To finish, note that it is possible to choose \( \epsilon \) such that we can cover any compact set by a finite number of open sets \( U \) with the property that either there is a \( k \) such that \( |\Psi^{(k)}(u^{(k)}) - u^{(k+1)}| > \epsilon \) on \( U \), either the jacobian of (24) is bounded from below by \( \epsilon \) on \( U \).

This lemma, and especially the fact that the kernel is integrable on the diagonal, is the first step towards showing that the star trace of an operator in \( K \in K^{SL}_s \cup K^{SR}_s \) is well defined. The second step is given in what follows.

Let \( K(x, y, dx, dy) \) have coefficient functions \( K: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R} \) that are continuous, except maybe at points \( (x, y) \) with \( \Psi(x) = y \) (\( \Psi \) belonging to a transversal family). We don’t put indices on the \( K(x, y) \)'s in order to avoid heavy notations. We ask the \( K(x, y) \)'s to have compact support in either \( x \) or \( y \), and \( K(x, x) \in L^1(dx) \).

For \( \xi > 0 \), let \( \chi_\xi(x, y) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n \) be a \( C^\infty \) function s.t.
\( \chi_\xi(x, y) = 0 \) in a \( \xi \)-neighborhood of the singularities of the \( K(x, y) \)'s, and \( \chi_\xi(x, y) = 1 \) outside of a \( 2\xi \)-neighborhood of the singularities of the \( K(x, y) \)'s, such that \( K(x, y) \cdot \chi_\xi(x, y) \) is continuous everywhere.

Let \( \phi_m(z) \) be a sequence of \( C^\infty \)-approximations of the Dirac delta \( \delta(z) \) (in the sense that \( \int f(z) \phi_m(z) \, dz \to f(0) \) for continuous \( f \)). We define \( \mu^{(m)}_k(x, y, dx, dy) \) to be the approximations of \( \rho_k \) obtained by using \( \phi_m(x - y) \) instead of \( \delta(x - y) \) in the definition (14) of \( \rho_k \).

**Lemma 4.2:**

\[
\lim_{\xi \to 0} \lim_{m \to \infty} \int_x \int_y \chi_\xi(x, y) \cdot K(x, y, dx, dy) \wedge \mu^{(m)}_k(x, y, dx, dy) = (-1)^{n(k+1)} \int x \cdot \text{Tr}^* K
\]

The proof uses again the transversality of the \( \Psi \)'s:

**Proof of Lemma 4.2**

By continuity,

\[
\int_x \int_y K(x, y) \chi_\xi(x, y) \cdot \phi_m(x - y) \, dx \, dy \to \int_x K(z, z) \chi_\xi(z, z) \, dz
\]

By transversality of \( \Psi \), and compacity of the support of the \( K(x, y) \)'s (in \( x \) or \( y \)), \( K(x, x) \) has only a finite number of singularities. If \( \xi \) is small enough, the support of \( 1 - \chi_\xi(x, x) \) (\( 1 \) being the constant function 1 on the diagonal) is contained in a finite number of balls of volume \( \leq C \cdot \xi \), \( C \) being a constant depending mainly on the derivative of \( \Psi \) at its fixed points. Since \( K(x, x) \) is in \( L^1(dz) \), it is a nonatomic finite measure with compact support. Hence,

\[
\int K(x, x) \cdot (1 - \chi_\xi(x, x)) \, dx \leq 2 \cdot \int_E K(x, x) \, dx
\]

where \( E \) is a finite union of balls of volume \( \leq C \cdot \xi \).

Since the measure of the limit of a sequence \( E_n \) of sets such that \( E_n \supset E_{n+1} \) is the limit of the measures of the \( E_n \), \( \int K(x, x) \cdot \chi_\xi(x, x) \, dx \to \int K(x, x) \, dx \) as \( \xi \to 0 \), and we conclude using

\[
(-1)^{n+\ell+n+\ell} \int_y \varphi(x, y, dx, dy) \wedge \rho_\ell(x, y, dx, dy) = \varphi(x, x, dx, dy)
\]

for any \( k \)-form in \( x \) and a \( \ell \)-form in \( y \). \qed

As a corollary, we get:

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Corollary 4.3: If $\mathcal{K} \in K^S_{\mathcal{L}} \cup K^S_{\mathcal{R}}$, then its star trace $\text{Tr}^*\mathcal{K} := \int_x K(x, x, dx, dx)$ is well defined, i.e. depends only on $\mathcal{K}$ as an operator acting on $C^\infty$ forms.

Proof of Corollary 4.3:

The two preceding lemmas show us that it is possible to approximate $\text{Tr}^*\mathcal{K} = \int_x K(x, x, dx, dx) = \int_x \int_y K(x, y, dx, dy) \wedge \rho (x, y, dx, dy)$ by

$$\int_x \int_y K(x, y, dx, dy) \wedge \mu^{(m)}(x, y, dx, dy).$$

Let us first recall that it is possible to approach any continuous function of $x$ and $y$ in $\mathbb{R}^n$ by a finite sum of terms $\varphi(x) \cdot \kappa(y)$ with $\varphi$ and $\kappa$ compactly supported and $C^\infty$.

Moreover,

$$\sum_{I(k), J(n-k)} \int_x \int_y K(x, y, dx, dy) \cdot \varphi(x) \kappa(y) dx \wedge dy$$

$$= \sum_{I(k), J(n-k)} \int_x \left( \int_y K(x, y, dx, dy) \cdot \kappa(y) dy \right) \wedge \varphi(x) dx.$$

Hence, $\int_x \int_y K(x, y, dx, dy) \wedge \mu^{(m)}(x, y, dx, dy)$ depends only on $\mathcal{K}$ as an operator acting on $C^\infty$ $k$-forms.

We will have to compute the traces of operators which are obtained with $d$, $S$ and $M$. Unfortunately, in general, these operators will be neither kernel operators, nor transversal transfer operators. However, by allowing a generalised kernel operator to have a kernel that is a current, we can also define the trace of such an operator to be the integral on the diagonal of the kernel if this restriction on the diagonal is a well defined current.

Definition 4.4: We define $K^S_k$ to be the vector space of operators acting on $A_k^\mathcal{C}$ which are linear combinations of finite products of $S_k \mathcal{M} d$, $dS_{k-1} \mathcal{M}$, $\mathcal{M} dS_{k-1}$, $\mathcal{M} S_k d$ (where $\mathcal{M}$ is any transversal transfer operator), with the restriction that an $S$ can not be adjacent to $dS$ or $S d$. By abuse of language, we call $S_k \mathcal{M} d$, $dS_{k-1} \mathcal{M}$, $\mathcal{M} dS_{k-1}$ and $\mathcal{M} S_k d$ the generators of $K^S_k$.

An operator in $K^S_k$ will be called generalised kernel operator.

Notice that $\mathcal{M}^{(k)}$ belongs to $K^S_k$ since $\mathcal{M}^{(k)} = \mathcal{M}^{(k)} dS_{k-1} + \mathcal{M}^{(k)} S_k d$.

Now, notice that although $d^y \sigma_k(x, y, dx, dy)$ is not a function but a distribution, we can integrate it with any compactly supported $C^1$ $(n-k)$-form using integration by parts:

$$\int_y d^y \sigma_k(x, y, dx, dy) \wedge \phi(y) = (-1)^{k+1} \int_y \sigma_k(x, y, dx, dy) \wedge d^y \phi(y).$$
By similarity with kernel operators, we would like to define the trace of a generalised kernel operator as the integral of the restriction to the diagonal of the kernel. This is done in the following lemma.

**Lemma 4.4:** If $K \in K^d_k$, with generalised kernel $K(x, y, dx, dy)$, then $\int K(x, x, dx, dx)$ is a well defined complex number, which we call the star trace $Tr^*K$ of $K$.

**Remark 4.5:** By a standard argument of approximations by $C^\infty$ maps (any distribution can be approximated by $C^\infty$ maps), we can use change of variables under the integral, and integration by part in the computation of traces. Thus, in the computations, there is no actual difference between a generalised kernel operator in $K^d_k$ and a kernel operator in $K^{SL}_k \cup K^{SR}_k$.

**Proof of Lemma 4.4:**

The proof is similar, in spirit, to the proof of Lemma 4.1.

Let us check the theorem on the generator $\mathcal{M}^{(k)}dS$. Its kernel is the current:

$$K(x, y, dx, dy) = \sum_{\omega} g_\omega(x) \cdot (\psi_\omega)_x^* d^n \sigma_k(x, y, dx, dy)$$

We would like to give a meaning to $\int K(x, x, dx, dx)$, even if at first sight, $K(x, x, dx, dx)$ is not defined.

Let us forget the forms for a moment, and consider only their coefficients. We introduce the following notations:

$$f_i : \mathbb{R}^n \to \mathbb{R} \quad \varphi : \mathbb{R}^{2n} \to \mathbb{R}^n \quad j : \mathbb{R}^n \to \mathbb{R}^{2n} \quad \Psi = (\Psi_1, \cdots, \Psi_n)$$

where $\Psi = (\Psi_1, \cdots, \Psi_n)$ is a transversal (local) diffeomorphism.

It follows directly from the definitions that the coefficient distributions of the kernel of $S^{(k)}d\mathcal{M}$ are sum of terms of the form $g(x) \cdot \partial_x (f_i \circ \varphi)(x, y)$ where $g(x)$ is a $C^2$ function compactly supported (it is a product of $g_\omega, \partial_x g_\omega, \partial_x \psi_\omega$).

Let us see that the restriction of $\partial_x (f_i \circ \varphi)$ is a distribution. In fact, by linearity of $j$ and a simple matrix computation,

$$(\partial_x (f_i \circ \varphi)) \circ j(x) = \sum_{\nu=1}^n \partial_x \Psi_{\nu}(x) \cdot (\partial_x f_i)(\varphi \circ j(x))$$

Away from the fixed points of $\Psi$, $(\partial_x f_i)(\varphi \circ j(x))$ is nothing but a $C^2$ function. Moreover, by transversality, $\varphi \circ j(x)$ is a diffeomorphism.
in a neighbourhood of the fixed points. Finally, \( \partial_{x_i} f_i \) is a distribution (of order one) since \( f_i \) is in \( L^1_{\text{loc}} \).

Hence, the integral \( \int (g(x) \cdot \partial_{x_i}(f_i \circ \varphi)) \circ j(x) \, dx \) is defined: Away from the fixed points, we integrate a \( \mathcal{C}^2 \) function with compact support, and near the fixed points, we apply the distribution of order one \( \partial_{x_i} f_i \) composed with the diffeomorphism \( \varphi \circ j(x) \) to the \( \mathcal{C}^2 \) function with compact support \( g(x) \cdot \partial_{x_k} \Psi_v(x) \).

The same type of arguments holds for the other generators.

Now, let \( \mathcal{K} \in K^i_k \) be the product of \( s \geq 2 \) generators \( \mathcal{K}_1, \cdots, \mathcal{K}_s \), and see what the coefficient distributions of its kernel are. The generators \( \mathcal{K}_1, \cdots, \mathcal{K}_s \) involve either \( d^x \sigma_k(x, y, dx, dy) \), either \( d^y \sigma_k(x, y, dx, dy) \), but by using formula (17), we can suppose that \( \mathcal{K} \) is a sum of products of at most \( s \) operators involving only \( d^x \sigma_k(x, y, dx, dy) \). For simplicity, let us say that \( s = 2 \), and that there is only one term in the sum (the general case being handled similarly).

It is easy to see that formally, the integral on the diagonal of the coefficient distributions of the kernel of \( \mathcal{K} \) is of the form:

\[
\int \int_{\mathbb{R}^{2n}} g(x) \hat{g}(y) \partial_{x_i}(f_{i_1} \circ \hat{\varphi})(x, y) \cdot \partial_{x_j}(f_{i_2} \circ \hat{\varphi})(y, x) \, dxdy
\]

\[
= \sum_{\nu, \gamma = 1}^n \int \int_{\mathbb{R}^{2n}} g(x) \hat{g}(y) \cdot \partial_{x_i} \hat{\Psi}_v(x) \cdot \partial_{x_i} \hat{\Psi}_s(x) \cdot \partial_{x_j} \hat{\varphi}(x, y) \cdot \partial_{x_j} \hat{\varphi}(y, x) \, dxdy
\]

(with \( \hat{\varphi}(x, y) = \hat{\Psi}(x) - y, \hat{\varphi}(y, x) = \hat{\Psi}(y) - x \), and \( g, \hat{g} \) being \( \mathcal{C}^2 \) functions compactly supported.) Again, we have to give a meaning to this integral. The trick is the same as in Lemma 4.1, i.e. we will make a change of variable near each point \( (x, y) \in \mathbb{R}^{2n} \) such that \( \hat{\Psi}(x) = y \) and \( \hat{\Psi}(y) = x \), which are the “annoying points”.

As in the proof of Lemma 4.1, we use the change of variable (24):
\( u = \hat{\varphi}(x, y), v = \hat{\varphi}(y, x) \). By transversality of \( \Psi \circ \hat{\Psi} \), this change of variable has Jacobian bounded away from zero in some neighbourhood of the points \( (x, y) \) described before.

After performing the change of variable (24), the \( \mathcal{C}^2 \) compactly supported function \( g(x) \hat{g}(y) \cdot \partial_{x_k} \hat{\Psi}_v(x) \partial_{x_i} \hat{\Psi}_s(x) \) becomes \( \tilde{g}_{\nu, \gamma}(u, v) \), and then the above integral is:

\[
\sum_{\nu, \gamma = 1}^n \int \int_{\mathbb{R}^{2n}} \tilde{g}_{\nu, \gamma}(u, v) \partial_{x_i} f_{i_1}(u) \partial_{x_j} f_{i_2}(v) \, dudv
\]

\[
= \sum_{\nu, \gamma = 1}^n \int \partial_{x_i} f_{i_1}(u) \left( \int \partial_{x_j} f_{i_2}(v) \tilde{g}_{\nu, \gamma}(u, v) \, dv \right) \, du
\]
and is therefore defined, since $\int \partial_x f_\gamma(v) \tilde{g}_{\gamma}(u, v) \, dv$ is a $C^2$ function compactly supported in $u$.

Notice that if $f_i \circ \phi$ is $C^3$ on $U$, then

$$\int \int_U K(x, y) = \int \int_U \tilde{g}(x, y) \cdot \partial_x (f_i \circ \phi)(y, x)$$

where $\tilde{g}$ is $C^2$, and this integral is defined by integration by parts. $\square$

Recall that $\mathcal{M}^{(k)} \in K_{ij}^d$, and we have thus two traces at disposal for $\mathcal{M}^{(k)}$. However, the next lemma shows that on the subset of $K_{ij}^d$ containing the transfer operators, the flat and the star trace coincide:

**Lemma 4.6:**

$$\text{Tr}^* \mathcal{M}^{(k)} d_{k-1} S_{k-1} + \text{Tr}^* \mathcal{M}^{(k)} S_k d_k = \text{Tr}^\dagger \mathcal{M}^{(k)}$$

**Proof of Lemma 4.6:**

$$\text{Tr}^* \mathcal{M}^{(k)} d_{k-1} S_{k-1} = \sum_\omega \int_x g_\omega(x) \cdot \text{RD} \left( (\psi_\omega)^* d_{k-1}^x \sigma(x, y, dx, dy) \right)$$

On the other hand,

$$\mathcal{M}^{(k)} S_k d_k \psi(x, dx) = \sum_\omega \int_y g_\omega(x) \cdot (\psi_\omega)^* d^y \sigma_{k-1}(x, y, dx, dy) \wedge \phi(y, dy)$$

Hence, using (17), and the fact that $\psi_\omega$ and $g_\omega$ are $C^3$,

$$\text{Tr}^* \mathcal{M}^{(k)} S_k d_k = (-1)^{n(k+1)} \sum_\omega \int_x g_\omega(x) \cdot \text{RD} \left( (\psi_\omega)^* d_{k-1}^x \rho_{k}(x, y, dx, dy) \right)$$

$$- \sum_\omega \int_x g_\omega(x) \cdot \text{RD} \left( (\psi_\omega)^* d^y \sigma_{k-1}(x, y, dx, dy) \right)$$

$$\text{Tr}^* \mathcal{M}^{(k)} d_{k-1} S_{k-1}$$

And we are done since

$$\sum_\omega \int_x g_\omega(x) \cdot \text{RD} \left( (\psi_\omega)^* d_{k-1}^x \rho_{k}(x, y, dx, dy) \right)$$

$$= \sum_\omega \sum_{l(k)} \int_x g_\omega(x) \cdot \delta(\psi_\omega(x) - x) \cdot \det \left( d_x \psi_\omega \right)^2$$

$$= \text{Tr}^\dagger \mathcal{M}^{(k)}$$

$\square$
5- Some commutations

In this section we show that in $K^a_{SL} \cup K^a_{SR}$ and $K^d_k$, we can perform
some commutation of operators without changing the traces $\text{Tr}^*$. By
Remark 4.5, it is unnecessary to make a difference between a kernel
operator and a generalised kernel operator in computations.

Definition 5.1: We call $G$-operator an operator in $K^a_{SL} \cup K^a_{SR} \cup K^d_k$.

The following lemma summarizes all the results on commutations.
The signs that appear in points b) and c) are the reason of the failure
of the proof of Theorem 6.1 in even dimension.

Lemma 5.1:

a) Let $\mathcal{H}: A_k \to A^C_k$ be of type $M$ (i.e. a transversal transfer operator).
If $K$ is a (generalised) kernel operator such that $K\mathcal{H}$ and $\mathcal{H}K$ are
$G$-operators, then $\text{Tr}^*K\mathcal{H} = \text{Tr}^*\mathcal{H}K$.

b) Let $\mathcal{H}: A_k \to A^C_{k+1}$ be of type $N$ (see (23)), and $K: A^C_{k+1} \to A_k$ be
a (generalised) kernel operator with kernel $K(x, y, dx, dy)$, such that
$\mathcal{H}K$ and $K\mathcal{H}$ are $G$-operators. Then, $\text{Tr}^*K\mathcal{H} = (-1)^{n-1} \text{Tr}^*\mathcal{H}K$.

c) Let $K_1: A^C_k \to A^C_k$ and $K_2: A^C_k \to A^C_k$ be (generalised) kernel operators
such that $K_2K_1$ and $K_1K_2$ are $G$-operators. Then,
$\text{Tr}^*K_1K_2 = (-1)^{(n+1)(k+\ell)} \text{Tr}^*K_2K_1$.

Note that c) does not imply a) even though we can view a transversal
transfer operator as a generalised kernel operator. For instance, if we
want to commute $\mathcal{M}^{(k)}$ with $\mathcal{S}$, we cannot use the equality $\mathcal{M}^{(k)} = \mathcal{M}^{(k)}(d\mathcal{S} + \mathcal{S}d)$, since we would have two adjacent $\mathcal{S}$s. Similarly, c)
does not imply b) either.

Proof of Lemma 5.1

a) Straightforward computations give:

$$\text{Tr}^*K\mathcal{H} = \sum_\omega \int_x g_\omega(x) \cdot \text{RD}((\psi_\omega)_x^*K(x, y, dx, dy))$$

$$\text{Tr}^*\mathcal{H}K = \sum_\omega e_\omega \int_x g_\omega \circ \psi_\omega^{-1}(x) \cdot \text{RD}((\psi_\omega^{-1})_y^*K(x, y, dx, dy))$$

$$= \sum_\omega \int_x g_\omega(x)(\psi_\omega)_x^* \text{RD}((\psi_\omega^{-1})_y^*K(x, y, dx, dy))$$

And we conclude via Lemma 1.1 a).

b) We use the notation (23) for $\mathcal{H}$. Notice that $K(x, y, dx, dy)$ is a
$k$-form in $x$, and a $n - k - 1$-form in $y$. Hence, alltogether, it is a
We have:

$$\text{Tr}^* \mathcal{H} \mathcal{K} = \sum_{\omega} \int_x \eta_\omega(x, dx) \wedge \text{RD}((\Psi_\omega)^* K(x, y, dx, dy))$$

as in point a).

If \( \phi(x, dx) \in A_k \), we have

$$\mathcal{K} \mathcal{H} \phi(x, dx) = (-1)^{n-1} \sum_{\omega} c_\omega \int_y (\Psi_\omega)^* (\eta_\omega(y, dy) \wedge K(x, y, dx, dy)) \wedge \phi(y, dy)$$

Computing its trace, we find:

$$\text{Tr}^* \mathcal{K} \mathcal{H} = (-1)^{n-1} \sum_{\omega} \int_x \Psi_\omega^* \left( \text{RD}((\Psi_\omega)^* (\eta_\omega(y, dy) \wedge K(x, y, dx, dy))) \right)$$

And we conclude using again Lemma 1.1 a).

\( c \) We leave the proof to the reader. One has to be careful with the signs and apply Fubini.

We can now be more precise and state a useful corollary.

**Corollary 5.2:**

a) \( \text{Tr}^* (d_{k-1} \mathcal{S}_{k-1} \mathcal{M}^{(k)})^m = \text{Tr}^* (\mathcal{M}^{(k)} d_{k-1} \mathcal{S}_{k-1})^m \forall m \geq 1. \)

b) If \( \mathcal{H} \) is of type \( N \), \( \text{Tr}^* ((\mathcal{H} \mathcal{S})^m) = (-1)^{n-1} \text{Tr}^* ((\mathcal{S} \mathcal{H})^m) \forall m \geq 1. \)

c) If \( \mathcal{K} \) is s.t. \( \mathcal{S}_k \mathcal{K} \) and \( \mathcal{K} \mathcal{S}_k \) satisfy the hypotheses of 5.1 c), then \( \text{Tr}^* \mathcal{S}_k \mathcal{K} = (-1)^{n+1} \text{Tr}^* \mathcal{K} \mathcal{S}_k. \)

d) \( \text{Tr}^* [(d_{k-1} \mathcal{S}_{k-1} \mathcal{M}^{(k)})^m] (\mathcal{S}_k \mathcal{M}^{(k+1)} d_k)^{m_2} = 0 \)

Point a) follows from the fact that \( (d_{k-1} \mathcal{S}_{k-1} \mathcal{M}^{(k)})^{m-1} d_{k-1} \mathcal{S}_{k-1} (m \geq 1) \) satisfies the hypotheses of 4.1 a). Points b) and c) are direct applications of 5.1 b) and 5.1 c). Point d) uses \( d^2 = 0 \) and 5.1 c).

We would also like to be able to commute \( \mathcal{S} \) and \( d \) separately around \( \mathcal{M}^{(k)} \). This is done in the next lemma.

**Lemma 5.3:** \( \text{Tr}^* (\mathcal{M}^{(k)} d_{k-1} \mathcal{S}_{k-1})^m = (-1)^{n+1} \text{Tr}^* (\mathcal{S}_{k-1} \mathcal{M}^{(k)} d_{k-1})^m. \)

**Proof of Lemma 5.3:**

For \( m \geq 2 \), it is an immediate corollary of 5.2 c).
For $m = 1$, let us denote by $K_{\mathcal{A}}(x, y, dx, dy)$ the kernel of $\mathcal{A}$. We have:

$$K_{\mathcal{M}(k)}(x, y, dx, dy) = \sum_{\omega} g_{\omega}(x) (\psi_{\omega})^{*}_{x} d^{x} \sigma_{k}(x, y, dx, dy)$$

$$K_{\mathcal{S}_{\mathcal{M}(k)}}(x, y, dx, dy) = (-1)^{n} \sum_{\omega} \epsilon_{\omega} (\psi_{\omega}^{-1})^{*}_{y} [g_{\omega}(y) \cdot d^{y} \sigma_{k}(x, y, dx, dy)]$$

$$+ (-1)^{n} \sum_{\omega} \epsilon_{\omega} (\psi_{\omega}^{-1})^{*}_{y} [d^{y} g_{\omega}(y) \wedge \sigma_{k}(x, y, dx, dy)]$$

Hence,

$$\text{Tr}^{*} \mathcal{M}^{(k)}_{d} S = \int_{x} \sum_{\omega} g_{\omega}(x) \text{RD}[(\psi_{\omega})^{*}_{x} d^{x} \sigma_{k}(x, y, dx, dy)]$$

$$\text{Tr}^{*} \mathcal{S}_{\mathcal{M}^{(k)}}_{d} = (-1)^{n} \sum_{\omega} \int_{x} g_{\omega}(x) \text{RD}[(\psi_{\omega})^{*}_{x} d^{y} \sigma_{k}(x, y, dx, dy)]$$

$$+ (-1)^{n+1} \sum_{\omega} \int_{x} g_{\omega}(x) \wedge d^{x} \text{RD}[(\psi_{\omega})^{*}_{y} \sigma_{k}(x, y, dx, dy)]$$

and we conclude using Lemma 1.1 b). □

6- Kneading operators and main theorem

We now have almost all the ingredients required to state and prove our main theorem. First, we need to give the definition of a formal determinant, and a formula for the resolvent of a linear operator. Suppose that we have a vector space of linear operators endowed with a (formal) trace ‘Tr’ (that is, a map from operators to $\mathbb{R}$ or $\mathbb{C}$ which is linear). We can then define the (formal) determinant of $(1 - z \mathcal{A})$ as follows:

$$\text{Det}(1 - z \mathcal{A}) = \exp \left\{ \sum_{n \geq 1} \frac{z^{n}}{n} \text{Tr} \mathcal{A}^{n} \right\} \quad \text{(26)}$$

(see, for instance, [Gro56] or [GGK00]). We define the sharp, flat and star (formal) determinants using (26) with the sharp, flat and star trace.

If $\mathcal{A}$ is a bounded operator acting on some Banach space $B$, and if $z$ is smaller than the inverse of a bound for $\mathcal{A}$, then $(1 - z \mathcal{A})$ is invertible on $B$, and we have the following formula:

$$(1 - z \mathcal{A})^{-1} = 1 + z \mathcal{A} + (z \mathcal{A})^{2} + (z \mathcal{A})^{3} + \cdots \quad \text{(27)}$$

Applying (27) to $\mathcal{M}^{(k)}$, we have that

$$(1 - z \mathcal{M}^{(k)})^{-1} = 1 + z \mathcal{M}^{(k)} + (z \mathcal{M}^{(k)})^{2} + (z \mathcal{M}^{(k)})^{3} + \cdots \quad \text{(28)}$$
and is therefore (formally) of type $M$. In fact, each term of the right-hand side of (28) (except the identity) is of type $M$, and since the trace is linear, we can formally compute the trace of each term individually and sum. We are then permitted to use Lemma 5.1, Corollary 5.2 and Lemma 5.3 termwise, and these formal computations have a meaning for $z$ small.

**Definition 6.1:** We call $\mathcal{D}_k(z) = zN_k(1 - zM(k))^{-1}S_k$ for $k = 0, \cdots, n - 1$ the kneading operators.

**Theorem 6.1:** If the dimension $n$ is odd, then, as power series in $z$,

$$\det^b(1 - zM) = \prod_{k=0}^{n} \det' (1 - zM(k))^{-1} = \prod_{k=0}^{n-1} \det^s(1 + \mathcal{D}_k(z))^{-1}^{k+1}$$

**Proof of Theorem 6.1:** The proof for $n$ even fails only at the last step, so for the moment, we do not have to assume that $n$ is odd.

The first equality is trivial from (26) and (11).

For the second equality, recall that $\det(1 - AB) = \det(1 - BA)$ whenever $\text{Tr}((AB)^m) = \text{Tr}((BA)^m)$ $\forall m$. A sufficient condition for $\det(1 - zA - zB + z^2AB) = \det(1 - zA)\det(1 - zB)$ is the following: for all $n$, putting $K_i = A$ or $K_i = B$, we must have that

$$\text{Tr}(K_1K_2 \cdots K_n) = \text{Tr}((\sigma_1K_{\sigma(1)}K_{\sigma(2)} \cdots K_{\sigma(n)})$$

for all circular permutations $\sigma$ (see for instance [Rue95], Appendix A). Since $\mathcal{N}_k(1 - zM(k))^{-1}$ is formally of type $N$, using Corollary 5.2 b), we get that

$$\det^s(1 + z\mathcal{N}_k(1 - zM(k))^{-1}S_k) = \det^s(1 + zS_k\mathcal{N}_k(1 - zM(k))^{-1})^{-1}$$

Notice that

$$1 + zS_k\mathcal{N}_k(1 - zM(k))^{-1} = (1 - z(M(k) - S_k\mathcal{N}_k))(1 - zM(k))^{-1}$$

Thus, using Lemma 5.1 a) and Proposition 2.2, we have that

$$\det^s(1 + \mathcal{D}_k(z)) = \left(\det^s(1 - z(M(k) - S_k\mathcal{N}_k))\det^s(1 - zM(k))^{-1}\right)^{(-1)^{n-1}}$$

We have now to see that the first terms in the righthandside cancel in the alternate product. This is done in the following lines (we dropped
the indices from \( S, N \) and \( d \):

\[
\begin{align*}
\det^* (1 - z (M^{(k)} - S N)) &= \det^* (1 - z (M^{(k)} - S d M^{(k)} + S M^{(k+1)} d)) \\
&= \det^* (1 - z (d S M^{(k)} + S M^{(k+1)} d)) \quad \text{(def. of } N) \\
&= \det^* (1 - z d S M^{(k)}) \det^* (1 - z S M^{(k+1)} d)) \quad \text{(17)} \\
&= \det^* (1 - z d S M^{(k)}) \cdot \det^* (1 - z d S M^{(k+1)})^{(-1)^{n-1}} \quad \text{5.2 a) and 5.3}
\end{align*}
\]

Hence, if \( n \) is odd, those terms will cancel in the alternate product. \( \square \)

In even dimension, one has the following less elegant lemma:

**Lemma 6.2:** If \( n \) is even, then

\[
\prod_{k=0}^{n} \det^* (1 + D_k (z)) = \frac{\prod_{k=0}^{n-1} \det^* (1 - z M^{(k)})}{\det^* (1 - z M^{(n)})}
\]

The proof is the same as Theorem 6.1.

The following computations show that Theorem 6.1 cannot be true in dimension 2. In fact, by the proof of Theorem 6.1, one has that

\[
\frac{\det^* (1 + D_0 (z))}{\det^* (1 + D_1 (z))} = \frac{\det^* (1 - z M^{(0)})}{\det^* (1 - z M^{(1)})} \cdot \frac{(\det^* (1 - z d S M^{(1)})^2}{\det^* (1 - z M^{(2)})}
\]

In order to obtain a formula analog to Theorem 6.1, the right-hand factor should be equal to \( \det^* (1 - z M^{(2)}) \), that is, for all \( j \geq 1 \),

\[
\text{Tr}^* (d S M^{(1)})^j = \text{Tr}^* (M^{(2)})^j.
\]

However, with \( |\Omega| = 1 \), \( \psi (x) = \theta \cdot x \) \((\theta < 1)\) and \( g \) such that \( g (0) = 1 \), one can show by straightforward computations that

\[
\text{Tr}^* (d S M^{(1)}) = -\frac{\theta}{(1 - \theta)^2} = -\frac{1}{2} \text{Tr}^* M^{(1)} \quad \text{and} \quad \text{Tr}^* M^{(2)} = \frac{\theta^2}{(1 - \theta)^2}
\]

causing the failure of the formula.

**7- Complements on the main theorem – Adjoints**

In this section, we define two type of adjoints, which will have interesting trace properties. The first one, the \( \wedge \)-adjoint of \( M \), namely \( \hat{M} \), will be a transversal transfer operator depending on \( M \). The \( \sim \)-adjoint of any (generalised) kernel operator will also be defined. Seen as a member of \( K_\Omega^\wedge \), hence as a generalised kernel operator, a simple relation between the \( \sim \) and the \( \wedge \) adjoint of \( M^{(k)} \) will be given in Lemma 7.4. This will give another version of Theorem 6.1.
Definition 7.1: Let $\mathcal{M}^{(k)}$ be a transversal transfer operator. We put $\widehat{\mathcal{M}}^{(k)}$ to be the operator acting on $A_k$ as

$$\widehat{\mathcal{M}}^{(k)}\phi(x, dx) = \sum_\omega \epsilon_\omega \cdot g_\omega (\psi_\omega^{-1}(x)) \cdot (\psi_\omega^{-1})^*(\phi(x, dx))$$

We also define $\widehat{\mathcal{N}}_k$ as $d\widehat{\mathcal{M}}^{(k)} d$, and $\widehat{\mathcal{D}}_k(z) = z\widehat{\mathcal{N}}_k (1 - z\widehat{\mathcal{M}}^{(k)})^{-1} S$.

Notice that a priori, $\widehat{\mathcal{M}}$ is not independent of the representant (8) of $\mathcal{M}^{(k)}$. This will however be of no importance, as the next lemma (very similar to Lemma 2.2 in [BR96]) shows.

Lemma 7.1:

$$Tr^\# \mathcal{M} = (-1)^n Tr^\# \widehat{\mathcal{M}}$$

The proof will follow from (11) and Lemma 7.4.

Let now $\mathcal{K} : A_k \to A_\ell$ be a (generalised) kernel operator with (generalised) kernel $K(x, y, dx, dy)$. We define

$$\widehat{\mathcal{K}} : A_{n-k} \to A_{n-\ell}
\widehat{\mathcal{K}} \varphi(x, dx) = \int_y K(y, x, dy, dx) \wedge \varphi(y, dy)$$

Notice that $K(y, x, dy, dx)$ is well defined. Unfortunately, the $\sim$-adjoint does not behave very well under composition, as the following lemma shows.

Lemma 7.2: Let $A_j \xrightarrow{\mathcal{K}_1} A_k \xrightarrow{\mathcal{K}_2} A_\ell$ be two (generalised) kernel operators. Then

$$\widehat{\mathcal{K}}_1 \mathcal{K}_2 = (-1)^{(j+k)(k+\ell)} \widehat{\mathcal{K}}_2 \mathcal{K}_1$$

The proof is by straightforward computations. Since the star trace of a (generalised) kernel operator and its $\sim$-adjoint are by definition the same, we obtain the following corollary:

Corollary 7.3: If $\mathcal{K} : A_k \to A_k$ is a (generalised) kernel operator, then $(\mathcal{K})^n = \mathcal{K}^n$, $Tr^\# \mathcal{K}^n = Tr^\# \mathcal{K}^n$ for all $n$, and

$$Det^a(1 + \mathcal{D}_k(z)) = Det^a(1 + \widehat{\mathcal{D}}_k(z))$$
We warn the reader that although $\sim$ and $\wedge$ are two involutions, one does not have for instance $\mathcal{D}_k(z) = \mathcal{D}_k(z)$. In fact, one can check that $\mathcal{D}_k(z) = -zS(1 - z\mathcal{M}^{(n-k)})^{-1}\mathcal{N}$. However, there is an easy relationship between the $\sim$-adjoint and the $\wedge$-adjoint for $\mathcal{M}^{(k)}$.

**Lemma 7.4:** $\widehat{\mathcal{M}}^{(k)} = \widehat{\mathcal{M}}^{(n-k)}$. Thus, $Tr^\dagger(\mathcal{M}^{(k)})^n = Tr^\dagger(\widehat{\mathcal{M}}^{(n-k)})^n$ for all $n$, and $Det^\dagger(1 - z\mathcal{M}^{(k)}) = Det^\dagger(1 - z\widehat{\mathcal{M}}^{(n-k)})$

*Proof of Lemma 7.4:*  

Seen as a generalised kernel operator, $\mathcal{M}^{(k)}$ has a kernel $\sum_{x,y} g_\omega(x) \cdot (\psi_\omega)_y^* \rho_k(x, y, dx, dy)$. Thus, the generalised kernel of $\widehat{\mathcal{M}}^{(k)}$ is $\sum_{x,y} g_\omega(y) \cdot (\psi_\omega)_x^* \rho_k(y, x, dy, dx)$. Since $\rho_k(y, x, dy, dx) = \rho_{n-k}(x, y, dx, dy)$, we have (with $\varphi(x, dx)$ a $n-k$-form):

\[
\widehat{\mathcal{M}}^{(k)} \varphi(x, dx) = \int_y g_\omega(y) \cdot (\psi_\omega)_y^* \rho_{n-k}(x, y, dx, dy) \wedge \varphi(y, dy) \\
= \int_y \rho_{n-k}(x, y, dx, dy) \wedge \left[ \mathcal{N}_\omega g_\omega(\psi_\omega^{-1}(y))(\psi_\omega^{-1})_y^* \varphi(y, dy) \right] \\
= \widehat{\mathcal{M}}^{(n-k)} \varphi(x, dx)
\]

As an immediate consequence, we get Proposition 2.2.

Theorem 6.1 can therefore be completed as follows:

**Theorem 6.1’:** If $n$ is odd, then as power series in $z$,

\[
Det^\#(1 - z\mathcal{M}) = Det^\#(1 - z\widehat{\mathcal{M}})^{-1} = \prod_{k=0}^{n-1} Det^\ast(1 + \mathcal{D}_k(z))^{-1}^{k+1} \\
= \prod_{k=0}^{n-1} Det^\ast(1 + \mathcal{D}_k(z))^{(-1)^k}
\]

8- $Det^\#(1 - z\mathcal{M})$ is holomorphic near 0

In this section, we prove, using Theorem 6.1, that the weighted Lefschetz $\zeta$-function $\zeta^\dagger(z) = 1/Det^\#(1 - z\mathcal{M})$ associated to a finite adapted and transversal family $\{\psi_\omega, g_\omega\}$ is meromorphic in some disk
\( \{ |z| < \epsilon \} \), and thus holomorphic in some (possibly smaller) disk (because 0 is not a pole for \( \text{Det}^\#(1 - zM) \)). The positive number \( \epsilon \) that we give here is not optimal. This result can be applied to families \( \{ \psi_\omega \} \) with superexponential growth of the number of periodic points, for which, a priori, \( \text{Det}^\#(1 - zM^{(k)}) \) does not converge in any disk. Kaloshin [Kal00] showed that there are transversal diffeomorphisms of a compact, connected manifold of dimension \( \geq 2 \) with superexponential growth of the number of fixed points. We will explain briefly how to obtain a local diffeomorphism of \( \mathbb{R}^n \) out of Kitaev’s diffeos (\( n \) will be strictly bigger than the dimension of the manifold, but we won’t try to obtain the sharpest results).

Let us suppose that \( n \) is odd. To obtain the meromorphicity of \( \text{Det}^\#(1 - zM) \) in some disk \( D \), it is sufficient (using Theorem 6.1) to show that each \( \text{Det}^\ast(1 + D_k(z)) \) is holomorphic in \( D \). In even dimension, we can use the following trick: define \( \bar{\psi}_\omega : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n \times \mathbb{R} \) as \( \bar{\psi}_\omega(x, y) = (\psi_\omega(x), \theta \cdot y) \) where \( \theta < 1 \) is a constant. We have that \( \psi_\omega \) is a transversal diffeo if \( \psi_\omega \) is; if \( x \) is a fixed point for \( \psi_\omega \), then \( (x, 0) \) is a fixed point for \( \bar{\psi}_\omega \) (and conversely).

\( L(x, \psi_\omega) = L((x, 0), \bar{\psi}_\omega) \). Choosing \( \bar{g}_\omega : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R} \), \( \bar{g}_\omega(x, y) = g_\omega(x) \cdot h(y) \), where \( h \) is a \( C^\infty \) map with compact support and \( h(0) = 1 \), we obtain that \( \text{Det}^\#(1 - zM) = \text{Det}^\#(1 - zM^\sim) \), where \( M^\sim \) is the transfer operator with family \( \{ \psi_\omega, \bar{g}_\omega \} \). We can therefore assume \( n \) to be odd and take Theorem 6.1 for granted.

So, our task is to show that \( \text{Det}^\ast(1 + D_k(z)) \) is holomorphic in some disk \( D \) for all \( k \). Notice first that if \( B \) is a Banach space such that \( SB \subset B \), then \( D_k(z) \) depends holomorphically on \( z \) when \( z^{-1} \) is not on the spectrum of \( M^{(k)} \) on \( B \) (this follows from the fact that it is the case for \( (1 - zM^{(k)})^{-1} \)). Now, if we denote by \( TC_m \) (\( m \in \mathbb{N} \)) the space of continuous linear operator \( K \) on \( B \) such that \( K^m \) is trace class, then the regularised determinant of order \( m \)

\[
\text{Det}_m(1 - K) = \exp - \sum_{\ell = m}^{\infty} \frac{1}{\ell} \text{Tr}_G(K^\ell) \quad (29)
\]

depends holomorphically on \( K \) in \( TC_m \) (see [GGK00]). The trace ‘\( \text{Tr}_G \)’ in (29) is the Grothendieck’s trace of a trace class operator.

Thus, the existence of a \( m \in \mathbb{N} \) such that the conditions a), b) and c) below hold will ensure that

\[
\text{Det}^\ast(1 + D_k(z)) = \left( \exp - \sum_{\ell = 1}^{m-1} (-1)^\ell \frac{1}{\ell} \text{Tr}^\#(D_k(z))^{\ell} \right) \cdot \text{Det}_m(1 + D_k(z))
\quad (30)
\]

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is holomorphic (and thus $\text{Det}^\#(1 - z\mathcal{M})$ meromorphic) in $D$:

a) There is a Banach (or Hilbert) space $B$ such that for all $z$ in $D$, $(\mathcal{D}_k(z))^m$ is trace class on $B$.
b) $\text{Tr}_G((\mathcal{D}_k(z))^\ell) = \text{Tr}\#((\mathcal{D}_k(z))^\ell)$ for all $\ell \geq m$ (where ‘$\text{Tr}_G$’ denotes the Grothendieck’s trace on $B$).
c) The $m - 1$ first terms of $\text{Det}\#(1 + \mathcal{D}_k(z))$ depend holomorphically on $z$ in $D$.

So, let $\mathcal{M}$ be a transversal transfer operator. Let $V, V'$ be open sets in $\mathbb{R}^n$ with compact closure such that $V' \supset V \supset V \cup \cup_{\omega} \text{supp}(g_{\omega})$, and $\chi : \mathbb{R}^n \to \mathbb{R}$ a $C^\infty$ map compactly supported in $V'$. One also checks that

$$
(1 - \chi(x))\phi(x, dx) + (1 - z\mathcal{M}^{(k)})^{-1}(\chi(x) \cdot \phi(x, dx))
$$

is a well defined and bounded if and only if $z^{-1}$ is not in the spectrum of $\mathcal{M}^{(k)}$ on $L^p(V, dx)$, and one verifies that (31) is the inverse of $(1 - z\mathcal{M}^{(k)})$ on $L^p(\mathbb{R}^n, dx)$.

Since $L^p(V, dx) \supset L^q(V, dx)$ if $\infty > p > q$, $\varrho_k^{(p)} < \varrho_k^{(q)}$ if $p > q$. Hence, if $z < \varrho_k^{(1)}$, $(1 - z\mathcal{M}^{(k)})$ has a bounded inverse on $L^p$ for all $1 \leq p < \infty$. In particular, $(1 - z\mathcal{M}^{(k)})^{-1}\sigma_k(x, y, dx, dy)$ is in $L^r(dx)$ (and in $L^r(dy)$) for all $r < \frac{n}{n-1}$.

It is obvious (by properties of the convolution) that $S_k(L^p_k(V, dx)) \subset L^1_k(V, dx)$. Indeed, $\sigma(x, dx) \in L^r(dx)$ for all $r < \frac{n}{n-1}$, and thus is in $L^1(dx)$. Since a coefficient function of $S_k\phi(x, dx)$ is a sum of convolutions of coefficient functions of $\sigma(x, dx)$ and $\phi(x, dx)$, it is in $L^p$ if $\phi(x, dx)$ is. One also checks that $\mathcal{N}_k(L^p_k(\mathbb{R}^n, dx)) \subset L^1_k(V, dx)$.

**Lemma 8.1:** Let $\gamma < 1 / \varrho_k^{(\frac{n}{n-1})}$. Then, $\text{Tr}^k((\mathcal{D}_k(z))^\ell)$ depends holomorphically on $z$ if $|z| < \gamma$. 

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Proof of Lemma 8.1: For \( \ell = 1 \), we use (28) to see that

\[
\text{Tr}^* D_k(z) = z \cdot \int_x \text{RD} \left( [N_k(1 - zM^{(k)})^{-1}]_x \sigma_k(x, y, dx, dy) \right) = \sum_{j \geq 0} z^{j+1} \int_x \text{RD} \left( ([N_k(M^{(k)})^j]_x \sigma_k(x, y, dx, dy) \right).
\]

(32)

Since \( |z| < \gamma < 1/\ell_k^{(z_{\infty})} \), (32) is true not only as power series in \( z \), but also as complex numbers (recall that \( \sigma_k(x, y, dx, dy) \in L^r(dx) \) \( \forall r < \frac{n}{n-1} \)), and thus \( \text{Tr}^* D_k(z) = \sum_j a_j z^{j+1} \) (with \( a_j \) as in (32)) is analytic on \( z \). The general case \( \ell \geq 1 \) is handled similarly.

This lemma shows that condition c) is fulfilled when \( |z| < 1/\ell_k^{(z_{\infty})} \).

We will now show that for some \( m \), \( D_k(z)^m \) is a kernel operator whose kernel is in \( L^2(dx \times dy) \). We can therefore apply the theory of Hilbert-Schmidt operators that says that \( D_k(z)^{2m} \) is a trace class operator on \( L^2(V, dx) \), proving condition a). For \( \ell \geq 2m \), the Grothendiek’s trace agrees automatically with our star trace \( \text{Tr}_\# \), that is, it is \( \int_x K(x, x, dx, dx) \) where \( K(x, y, dx, dy) \) is the kernel of \( D_k(z)^\ell \), proving condition b). See [GGK00] or [Sim79] for an account on Hilbert Schmidt theory.

Lemma 8.2: Let \( \gamma > 0 \) be small, \( |z| < 1/\ell_k^{(z_{\infty})} \), and \( i \in \mathbb{N} \). Then, \( D_k(z)^m \) has a kernel in \( L^2(dx \times dy) \) if \( m > \frac{n}{1+i} \).

Proof of Lemma 8.2: We denote also by \( \chi : L^p_k(\mathbb{R}^n, dx) \to L^p_k(V, dx) \) the operator \( \phi \mapsto \chi \cdot \phi \). Thus, \( \chi \) is the identity on \( L^p_k(V, dx) \). Up to replace \( D_k(z) \) with \( D_k(z) \chi \), we can assume that \( D_k(z)^\ell \) has a kernel \( K(x, y, dx, dy) \) that is compactly supported in \( y \). For all \( y \), we have \( \sigma_k(x, y, dx, dy) \) is in \( L^r(dx) \) for all \( r < \frac{n}{n-1} \). Let us fix this a \( r \) such that \( \frac{n}{n-1} - \gamma - \epsilon < r < \frac{n}{n-1} - \gamma \) (with \( \epsilon, \gamma \) small).

The kernel of \( D_k(z) \) is \([N_k(1 - zM^{(k)})^{-1}]_x \sigma_k(x, y, dx, dy) \). Recall that the convolution of an \( L^p \) function with an \( L^q \) function is in \( L^s \) for \( \frac{1}{s} = \frac{1}{p} + \frac{1}{q} - 1 \) (see for instance [Sch66]). If \( K \) is a kernel operator with kernel \( K(x, y, dx, dy) \) whose coefficients functions are, for \( y \) fixed, in \( L^q \) for some \( q > \frac{n}{n-1} - \epsilon \), the kernel of \( SM_k(z) \) is the convolution (on the variable \( x \)) of \( \sigma_k(x, y, dx, dy) \) with \( K(x, y, dx, dy) \). Hence, for all \( y \in V \), the kernel of \( SM_k \) has coefficient functions in \( L^s(V, dx) \) with \( \frac{1}{s} = \frac{1}{r} + \frac{1}{q} - 1 \). In particular, if \( \epsilon \) and \( \gamma \) are small enough, \( s > \frac{n}{n-1} > r \), and thus \( [N_k(1 - zM^{(k)})^{-1}]_x \) is a bounded operator on \( L^s(\mathbb{R}^n, dx) \), (recall that \( |z| < 1/\ell_k^{(z_{\infty})} \)), therefore the kernel of \( D_k(z)K \) is also in \( L^s(V, dx) \), and is compactly supported in \( x \) (because of \( N_k \)).

We can thus proceed by recurrence and define the sequence \( p_k \), such
that

\[ p_i = r, \quad \frac{1}{p_i} = \frac{1}{r} + \frac{1}{p_i-1} \]  

(33)

and for \( y \) fixed, \( D_k(z) \) has a kernel with coefficient functions in \( L^{p_i}(V, dx) \). From (33) it follows that \( \frac{1}{p_i} = \frac{1}{r} + (i-1)(\frac{1}{r} - 1) \). If we impose \( p_i = 2 \), we obtain \( i = \frac{1}{2}(r + (i-1)(\frac{1}{r} - 1)) \). With \( r = \frac{n}{n-1} - \epsilon \), we thus have

\[ i = \frac{2}{n - 1} - \frac{1}{r - 1} \cdot \frac{n - 1}{n - 1} - \epsilon \cdot \frac{n - 1}{r - 1} \cdot \frac{n - 1}{r - 1} \]. Since \( \epsilon \) is arbitrarily small, taking \( i = [n/2] + 1 \) (where \( [\cdot] \) denotes the entire part) yields \( p_i \geq 2 \).

We thus proved that for \( y \) fixed, the kernel of \( D_k(z)^m \) is in \( L^2(V, dx) \) if \( m > \frac{2}{n} \) (in fact, it is also in \( L^2(\mathbb{R}^n, dx) \), because of \( N_k \)). Moreover, as noted at the beginning of the proof, we can suppose that this kernel is compactly supported in \( y \). Thus, the kernel of \( D_k(z)^m \) is in \( L^2(V, dx \times dy) \) if \( m > \frac{2}{n} \).

\[ \square \]

**Corollary 8.3:** Let \( g_k(n) = \sup_{r \leq n/(n-1)} g_k^{(r)} \). Then, when \( |z| < 1/g_k(n) \) and \( n \) odd, \( D_k(z)^{n+1} \) is trace class on \( L^2_{k+1}(V, dx) \). If \( n \) is even, one has to take \( D_k(z)^{n+2} \).

\[ \square \]

Consequently, we have the following theorem:

**Theorem 8.4:** Let \( M \) be a transversal transfer operator (see Definition 2.5). Then, its weighted Lefschetz \( \zeta \)-function

\[ \zeta^\#(z) = \frac{1}{\text{Det}^\#(1 - zM)} \]

is meromorphic in the disk \( \{|z| < \min_k 1/g_k(n)\} \). Moreover, there is a (possibly smaller) open disk where \( \zeta^\#(z) \) is holomorphic.

**Proof of theorem 8.4:**

The assumption about \( \zeta^\#(z) \) being meromorphic follow from the conditions a), b) and c) which are fulfilled thanks to the preceding lemmas. Moreover, since \( \text{Det}^\#(1 + D_k(0)) = 1 \) for each \( k \), by continuity there is an open disk \( D \) centered at 0 such that \( \text{Det}^\#(1 + D_k(z)) \neq 0 \) for \( z \in D \) and thus we have that \( \zeta^\#(z) \) is holomorphic in \( D \).

\[ \square \]

We will now give an example of family of diffeomorphisms for which Theorem 8.4 is needed to prove that \( \zeta^\# \) is holomorphic near zero (or, at least, the result does not trivially follows from the properties of the family \( \{\psi_{\omega_i}\} \)). Let \( M \) be a \( d \)-dimensional real compact and \( C^r \) manifold \((r \geq 3)\). We denote by \( D^r(M) \) the space of \( C^r \) diffeomorphisms of \( M \). If
If \( f \in D^r(M) \), we denote by \( P_f(n) \) the number of its periodic points of period \( n \) (if this number is finite). Kaloshin showed [Kal00] that in the \( C^r \) topology, there is an open set \( U \) such that for any sequence \( a = (a_\ell)_{\ell \in \mathbb{N}} \) of real numbers, the set \( R_a \) of transversal diffeomorphisms \( f \) with
\[
\limsup_{\ell \to \infty} P_f(n)/a_n = \infty
\]
is residual in \( U \), meaning that it contains a countable intersection of dense open sets (for the \( C^r \) topology).

So, let \( f \) be such a diffeomorphism. One can define the Lefschetz sign of any fixed point of \( f \) using charts, and this does not depend on the choice of our charts. Since \( M \) is a \( C^r \) manifold and \( r \geq 3 \), it is possible to \( C^r \)-embed \( M \) into \( \mathbb{R}^n \), with \( n = 2d + 1 \), and to find a tubular neighbourhood \( V \) of \( M \) in \( \mathbb{R}^n \) (see for instance [Hir76]). A tubular neighbourhood is just a local trivialisation of the normal bundle of \( M \) in \( \mathbb{R}^n \). We define \( \psi : V \to V \) to be \( f \) on \( M \), and to contract linearly (by a factor \( \theta < 1 \)) along the direction normal to \( M \). Up to making a change of coordinates, the derivative of \( \psi \) at \( x \) is just
\[
d_x \psi = \begin{pmatrix}
d_x f & 0 \\
0 & \theta \cdot id_{\mathbb{R}^n-d}
\end{pmatrix}.
\]

Hence, \( \psi \) is a \( C^r \) local diffeomorphism, and is transversal if \( f \) is. The periodic points of \( f \) and \( \psi \) are the same, and moreover their Lefschetz sign does not change: \( L(x, \psi^\ell) = L(x, f^\ell) \) for all \( \ell \geq 1 \). We take now any \( C^3 \) compactly supported weight \( g : V \to \mathbb{C} \). We construct \( \mathcal{M} \) with \( \psi \) and \( g \), and Theorem 8.4 tells us then that the weighted Lefschetz \( \zeta \)-function for \( \psi \) (or for \( f \)) \( \zeta^\#(z) \) is holomorphic near zero. However, at first sight, the series \( \sum \ell \zeta^\#(\mathcal{M})^\ell \) is not trivially convergent near zero, because of (34) (take \( a_\ell > \ell^4 \), for instance). In addition, the series \( \text{Det}^1(1 - z\mathcal{M}^\ell) \) have probably no radius of convergence.

The same arguments also apply to finite transversal families of diffeomorphisms with the property (34) as well, for which the convergence of the weighted Lefschetz \( \zeta \)-function is even less obvious.

References


