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Abstract

We study the space \( N^m_d \) of clouds in \( \mathbb{R}^d \) (ordered sets of \( m \) points modulo the action of the group of affine isometries). We show that \( N^m_d \) is a smooth space, stratified over a certain hyperplane arrangement in \( \mathbb{R}^m \). We give an algorithm to list all the chambers and other strata (this is independent of \( d \)). With the help of a computer, we obtain the list of all the chambers for \( m \leq 9 \) and all the strata when \( m \leq 8 \). As the strata are the product of a polygon spaces with a disk, this gives a classification of \( m \)-gon spaces for \( m \leq 9 \). When \( d = 2, 3 \), \( m = 5, 6, 7 \) and modulo reordering, we show that the chambers (and so the different generic polygon spaces) are distinguished by the ring structure of their mod 2-cohomology.

1 Introduction

Let \( E \) be an oriented finite-dimensional Euclidean space. Let \( N^m_E \) be the space of ordered sets of \( m \) points in \( E \), modulo the group of rigid motions of \( E \); more precisely,

\[
N^m_E := G(E)\backslash E^m,
\]

where the Lie group \( G(E) \) is the semi-direct product of the translation group of \( E \) by \( SO(E) \), the group of linear orientation-preserving isometries of \( E \), and the group \( G(E) \) acts diagonally on \( E^m \). We shall occasionally consider the space \( \bar{N}^m_E = \bar{G}(E)\backslash E^m \), where \( \bar{G} \) is the group of all affine isometries of \( E^m \). Observe that \( \bar{N}^m_E \) is a subspace of \( N^m_E \) when \( E \) is a proper subspace of \( E' \). An element of \( N^m_E \) will be called a cloud of \( m \) points in \( E \). (The letter \( N \) stands for “nuage”, meaning “cloud” in French.) We abbreviate \( N^m_E \) to \( N^m_d \). Observe that \( N^m_E \) is canonically homeomorphic to \( \bar{N}^m_d \), when \( d = \dim E \).

The space \( N^m_d \) plays a natural role in celestial mechanics, at least for \( d = 2 \) or 3 (see, for instance, [AC]). Moreover, its importance was recognized especially in statistical shape theory, a subject which has developed rapidly during the last two decades (see [Sm] and [KBCL] for a history). There, the space \( N^m_d \) is called the size-and-shape space and is denoted by \( S\Sigma^m_d \) [KBCL, §11.2]. This
terminology and notation emphasizes that $N_d^m$ is the cone, with vertex $N_0^m$, over the shape space $\Sigma_d^m$, defined as the quotient of $N_d^m - N_d^0$ by the homotheties. A great amount is known about the homotopy type of shape space $\Sigma$. For instance, in [KBCL], Kendall, Barden, Carne and Li show that $\Sigma^m$ admits cellular decompositions leading to a complete computation of its homology groups.

In this paper, we present an alternative decomposition of the space $N_d^m$. It is based on polygon spaces, a subject which has also encountered a rich development during the last decade, in connection with Hamiltonian geometry. This approach is completely different from that of statistical shape theory and this paper is essentially self-contained.

First of all, the point set topology of $N_d^m$ is well behaved and $N_d^m$ is endowed with a smooth structure. More precisely, the translations act freely and properly on $E^m$ with quotient diffeomorphic to the vector subspace $K(E^m) = \{ (z_1, \ldots, z_m) \in E^m | \sum z_i = 0 \}$. Being therefore the quotient of $K(E^m)$ by the action of the compact group $SO(E)$, the space of clouds $N_d^m$ is locally compact (in particular Hausdorff). Classical invariant theory provides a proper topological embedding $\varphi$ of $N_d^m$ into an Euclidean space $\mathbb{R}^N$ (see 2.2). This embedding makes $N_d^m$ a smooth space, i.e. a topological space together with an algebra $C^\infty(X)$ of smooth functions (with real values): those functions which are locally the composition of $\varphi$ with a $C^\infty$-function on $\mathbb{R}^N$. One can prove that $f \in C^\infty(N_d^m)$ if and only if $f \circ \pi$ is smooth on $E^m$, where $\pi : E^m \to N_d^m$ is the natural projection (Proposition 2.3). Any subspace of of a smooth space naturally inherits a smooth structure and, together with smooth maps (see 2.1), smooth spaces form a category whose equivalences are called diffeomorphisms. Finally, we mention that the space $N_3^m$ has the special feature that the smooth maps admit a Poisson bracket (see 2.6).

Our main tool for stratifying the space $N_d^m$ is the map $\ell : N_d^m \to \mathbb{R}^m$ defined on $\rho = (\rho_1, \ldots, \rho_m) \in E^m$ by

$$ \ell(\rho) := (|\rho_1 - b(\rho)|, \ldots, |\rho_m - b(\rho)|), $$

where $b(\rho) = \frac{1}{m} \sum \rho_i$ is the barycentre of $\rho$. This map $\ell$ is continuous and is smooth on $\ell^{-1}((\mathbb{R}_{>0})^m)$, the open subset of points $\rho \in N_d^m$ such that no $\rho_i$ is equal to $b(\rho)$.

We shall prove in Section 3 that the critical points of $\ell$ are the one-dimensional clouds $\bar{N}_1^m \subset N_d^m$. The space of critical values is then an arrangement of hyperplanes in $\mathbb{R}^m$ that we shall describe now. Let $m := \{1, 2, \ldots, m\}$ and denote by $P(m)$ the family of subsets of $m$. For $I \in P(m)$, let $H_I$ be the hyperplane of $\mathbb{R}^m$ defined by

$$ H_I := \{ (a_1, \ldots, a_m) \in \mathbb{R}^m | \sum_{i \in I} a_i = \sum_{i \notin I} a_i \}. $$

We call these hyperplanes walls. They determine a stratification $\mathcal{H}(\mathbb{R}^m)$ of $\mathbb{R}^m$,
i.e. a filtration

\[ \{0\} = \mathcal{H}^{(0)}(\mathbb{R}^m) \subset \mathcal{H}^{(1)}(\mathbb{R}^m) \subset \cdots \subset \mathcal{H}^{(m)}(\mathbb{R}^m) = \mathbb{R}^m, \]

with \( \mathcal{H}^{(k)}(\mathbb{R}^m) \) being the subset of those \( a \in \mathbb{R}^m \) which belong to at least \( m - k \) distinct walls \( \mathcal{H}_k \). A stratum of dimension \( k \) is a connected component of \( \mathcal{H}^{(k)}(\mathbb{R}^m) - \mathcal{H}^{(k-1)}(\mathbb{R}^m) \). Note that a stratum of dimension \( k \geq 1 \) is an open convex cone in a \( k \)-plane of \( \mathbb{R}^m \). Strata of dimension \( m \) are called chambers. We denote by \( \text{Str}(a) \) the stratum of \( a \in \mathbb{R}^m \). If \( \text{Str}(a) \) is a chamber, the \( m \)-tuple \( a \) is called generic and we will often denote \( \text{Str}(a) \) by \( \text{Ch}(a) \).

The stratification \( \mathcal{H}(\mathbb{R}^m) \) induces a stratification of any subset \( U \) of \( \mathbb{R}^m \), in particular of \( U = (\mathbb{R}_{>0})^m \). Write \( \text{Str}(U) \) for the set of all the strata of \( U \). Denote by \( \mathcal{N}_E^m(a) \) the preimage \( \ell^{-1}(\{a\}) \) of \( a \in \mathbb{R}^m \). Strengthening results of [HK2], we shall prove the following theorem in Section \( \S \).

**Theorem A** Let \( a \in (\mathbb{R}_{>0})^m \). Then there is a diffeomorphism from \( \ell^{-1}(\text{Str}(a)) \) onto \( \mathcal{N}_E^m(a) \times \text{Str}(a) \) intertwining the map \( \ell \) with the projection to \( \text{Str}(a) \).

In the proof of Theorem A, we actually construct, when \( \text{Str}(a) = \text{Str}(b) \), a diffeomorphism \( \psi_{ba} : \mathcal{N}_E^m(a) \overset{\approx}{\rightarrow} \mathcal{N}_E^m(b) \). One has \( \psi_{ab} = \psi_{ba}^{-1} \) and \( \psi_{aa} = \text{id} \). For \( \alpha \in \text{Str}((\mathbb{R}_{>0})^m) \), we will sometimes use the notation \( \mathcal{N}_E^m(\alpha) \) for any of the spaces \( \mathcal{N}_E^m(a) \) with \( a \in \alpha \). This is in fact ambiguous because, in general, \( \psi_{ca} \neq \psi_{cb} \circ \psi_{ba} \), so one cannot use the maps \( \psi_{ba} \) to define an equivalence relation on \( \ell^{-1}(\text{Str}(a)) \) giving the points of \( \mathcal{N}_E^m(\alpha) \). However, \( \psi_{ca} \) is isotopic to \( \psi_{cb} \circ \psi_{ba} \), and so the homotopy invariants of \( \mathcal{N}_E^m(\alpha) \), for instance the elements of its cohomology ring, are well defined.

Theorem A may provide good local models for describing the evolution of a cloud. This is especially likely when \( \dim E = 2, 3 \), where, for generic \( a \), the spaces \( \mathcal{N}_E^m(a) \) and thus \( \ell^{-1}(\text{Str}(a)) \) are smooth manifolds (see below).

Theorem A shows that \( \mathcal{N}_E^m \) is obtained by gluing together pieces of the form \( \mathcal{N}_E^m(\alpha) \times \alpha \) for various \( \alpha \in \text{Str}((\mathbb{R}_{>0})^m) \). From this point of view, the following questions are natural.

1. Describe the set of all strata of \( (\mathbb{R}_{>0})^m \), in particular the set of chambers. This combinatorial problem does not depend on \( E \).
2. Describe \( \mathcal{N}_E^m(\alpha) \) for all \( \alpha \in \text{Str}((\mathbb{R}_{>0})^m) \).
3. Describe how a stratum of \( \mathcal{H}(\mathcal{N}_E^m) \) is attached to its bordering strata of lower dimension.

The main issue of this paper is to answer Question 1 and, partly Question 2 above. It is convenient to take advantage of the right action of the symmetric group \( \text{Sym}_m \) on \( \mathcal{N}_E^m \) and on \( \mathbb{R}^m \) by permutation of the coordinates (to deal directly with the smooth space \( \mathcal{N}_E^m / \text{Sym}_m \) and get a corresponding statement of
Theorem A, see (2.5). This action permutes the strata of \( \mathcal{H}(\mathbb{R}_{>0}^m) \), and \( \mathcal{N}_E^m(\alpha) \) is diffeomorphic to \( \mathcal{N}_E^m(\alpha^\sigma) \) for \( \sigma \in \text{Sym}_m \). The map \( \ell \) is equivariant and each \( a \in (\mathbb{R}_{>0})^m \) has a unique representative in \( R^m \), where

\[
\mathbb{R}^m := \{(a_1, \ldots, a_m) \in \mathbb{R}^m \mid 0 < a_1 \leq \cdots \leq a_m\}.
\]

Therefore, the set \( \text{Str}((\mathbb{R}_{>0})^m)/\text{Sym}_m \) is in bijection with the set \( \text{Str}(R^m) \).

In Sections 4 and 5, we show how to obtain a complete list of the elements of \( \text{Ch}(R^m) \) and \( \text{Str}(R^m) \). For this, we first show that the set of inequalities defining a chamber \( \alpha \) of \( R^m \) can be recovered from some very concentrated information which we call the \textit{genetic code} of \( \alpha \). Abstracting some properties of these genetic codes gives rise to the combinatorial notion of a \textit{virtual genetic code}. We design an algorithm to find all virtual genetic codes, with the help of a computer (the program in C++ is available at [HR Web]). Deciding which virtual genetic code is the genetic code of a chamber (realizability) is essentially done using the simplex algorithm of linear programming. We thus obtain the list of all the chambers of \( R^m \), with the restriction \( m \leq 9 \) due to the computer’s limited capacities. The set \( \text{Str}(R^{m-1}) \) is determined using an injection of \( \text{Str}(R^{m-1}) \) into \( \text{Ch}(R^m) \) (see §5). The number of elements of these sets is

<table>
<thead>
<tr>
<th>( m )</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td>(</td>
<td>\text{Ch}(R^m)</td>
<td>)</td>
<td>2</td>
<td>3</td>
<td>7</td>
<td>21</td>
<td>135</td>
</tr>
<tr>
<td>(</td>
<td>\text{Str}(R^m)</td>
<td>)</td>
<td>3</td>
<td>7</td>
<td>21</td>
<td>117</td>
<td>1506</td>
</tr>
</tbody>
</table>

(1.1)

It turns out that, for \( m \leq 8 \), all virtual genetic codes are realizable, but not for \( m = 9 \): only 175428 out of 319124 are realizable. The non-realizable ones might well be of interest (see Remark 7.13).

Our algorithms produce, in each chamber \( \alpha \), a distinguished element \( a_{\text{min}}(\alpha) \in R^m \) with integral coordinates and with \( \sum a_i \) minimal. Several theoretical questions about these elements \( a_{\text{min}}(\alpha) \) remain open (see §4).

To describe the spaces \( \mathcal{N}_E^m(a) \) (Question 2 above), we note that

\[
\mathcal{N}_E^m(a) = SO(E) \backslash \{\rho \in E^m \mid \sum_{i=1}^m \rho_i = 0 \text{ and } |\rho_i| = a_i\}.
\]

The condition \( \sum_{i=1}^m \rho_i = 0 \) suggests the picture of a closed \( m \)-step piecewise-linear path in \( E \), whose \( i \)th step has length \( a_i \). Therefore, the space \( \mathcal{N}_E^m(a) \) is often called the \( m \)-gon space (in \( E \)) of type \( a \) (we could call it the space of clouds “calibrated at \( a \)”). These polygon spaces have been studied in different notations, especially for \( \dim E = 2 \) and 3 where, for generic \( a \), they are manifolds: see, for instance, \([K]\), \([KM]\), \([HK1]\), \([HK2]\). For \( \dim E > 3 \) or for a non-generic, see \([Ka1]\) and \([Ka2]\).

The classification of the polygon spaces \( \mathcal{N}_E^m(a) \), for generic \( a \), was previously known when \( \dim E = 2,3 \) and \( m \leq 5 \) (see, for instance \([HK1\text{, §6}]\)). The genetic
codes introduced in this paper extend this classification up to \( m = 9 \). In §\( \textcircled{3} \), we give handle-decomposition information about the 6-gon spaces \( N^6_m \) for the 21 chambers of \( \mathbb{R}^6 \). This type of method could be applied to any space \( N^m_E(a) \) for generic \( a \). In addition to these geometric descriptions, algorithms were previously found which compute cohomological invariants of the spaces \( N^m_3(a) \), for example their Poincaré polynomial ([Kl, Th. 2.2.4], [HK2, Cor. 4.3]). This enables us, in Section 6, to compute the Betti numbers of the spaces \( N^m_3(a) \) for \( m \leq 9 \). Moreover, presentations of the cohomology ring of \( N^m_3(a) \) for any coefficients were given in [HK2, Th. 6.4]. This permits us to compute some invariants of the ring \( H^*(N^m_3(a); \mathbb{F}_2) \) and prove in 7.10 the following result:

**Proposition B** For \( 5 \leq m \leq 7 \), the spaces \( N^m_3(a) \) (or \( N^m_2(a) \)), for distinct chambers \( a \) of \( \mathbb{R}^m \), have non-isomorphic mod 2-cohomology rings.

Here, the ring structure of \( H^*(N^m_3; \mathbb{Z}_2) \) is important: the Betti numbers alone do not distinguish the spaces. Interestingly enough, the virtual genetic codes which are not realizable also give rise to non-trivial graded rings. We do not know if these rings are cohomology rings of a space, or of a manifold (see Remark 7.13).

The paper is organized as follows. In Section 2, we set the background of the smooth structure on \( N^m_E \) which is used in Section 3 to prove Theorem A. In Section 4, we introduce the genetic code of a chamber and show how to obtain the list of all chambers of \( \mathbb{R}^m \) for \( m \leq 9 \). In Section 5 we study the injection \( \text{Str}(\mathbb{R}^{m-1}) \) into \( \text{Str}(\mathbb{R}^m) \) and show how to obtain the list of all strata of \( \mathbb{R}^m \) for \( m \leq 8 \). Section 6 contains our information on the spaces \( N^m_3(a) \) and \( N^m_2(a) \) for generic \( a \). Section 7 is devoted the cohomology invariants of the polygon spaces. Finally, the results of Sections 6 and 7 are applied in Section 8 to the case of hexagon spaces.

**Acknowledgments:** Both authors thank the Swiss National Fund for Scientific Research for its support. We are indebted to R. Bacher for suggesting the cohomology invariant \( s(\alpha) \) of Proposition 7.8.

2 The smooth structure on \( N^m_E \)

2.1 Smooth spaces and maps. For \( X \) a topological space, denote by \( C^0(X) \) the \( \mathbb{R} \)-algebra of continuous functions on \( X \) with real values. If \( h : X \to Y \) is a continuous map, denote by \( h^* : C^0(Y) \to C^0(X) \) the map \( h^*(f) = f \circ h \). Let \( X \) be a subspace of \( \mathbb{R}^N \). A map \( f : X \to \mathbb{R} \) is smooth if, for each \( x \in X \) there exists an open set \( U \) of \( \mathbb{R}^N \) containing \( x \) and a \( C^\infty \) map \( F : U \to \mathbb{R} \) which coincides with \( f \) throughout \( U \cap X \) (compare [Mi, §1]). The smooth maps on \( X \) constitute a subalgebra \( C^\infty(X) \) of \( C^0(X) \).

More generally, if \( \varphi : X \to \mathbb{R}^N \) is a topological embedding of a space \( X \) into
The kernel is the diagonal $\Delta$ in $\mathbb{E}_\mathbb{R}^n$ space if $h: \mathbb{E}_\mathbb{R}^n \rightarrow \mathbb{E}_\mathbb{R}^n$ is smooth structure on $\mathbb{E}_\mathbb{R}^n$ and $h$ and $h^{-1}$ are smooth. A smooth embedding is thus a diffeomorphism onto its image.

2.2 The smooth structure on $\mathcal{N}_E^m$. Let $\kappa: E^m \rightarrow E^m$ be the linear projection

$$\kappa(z_1, \ldots, z_m) = (z_1 - b(z), \ldots, z_m - b(z)),$$

where $b(z) = \frac{1}{m} \sum z_i$ is the barycentre of $z$. The image of $\kappa$ is $\mathcal{K}(E^m)$ and its kernel is the diagonal $\Delta$ in $E^m$.

The normal subgroup $\mathbb{E}$ in $G(E)$ of translations acts freely and properly on $E^m$ and the quotient space $E/E^m$ is the same as the quotient vector space $E/\Delta$. The projection $\kappa$ descends to a linear isomorphism $\bar{\kappa}: E/E^m \rightarrow \mathcal{K}(E^m)$. The space $\mathcal{N}_E^m$ is now the quotient of $\mathcal{K}(E^m)$ by the action of the compact group $SO(E)$. Therefore $\mathcal{N}_E^m$ is a locally compact Hausdorff space.

Consider the $m^2$ polynomial functions on $E^m$ given by $z \mapsto \langle \kappa(z), \kappa(z) \rangle$, where $\langle \cdot, \cdot \rangle$ denotes the scalar product on $E$. Choose an orientation on $E$. The determinants $|\kappa(z_{i_1}), \ldots, \kappa(z_{i_k})|$ with $i_1 < \cdots < i_k$ ($k = \dim E$) are another family of $\binom{m}{k}$ polynomial functions on $E^m$. All these functions are $G(E)$-invariant and produce a continuous map $\varphi: \mathcal{N}_E^m \rightarrow \mathbb{R}^N$ with $N = m^2 + \binom{m}{k}$. It is an exercise to prove that $\varphi$ is injective and proper. As $\mathcal{N}_E^m$ is locally compact, the map $\varphi$ is a topological embedding of $\mathcal{N}_E^m$ into $\mathbb{R}^N$ and its image is closed (for a family of inequalities defining $\varphi(\mathcal{N}_E^m)$ as a semi-algebraic set, see [FS]).

The embedding $\varphi$ endows $\mathcal{N}_E^m$ with a smooth structure. The following proposition identifies $\mathcal{C}^\infty(\mathcal{N}_E^m)$ with the smooth functions on $E^m$ which are $G(E)$-invariant.

**Proposition 2.3** Let $f \in C^0(\mathcal{N}_E^m)$. The following are equivalent:

(A) $f \in C^\infty(\mathcal{N}_E^m)$.

(B) There is a global $C^\infty$-function $F: \mathbb{R}^N \rightarrow \mathbb{R}$ such that $f = F \circ \varphi$.

(C) The map $f \circ \pi: E^m \rightarrow \mathbb{R}$ is $C^\infty$, where $\pi: E^m \rightarrow \mathcal{N}_E^m$ denotes the natural projection.

**Proof:** It is clear that (B) implies (A). Conversely, let $f \in C^\infty(\mathcal{N}_E^m)$. For every $\rho \in \mathcal{N}_E^m$, one has an open set $U_{\rho}$ of $\mathbb{R}^N$ containing $\varphi(\rho)$ and a smooth function $F_{\rho}: U_{\rho} \rightarrow \mathbb{R}$ with $f_{\rho} \circ \varphi = f$ on $\varphi^{-1}(U_{\rho})$. Call $U_{\infty} = \mathbb{R}^N - \varphi(\mathcal{N}_E^m)$ and $F_{\infty}: U_{\infty} \rightarrow \mathbb{R}$.
the constant map to 0. As \( \varphi(N^m_E) \) is closed, the family \( U := \{ U_\rho \}_{\rho \in N^m_E \cup \{ \infty \}} \) is an open covering of \( \mathbb{R}^N \). Let \( \mu_\rho : \mathbb{R}^N \to \mathbb{R} \) be a smooth partition of the unity subordinated to \( U \). Then \( F(x) = \sum_{\rho \in N^m_E \cup \{ \infty \}} F_\rho(x) \) satisfies (B).

Statement (B) is obviously stronger than (C) (which, incidentally, implies that \( \pi \) is a smooth map). For the converse, one uses that the components of \( \varphi \) constitute a generating set for the algebra of \( SO(E) \)-invariant polynomial functions on \( \mathcal{K}(E^m) \) \([1], \S \ I I .9 \). Then, any \( SO(E) \)-invariant smooth function on \( \mathcal{K}(E^m) \) is of the form \( F \circ \varphi \) by the Theorem of G. Schwarz \([S c h] \).

2.4 The smooth structure on the space \( \bar{N}^m_E = G(E) \backslash E^m \) is obtained as in 2.2. The embedding \( \varphi : \bar{N}^m_E \to \mathbb{R}^{m^2} \) is given by the polynomial function \( \rho \mapsto \langle \rho_i \rho_j \rangle \).

Proposition 2.3 holds true.

2.5 Clouds of unordered points. On \( E^m = \{ \rho : m \to E \} \), the symmetric group \( \text{Sym}_m \) acts on the right, by pre-composition (or by permuting the coordinates). This action descends on \( N^m_E \).

As in 2.2, the space \( N^m_E / \text{Sym}_m = G(E) \backslash E^m / \text{Sym}_m \) has a smooth structure, via a topological embedding \( \varphi : N^m_E / \text{Sym}_m \to \mathbb{R}^N \) given by a generating set of the algebra of polynomial functions on \( \mathcal{K}(E^m) \) which are \( SO(E) \times \text{Sym}_m \)-invariant. Proposition 2.3 holds true accordingly.

The space \( \mathbb{R}^m / \text{Sym}_m \) also has a smooth structure via the smooth embedding \( \varphi : \mathbb{R}^m / \text{Sym}_m \to \mathbb{R}^m \) given by the \( m \) elementary symmetric polynomials. The map \( \ell \) descends to a continuous map \( \bar{\ell} : N^m_E / \text{Sym}_m \to \mathbb{R}^m / \text{Sym}_m \) which is smooth away from \( \ell^{-1}(\{0\}) \). The composition \( \psi : \mathbb{R}^m \subset (\mathbb{R}_{>0})^m \to (\mathbb{R}_{>0})^m / \text{Sym}_m \) is a smooth homeomorphism. The stratification \( \mathcal{H}(\mathbb{R}^m) \) can be transported via \( \psi \) to \( (\mathbb{R}_{>0})^m / \text{Sym}_m \), giving rise to a stratification \( \mathcal{H}((\mathbb{R}_{>0})^m / \text{Sym}_m) \). The map \( \bar{\ell} \) is stratified and Theorem A holds true for \( \bar{\ell} \). Indeed, the diffeomorphisms constructed in the proof of Theorem A given in \S \ 3 are natural with respect to the action of \( \text{Sym}_m \).

We must be careful that the smooth homeomorphism \( \psi : \mathbb{R}^m \to (\mathbb{R}_{>0})^m / \text{Sym}_m \) is not a diffeomorphism: the projection onto the first coordinate is smooth on \( \mathbb{R}^m \) but not on \( (\mathbb{R}_{>0})^m / \text{Sym}_m \).

2.6 Poisson structures on \( \mathcal{N}^m_3 \). Recall that a Poisson structure on a smooth manifold \( X \) is a Lie bracket \( \{ , \} \) on \( C^\infty(X) \) satisfying the Leibnitz rule: \( \{ fg, h \} = f \{ g, h \} + \{ f, h \} g \). See \([MR] \) for properties of Poisson manifolds. The same definition makes sense on a smooth space.

The Euclidean space \( E = \mathbb{R}^3 \) has a standard smooth structure by

\[ \{ f, g \}(x) := \langle \nabla f \times \nabla g, x \rangle. \]
We endow the product space \( E^m \) with the product Poisson structure. If \( f, g : E^m \to \mathbb{R} \) are \( SO(E) \)-invariant, so is the bracket \( \{ f, g \} \). Thus, the quotient space \( SO(E) \backslash E^m \) inherits a Poisson structure.

Using a canonical identification of \( \mathbb{R}^3 \) with \( so(3)^* \), the above Poisson bracket on \( \mathbb{R}^3 \) corresponds, up to sign, to the classical Poisson structure on \( so(3)^* \) [MR, p. 287]. The map \( \mu : z \mapsto \sum_i \zeta_i \), from \( \mathbb{R}^3 = so(3)^* \) to \( E \) is the moment map for the diagonal action of \( SO(E) \). Let \( \xi : \mathbb{R}^3 \to \mathbb{R} \) be a linear map. By the Theorem of Noether [MR, Th. 11.4.1], if \( f : E^m \to \mathbb{R} \) is a smooth \( SO(3) \)-invariant map, then \( \{ f, \xi \} = 0 \). This proves that \( \{ f, g \} = 0 \) for all \( g \in C^\infty(E^m) \) such that \( g_{|K(E^m)} = 0 \). Thus, the space \( N_3^m \) inherits a Poisson structure so that the inclusion \( N_3^m \subset SO(E) \backslash E^m \) is a Poisson map.

When \( a \in (\mathbb{R}_{>0})^m \) is generic, the spaces \( N_3^m(a) \) are manifolds and are the symplectic leaves of \( \ell^{-1}(\text{Str}(a)) \). This accounts for the symplectic structures on the polygon spaces in \( \mathbb{R}^3 \) studied in [Kl], [KM] and [HK1 and 2].

### 3 Proof of Theorem A

Throughout this section, the Euclidean space \( E \) and the number of points are constant. Denote by \( \hat{K} \) the subset of \( m \)-tuples \( \rho = (\rho_1, \ldots, \rho_m) \in E^m \) such that \( \rho_i \neq 0 \) and \( \sum_i \rho_i = 0 \). Define the map \( \hat{\ell} : \hat{K} \to \mathbb{R}^m \) by \( \hat{\ell}(\rho) := (|\rho_1|, \ldots, |\rho_m|) \).

An element \( \rho = (\rho_1, \ldots, \rho_m) \in E^m \) is called 1-dimensional if the vector subspace of \( E \) spanned by \( \rho_1, \ldots, \rho_m \) is of dimension 1 (therefore, \( \rho \) represents an element of \( \hat{N}_1^m \subset \hat{N}_E^m \)). These are precisely the singularities of the map \( \hat{\ell} : \hat{K} \to \mathbb{R}^m \). Indeed:

**Lemma 3.1** Suppose that \( \rho \in \hat{K} \) is not 1-dimensional. Then \( T_{\rho} \hat{\ell} \) is surjective.

**Proof:** Let \( (a_1, \ldots, a_m) = \hat{\ell}(\rho) \). As \( \rho \) is not 1-dimensional, there are two vectors among \( \rho_2, \ldots, \rho_m \) which are linearly independent. The orthogonal complements to these two vectors then span \( E \). Thus, there are curves \( \rho_i(t) \) for \( i = 2, \ldots, m \) such that \( |\rho_i(t)| = a_i \) and

\[
\sum_{i=2}^m \rho_i(t) = -(1 + \frac{t}{a_1})\rho_1.
\]

Therefore the map

\[
t \mapsto \left( (1 + \frac{t}{a_1})\rho_1, \rho_2(t), \ldots, \rho_m(t) \right)
\]

represents a tangent vector \( v \in T_{\rho} \hat{K} \) with \( T_{\rho} \hat{\ell}(v) = (1, 0, \ldots, 0) \). The same can be done for the other basis vectors of \( \mathbb{R}^m \) proving that \( T_{\rho} \hat{\ell} \) is surjective. □

Let \( \rho \in \hat{K} \) be 1-dimensional. One thus has \( \rho_i = \lambda_i \rho_m \) with \( \lambda_i \in \mathbb{R} \setminus \{0\} \). Let \( I(\rho) \in \mathcal{P}(m) \) defined by \( i \in I(\rho) \) if and only if \( \lambda_i < 0 \). It is obvious that \( \hat{\ell}(\rho) \) belongs to the wall \( \mathcal{H}_{I(\rho)} \).
Lemma 3.2 Suppose that $\rho \in \hat{K}$ is 1-dimensional. Then the image of $T_{\rho} \tilde{\ell}$ is $\mathcal{H}_{I(\rho)}$.

PROOF: Let $I = I(\rho)$. One has $\sum_{i \in I} \rho_i = -\sum_{i \notin I} \rho_i$. The components $\rho_i(t)$ of a curve $\rho(t) \in E^m$ with $\rho(0) = \rho$ are of the form

$$\rho_i(t) = (1 + \frac{c_i(t)}{a_i})\rho_i + w_i(t),$$

with $c_i(0) = 0$ and $w_i(0) = 0$, where $c_i(t) \in \mathbb{R}$, and $w_i(t)$ is in the orthogonal complement of $\rho_i$. The curve $\rho(t)$ is in $\hat{K}$ if and only if $\sum_{i=1}^m w_i(t) = 0$ and

$$\sum_{i \in I}(1 + \frac{c_i(t)}{a_i})\rho_i = -\sum_{i \notin I}(1 + \frac{c_i(t)}{a_i})\rho_i.$$  \hfill (3.2)

Let $c(t) = (c_1(t), \ldots, c_m(t))$. The vector $\frac{\rho_i}{a_i}$ is constant when $i \in I$ and $\frac{\rho_i}{a_i} = -\frac{\rho_j}{a_j}$ if $i \in I$ and $j \notin I$. Therefore, Equation (3.2) is equivalent to $c(t) \in \mathcal{H}_I$. Finally, a direct computation shows that the tangent vector $v \in T_{\rho} \hat{K}$ represented by $\rho(t)$ satisfies $T_{\rho} \tilde{\ell}(v) = \dot{c}(0)$. This proves the lemma. \[ \square \]

PROOF OF THEOREM A: Let $a, b \in (\mathbb{R}_{>0})^m$ be in the same stratum $\alpha$. Let $X \subset \alpha$ be the segment joining $a$ to $b$. For $\delta > 0$, write $U_\delta := \{x \in \mathbb{R}^m \mid d(x, X) < \delta\}$, where $d(x, X)$ is the distance from $x$ to the segment $X$. We choose $\delta$ small enough so that the walls meeting $U_\delta$, if any, are only those containing $\alpha$. Let $\tilde{U}_\delta := \tilde{\ell}^{-1}(U_\delta) \subset \hat{K}$.

Let $V^b$ be a vector field on $U_\delta$ of the form $V^b_x = \lambda(x)(b - a)$, where $\lambda : U_\delta \rightarrow [0, 1]$ is a smooth function equal to 1 on $U_{\delta/3}$ and to 0 out of $U_{2\delta/3}$.

Put on $\hat{K}$ and $\mathbb{R}^m$ the standard Riemannian metrics. For $\rho \in \hat{K}$, define the vector subspace $\Delta_\rho$ of $T_{\rho} \hat{K}$ by $\Delta_\rho := (T_{\rho} \tilde{\ell})^*(T_{\ell(\rho)}(\mathbb{R}^m))$, where $(T_{\rho} \tilde{\ell})^*$ is the adjoint of $T_{\rho} \tilde{\ell}$. The vector spaces $\delta_\rho$ form a smooth distribution (of non-constant rank) on $\hat{K}$.

The tangent map $T_{\rho} \tilde{\ell}$ sends $\Delta_\rho$ isomorphically onto the image of $T_{\rho} \tilde{\ell}$. Since $X$ lies in $\alpha$, Lemmas 3.1 and 3.2 show that $V^b_{\tilde{\ell}(z)}$ is in the image of $T_{\rho} \tilde{\ell}$ for all $z \in \tilde{U}_\delta$. Therefore, there exists a unique vector field $W^b$ on $\tilde{U}_\delta$ such that, for each $z \in \tilde{U}_\delta$, one has $W^b_z \in \Delta_z$ and $T_{\rho} \tilde{\ell}(W^b_z) = V^b_{\tilde{\ell}(z)}$. The map $\tilde{\ell}$ being proper, the vector field $W^b$ has compact support, so its flow $\Phi_t$ is defined for all times $t$. Therefore, $z \mapsto \Phi_1(z)$ gives a diffeomorphism $\psi_{ba} : \tilde{\ell}^{-1}(b) \xrightarrow{\sim} \tilde{\ell}^{-1}(a)$.

As its notation suggests, the map $\psi_{ba}$ depends only on $b$ and not on the choices involved in the definition of $V^b$ (\delta and \lambda). One can thus define $\psi : \tilde{\ell}^{-1}(a) \rightarrow N^m_E(a) \times \alpha$ by $\psi(z) := (\tilde{\ell}(z), \psi_{\tilde{\ell}(z)a}(z))$. The vector fields $V^b$ and $W^b$ depending smoothly on $b \in \alpha$, the map is smooth as well as its inverse $(x, u) \mapsto \psi_{ax}(u)$. Therefore, $\psi$ is a diffeomorphism. As the Riemannian metric on $\hat{K}$ and the map $\tilde{\ell}$ are invariant with respect to the action of $SO(E) \times \text{Sym}_m$ the map $\psi$ descends to a diffeomorphism $\psi : \tilde{\ell}^{-1}(a) \xrightarrow{\sim} N^m_E(a) \times \alpha$, which proves Theorem A. Actually, each diffeomorphism $\psi_{ba}$ descends to a diffeomorphism $\psi_{ba} : N^m_E(b) \xrightarrow{\sim} N^m_E(a)$.
Remark 3.3 Theorem A is also true for the spaces $\mathcal{N}_E^m$.

4 The genetic code of a chamber

Let $a \in (\mathbb{R}_{\geq 0})^m$. Following [HK2, §2], we define $S(a) \subset \mathcal{P}(m)$ by

$$I \in S(a) \iff \sum_{i \in I} a_i \leq \sum_{i \notin I} a_i. \quad (4.1)$$

The very definition of the stratification $\mathcal{H}$ implies that $S(a) = S(a')$ if and only if $\text{Str}(a) = \text{Str}(a')$. Thus, for $\alpha$ a stratum of $(\mathbb{R}_{>0})^m$, we shall write $S(\alpha)$ for the common set $S(a)$ with $a \in \alpha$.

When $\alpha$ is a chamber, the inequalities in (4.1) are all strict. The elements of $S(\alpha)$ are then, as in [HK2, §2], called short subsets of $m$. Observe that $A \in m$ is short if and only if its complement $\overline{A}$ is not short. Therefore, if $\alpha$ is a chamber, the set $S(\alpha)$ contains $2^{m-1}$ elements.

Define $S_m(\alpha) := S(\alpha) \cap \mathcal{P}_m(m)$, where $\mathcal{P}_m(m) := \{X \in \mathcal{P}(m) \mid m \in X\}$.

Lemma 4.2 Let $\alpha \in \text{Ch}((\mathbb{R}_{>0})^m)$. Then $S(\alpha)$ is determined by $S_m(\alpha)$.

Proof: One has

$$I \in S(\alpha) \iff \begin{cases} m \in I \text{ and } I \in S_m(\alpha) \\ m \notin I \text{ and } \overline{I} \notin S_m(\alpha). \end{cases} \quad (4.2)$$

Let us now restrict ourselves to chambers of $\mathbb{R}_m^n$. We shall determine them by a very concentrated information called their “genetic code”. Define a partial order “$\hookrightarrow$” on $\mathcal{P}(m)$ by saying that $A \hookrightarrow B$ if and only if there exits a non-decreasing map $\varphi : A \to B$ such that $\varphi(x) \geq x$. For instance $X \hookrightarrow Y$ if $X \subseteq Y$ since one can take $\varphi$ being the inclusion. The genetic code of $\alpha$ is the set of elements $A_1, \ldots, A_k$ of $S_m(\alpha)$ which are maximal with respect to the order “$\hookrightarrow$”. By Lemma 4.2, the chamber $\alpha$ is determined by its genetic code; we write $\alpha = \langle A_1, \ldots, A_k \rangle$ and call the sets $A_i$ the genes of $\alpha$. Thanks to (4.2), the explicit reconstruction of $S(\alpha)$ out of its genetic code is given by the following recipe.

Lemma 4.3 Let $\alpha = \langle A_1, \ldots, A_k \rangle \in \text{Ch}(\mathbb{R}_m^n)$. Let $I \in \mathcal{P}(m)$. Then

$$I \in S(\alpha) \iff \begin{cases} m \in I \text{ and } \exists j \in k \text{ with } I \hookrightarrow A_j \\ m \notin I \text{ and } \overline{I} \not\hookrightarrow A_j \forall j \in k. \end{cases} \quad \square$$
Example 4.4 To unburden the notations, a subset $A$ of $\mathbb{m}$ is denoted by the number whose digits are the elements of $A$ in decreasing order; example: $531 = \{5, 3, 1\}$. In $\mathbb{R}_3^r$, there are 2 chambers. One of them, say $\alpha_0$, contains points such as $(1, 1, 3)$ which are not in the image of $\ell : N_3^3 \rightarrow \mathbb{R}^3$. One has $S_3(\alpha_0) = \emptyset$. Its genetic code is empty and one has 

$$\alpha_0 = \langle \rangle ; \quad S(\alpha_0) = \{\emptyset, 1, 2, 21\}.$$ 

The other, $\alpha_1$ contains $(1, 1, 1)$, and one has 

$$\alpha_1 = \langle 3 \rangle ; \quad S(\alpha_1) = \{\emptyset, 1, 2, 3\}.$$ 

Let us now figure out which subset $A \subset P_m(\mathbb{m})$ is the genetic code of a chamber of $\mathbb{R}_m^r$. To reduce the number of trials, observe that if $\alpha = \langle A_1, \ldots, A_k \rangle$, then

(a) $A_i \not\subset A_j$ for all $i \neq j$ and

(b) $\bar{A}_i \not\subset A_j$ for all $i, j$.

Indeed, one has Condition (a) since the sets $A_i$ are maximal (and we do not write them twice). For Condition (b), if $\bar{A}_i \subset A_j$, then $A_i$ would be both short and not short and Inequalities (4.1) would have no solution. A finite set $\{A_1, \ldots, A_k\}$, with $A_i \in P_m(\mathbb{m})$ satisfying Conditions (a) and (b) is called a virtual genetic code (of type $m$), and we keep writing it by $\langle A_1, \ldots, A_k \rangle$. Let $G_m$ be the set of virtual genetic codes and $G_m^{(k)}$ the subset of those virtual genetic codes containing $k$ genes.

The determination of $G_m$ is algorithmic:

1. $G_m^{(0)} = \{\langle \rangle \}$.

2. Each $A \in P_m(\mathbb{m})$ satisfying $\bar{A} \not\subset A$ gives rise to a virtual genetic code $\langle A \rangle$. This gives the set $G_m^{(1)}$.

3. Suppose, by induction, that we know the set $G_m^{(k)}$. Then, each $\langle (A_1, \ldots, A_k), (A_{k+1}) \rangle$ in $G_m^{(k)} \times G_m^{(1)}$, so that $\{A_1, \ldots, A_{k+1}\}$ satisfies Conditions (a) and (b), gives rise to an element of $G_m^{(k+1)}$.

When $G_m^{(k+1)} = \emptyset$, the process stops and $G_m = \bigcup_{i=0}^{m} G_m^{(i)}$.

Examples 4.5 For $m = 3$, the family $P_3(3)$ contains the sets 3, 31, 32 and 321 (with the notations introduced in Example 4.4). Only 3 satisfies $\bar{3} = 21 \not\subset 3$. Thus $G_3^{(1)} = \{3\}$ while $G_3^{(2)}$ is empty. We deduce that $G_3 = \{\langle \rangle, \langle 3 \rangle \}$. They correspond to the two chambers of $\mathbb{R}_3^r$ found in Example 4.4. In the same way, we easily find the following table:
Having found the virtual genetic codes of type \( m \), the next question is which of them are \textit{realizable}, that is, which of them is the genetic code of a chamber of \( \mathbb{R}^m \). We proceed as follows. Each virtual genetic codes \( \langle A_1, \ldots, A_k \rangle \) of type \( m \) determines, a subset \( S_{\langle A_1, \ldots, A_k \rangle} \) by the recipe of Lemma 4.3. Define the open polyhedral cone \( P := P_{\langle A_1, \ldots, A_k \rangle} \) by

\[
P := \left\{ x \in \mathbb{R}^m \mid \sum_{i \in I} x_i < \sum_{i \notin I} x_i, \forall I \in S_{\langle A_1, \ldots, A_k \rangle} \right\}.
\]

If there exists \( \alpha \in \text{Ch}(\mathbb{R}^m) \) with \( \alpha = \langle A_1, \ldots, A_k \rangle \), then \( \alpha = P_{\langle A_1, \ldots, A_k \rangle} \). The realization problem is thus equivalent to \( P \) being not empty. To find a point inside \( P \), we “push” its walls and consider:

\[
P_1 := \left\{ x \in \mathbb{R}^m \mid \sum_{i \in I} x_i \leq \sum_{i \notin I} x_i - 1, \forall I \in S_{\langle A_1, \ldots, A_k \rangle} \right\} \subseteq P \tag{4.5}
\]

As \( P \) is an open cone in \( \mathbb{R}^m \), then \( P \) is not empty if and only if \( P_1 \) is not empty. Indeed, if \( P \) is not empty, then \( \emptyset \neq P \cap \mathbb{Z}^m \subset P_1 \). We then use the \textit{simplex algorithm} of linear programming to minimize the \( \ell_1 \)-norm \( \sum_{i=1}^{m} x_i \) on \( P_1 \). This algorithm either outputs an optimal solution, which is a vertex of \( P_1 \), or concludes that \( P_1 \) is empty [Ch].

A program in C++ was designed, following the above algorithms (comments on this program and the source code can be found in [HRWeb]). A computer could thus list all the chambers of \( \mathbb{R}_+^m \) for \( m \leq 9 \). Each chamber \( \alpha \) is given by a distinguished element \( a_{\min}(\alpha) \in \mathbb{Z}_+^m \) with minimal \( \sum_{i=1}^{m} a_i \). The number of these chambers, \( |\text{Ch}(\mathbb{R}^m)| = |\text{Ch}((\mathbb{R}_+)^m)/\text{Sym}_m| \) is the one given in the first line of Table (1.1) in the introduction.

Experimentally, it turned out that, for \( m \leq 8 \), all virtual genetic codes are realizable. This is not true for \( m = 9 \):

\textbf{Lemma 4.6} The virtual genetic code \( \langle 9642 \rangle \in \mathcal{G}_9 \) is not realizable.

\textbf{Proof:} Let \( S := S_{\langle 9642 \rangle} \). As 9531 \( \not\leftrightarrow 9642 \), one has 9531 \( \not\in S \). On the other hand, \( 9642 = 87531 \not\in S \). If \( S = S(a) \) for some generic \( a \in \mathbb{R}_+^9 \), we would have \( a_7 + a_8 > a_9 \). Now 965 \( \not\in S \) by Lemma 4.3, therefore 965 = 874321 \( \in S \). By the above inequality on the \( a_i \)'s, this would imply that 94321 \( \not\in S \) which contradicts 94321 \( \not\leftrightarrow 9642 \). []

Our algorithm found 319124 elements in \( \mathcal{G}_9 \), out of which 175248 are realizable.
The list of all the chambers $\alpha$ of $\mathbb{R}^m$ with their representative $a_{\text{min}}(\alpha)$ can be found further in this paper for $m \leq 6$ (Sections 3 and 8) and on the WEB page \[HR Web\] for $m = 7, 8, 9$.

Several theoretical questions about $a_{\text{min}}(\alpha)$ remain open. For example, why $a_{\text{min}}(\alpha)$ has integral coordinates (with the $\ell_1$-norm $|a|_1 = \sum |a_i|$ odd)? \textit{A priori}, the vertices of $P_1$ should only be in $\mathbb{Q}^m$. Is $a_{\text{min}}(\alpha)$ always unique? This suggests the following

4.7 Conjectures:

a) any stratum of $\alpha \in \mathcal{H}(\mathbb{R}^m)$ contains a unique element $a_{\text{min}}(\alpha) \in \mathbb{Z}^m$ with minimal $\ell_1$-norm.

b) $\alpha$ is a chamber if and only $|a_{\text{min}}(\alpha)|_1$ is an odd integer.

c) All vertices of $P_1(S)$ have integral coordinates.

Conjecture b) is supported by the following evidences. First, it is obvious that an element $a \in \mathbb{Z}^m$ with $|a|_1$ odd is generic. On the other hand, it is experimentally true for $m \leq 9$. Conjecture a) for non generic strata is experimentally true for $m \leq 8$ (see Section 5). Conjecture c) has been checked for $m \leq 8$.

4.8 Cuts: One can prove that the set $\mathcal{G}_m$ of virtual genetic code of type $m$ is in bijection with the set of “cuts” on $m$ (the name is given in analogy with the Dedekind cuts of the rationals). A subset $S$ of $\mathcal{P}(m)$ is a cut if, for all $I, J \subset m$ the two following conditions are fulfilled:

(A) $I \in S \iff \bar{I} \notin S$.

(B) if $I \in S$ and $J \hookrightarrow I$, then $J \in S$.

The bijection sends a cut $S$ of $m$ to the set of maximal elements (with respect to the order “$\hookrightarrow$”) of $S_m$. For details, see \[HR Web\].

5 Non generic strata

If $a \in \mathbb{R}^m$ is not generic, some inequalities of (4.1) are equalities. Thus, an element $I \in S(a)$ is either a short subset of $m$ (strict inequality) or an almost short subset. As in Lemma 1.2, $S(a)$ is determined by $S_m(a) = S(a) \cap \mathcal{P}_m(m)$ and the latter is determined by those elements which are maximal with respect to the order “$\hookrightarrow$” (the genes of $S(a)$). We denote the genes which are short subsets by $A_1, \ldots, A_k$ and those which are almost short by $B_1^=, \ldots, B_l^=$. For instance, when $m = 3$, one writes $S(1, 1, 1) = \langle 3 \rangle$ and $S(1, 1, 2) = \langle 3^- \rangle$. To be more precise on our conventions, let $I^= \in S(a)$ be an almost short gene of $S(a)$ and $J \hookrightarrow I$. If $|J| < |I|$, then $J$ is automatically short (since $a \in \mathbb{R}^m \subset (\mathbb{R}_{>0})^m$). If $|J| = |I|$, then $J$ is supposed to be almost short unless there is a short gene $K$ with $J \hookrightarrow K$. For instance, $S(1, 2, 2, 3, 4) = \langle 51, 53^= \rangle$. 

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The set \( \text{Str}(\mathbb{R}_>) \) of all the strata of \( \mathcal{H}(\mathbb{R}_>) \) will be studied via a map \( \alpha \mapsto \alpha^+ \) from \( \text{Str}(\mathbb{R}_>) \) to \( \text{Ch}(\mathbb{R}_>) \) which we define now. Let \( a \in \mathbb{R}_> \). If \( \varepsilon \) is small enough, the \( m \)-tuple \( \alpha^+ := (\delta, a_1, \ldots, a_{m-1}) \) is a generic element of \( \mathbb{R}_> \) for \( \delta < \varepsilon \) and \( \alpha^+ := \text{Ch}(\alpha^+) \) depends only on \( \alpha = \text{Str}(a) \).

If \( \beta = \alpha^+ \), we denote \( \alpha = \beta^- \). This makes sense because of the following lemma.

**Lemma 5.1** The map \( \alpha \mapsto \alpha^+ \) is injective.

**Proof:** Let \( a, b \in \mathbb{R}_>m \) such that \( \text{Str}(a) \neq \text{Str}(b) \). The segment joining \( a \) to \( b \) will then cross a wall \( \mathcal{H}_I \) which does not contain \( \text{Str}(a) \). But then, the segment joining \( a^+ \) and \( b^+ \) will also cross \( \mathcal{H}_I \), showing that \( \text{Ch}(a^+ \neq \text{Ch}(b^+) \). [ □ ]

The correspondence \( \alpha \mapsto \alpha^+ \) can easily be described on the genetic codes. The genetic code of \( a^+ \) has the same number of genes than that of \( a \). The correspondence goes as follows. If \( \{p_1 \ldots p_r\} \) is a gene of \( S(a) \) which is short, then \( \{p_1^+ \ldots p_r^+, 1\} \) is a gene of \( S(a^+) \), where \( p_i^+ = p_i + 1 \) (the genes of \( S(a^+) \) are all short). If \( \{p_1 \ldots p_r\} = \) is an almost short gene of \( S(a) \), then \( \{p_1^+ \ldots p_r^+\} \) is a gene of \( S(a^+) \).

The following convention will be useful.

**5.2** Let \( a = (a_1, \ldots, a_k) \) be a generic element of \( \mathbb{Z}_k > \). For \( m \geq k \), the \( m \)-tuple \( \hat{a} = (0, \ldots, 0, a_1, \ldots, a_k) \) determines a chamber \( \hat{\alpha} \) represented by \( (\delta_1, \ldots, \delta_{m-k}, a_1, \ldots, a_k) \), where \( \delta_i > 0 \) and \( \sum \delta_i < 1 \). We say that \( \hat{a} \) is a conventional representative of \( \hat{\alpha} \). For instance, \( \langle 521 \rangle \) having the conventional representative \((0, 0, 1, 1, 1) \) shows that \( \langle 521 \rangle = \langle 321 \rangle^{++} \).

**Lemma 5.3** Let \( \alpha \) be a chamber of \( \mathbb{R}_>m \). Then \( \alpha = \beta^+ \) if and only if one (at least) of the two following statement holds:

1. \( \alpha \) has a conventional representative \((0, a_2, \ldots, a_m) \).
2. there exists \( (a_1, \ldots, a_m) \in \alpha \cap \mathbb{Z}^m \) with \( \sum a_i \) odd and \( a_1 = 1 \).

**Proof:** It is clear that either 1. or 2. implies \( \alpha = \text{Str}(a_2, \ldots, a_m)^+ \). Also, if \( \alpha = \beta^+ \) for \( \beta \) generic, then \( \alpha \) admits a conventional representative. It remains to show that, if \( \alpha = \beta^+ \) with \( \beta \) non-generic, then Statement 2. holds true.

Observe that, as the walls \( \mathcal{H}_I \) are defined by linear equations with integral coefficients, then \( \beta \cap \mathbb{Q}_m \) is dense in \( \beta \). As \( \beta \) is a cone, it must contain a point in \( b \in \mathbb{Z}_k > \). As \( b \) is not generic, then \( \sum b_i \) must be even and the \( m \)-tuples \( (\delta, b_1, \ldots, b_{m-1}) \) are all in \( \alpha \) for \( \delta < 2 \). [ □ ]

Tables III-V of Section [3] and Table VI of Section [8] show that \( a_{\text{min}}(\alpha) \) satisfies the above conditions for all \( \alpha \in \text{Ch}(\mathbb{R}_>) \) when \( m \leq 6 \). This proves the following
Proposition 5.4  The correspondence $\alpha \leftrightarrow \alpha^+$ gives a bijection $\text{Str}(\mathbb{R}^{m-1}) \rightarrow \text{Ch}(\mathbb{R}^m)$ for $m \leq 6$.

Tables I and II below make the bijection $\alpha \leftrightarrow \alpha^-$ from $\text{Ch}(\mathbb{R}^m)$ to $\text{Str}(\mathbb{R}^{m-1})$ explicit (we put a conventional $a_{\min}(\alpha)$ when there exists one).

**The bijection** $\text{Ch}(\mathbb{R}^4) \xrightarrow{\approx} \text{Str}(\mathbb{R}^3)$

<table>
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<tr>
<th>$\alpha$</th>
<th>$a_{\min}(\alpha)$</th>
<th>$\mathcal{N}_2^4(\alpha)$</th>
<th>$\alpha^-$</th>
<th>$a_{\min}(\alpha^-)$</th>
<th>$\mathcal{N}_2^4(\alpha^-)$</th>
</tr>
</thead>
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<td>$\emptyset$</td>
<td>$\langle \rangle$</td>
<td>(0, 0, 1)</td>
<td>$\emptyset$</td>
</tr>
<tr>
<td>$\langle 4 \rangle$</td>
<td>(1, 1, 1, 2)</td>
<td>$S^1$</td>
<td>$\langle 3^- \rangle$</td>
<td>(1, 1, 2)</td>
<td>1 point</td>
</tr>
<tr>
<td>$\langle 41 \rangle$</td>
<td>(0, 1, 1, 1)</td>
<td>$S^1 \cup S^1$</td>
<td>$\langle 3 \rangle$</td>
<td>(1, 1, 1)</td>
<td>2 point</td>
</tr>
</tbody>
</table>

**The bijection** $\text{Ch}(\mathbb{R}^5) \xrightarrow{\approx} \text{Str}(\mathbb{R}^4)$

<table>
<thead>
<tr>
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<th>$\mathcal{N}_2^5(\alpha)$</th>
<th>$\alpha^-$</th>
<th>$a_{\min}(\alpha^-)$</th>
<th>$\mathcal{N}_2^5(\alpha^-)$</th>
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</thead>
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<td>$\langle \rangle$</td>
<td>(0, 0, 0, 1)</td>
<td>$\emptyset$</td>
</tr>
<tr>
<td>$\langle 5 \rangle$</td>
<td>(1, 1, 1, 1, 3)</td>
<td>$S^2$</td>
<td>$\langle 4^- \rangle$</td>
<td>(1, 1, 1, 3)</td>
<td>1 point</td>
</tr>
<tr>
<td>$\langle 51 \rangle$</td>
<td>(0, 1, 1, 1, 2)</td>
<td>$T^2$</td>
<td>$\langle 4 \rangle$</td>
<td>(1, 1, 1, 2)</td>
<td>$S^3$</td>
</tr>
<tr>
<td>$\langle 52 \rangle$</td>
<td>(1, 1, 2, 2, 3)</td>
<td>$\Sigma_2^\text{or}$</td>
<td>$\langle 41^- \rangle$</td>
<td>(1, 2, 2, 3)</td>
<td>$S^1 \cup S^1$</td>
</tr>
<tr>
<td>$\langle 521 \rangle$</td>
<td>(0, 0, 1, 1, 1)</td>
<td>$T^2 \cup T^2$</td>
<td>$\langle 41 \rangle$</td>
<td>(0, 1, 1, 1)</td>
<td>$S^3 \cup S^1$</td>
</tr>
<tr>
<td>$\langle 53 \rangle$</td>
<td>(1, 1, 1, 2, 2)</td>
<td>$\Sigma_3^\text{or}$</td>
<td>$\langle 42^- \rangle$</td>
<td>(1, 1, 2, 2)</td>
<td>$\bigcirc$</td>
</tr>
<tr>
<td>$\langle 54 \rangle$</td>
<td>(1, 1, 1, 1, 1)</td>
<td>$\Sigma_4^\text{or}$</td>
<td>$\langle 43^- \rangle$</td>
<td>(1, 1, 1, 1)</td>
<td>$\bigtriangledown$</td>
</tr>
</tbody>
</table>

Here, $\Sigma_g^\text{or}$ stands for the orientable surface of genus $g$ and the two graphs in the last column are the 2 and 3-fold covers of $S^1 \cup S^1$ without loops. The same work with the bijection $\text{Ch}(\mathbb{R}^6) \xrightarrow{\approx} \text{Str}(\mathbb{R}^5)$ gives the classification of all the 21 pentagon spaces (not necessarily generic) obtained by A. Wenger [Wc].

In the above two tables, one sees that $\mathcal{N}_2^m(\alpha)$ is the boundary of a regular neighborhood (here in $\mathbb{R}^3$) of $\mathcal{N}_2^m(\alpha^-)$. This reflects the following fact. Let $a_0 \in \alpha$ without zero coordinate. For any $a \in \alpha$, the Riemannian manifold $\mathcal{N}_E^m(a)$ is canonically diffeomorphic to $\mathcal{N}_E^m(a_0)$ by Theorem A and its proof. This produces a family of Riemannian metrics $g_a$ on $\mathcal{N}_E^m(a_0)$, indexed by $a \in \alpha$. When $a$ tends to a point $a^- \in \alpha^-$, the Riemannian manifold $(\mathcal{N}_E^m(a_0), g_a)$ converges, for the Gromov-Hausdorff metric, to the metric space $\mathcal{N}_E^m(a^-)$.

On the other hand, the map $\alpha \mapsto \alpha^+$ is not surjective when $m \geq 7$. For instance, $\langle 764 \rangle$, with $a_{\min} = (2, 2, 2, 2, 3, 3, 3)$, is not of the form $(\alpha^-)^+$. For, $\alpha^-$ would be $\langle 653^- \rangle$. As $421 = 653 \leftrightarrow 653$, this would imply that all $a_i^-$ are equal and $\alpha^- = \langle 654^- \rangle$. But $(654^-)^+ = \langle 765 \rangle \neq \langle 764 \rangle$.

The table of $\text{Ch}(\mathbb{R}^7)$ (giving the 135 7-gon spaces) shows 18 chambers with the first coordinate $a_{\min}$ not equal to 0 or 1 (see [HRW]). One might ask whether
there are other $m$-tuples $a$ in these chambers with $a_1 = 0, 1$. But, by applying the simplex algorithm to minimize $a_1$ on the polytope $P_1$ of (4.3), we saw that this is not the case. Therefore $|\text{Str}(\mathbb{R}^6)| = 118$. The same procedure succeeded for $m = 8$ and $9$, giving the cardinality of $\text{Str}((\mathbb{R}_{>0})^m)/\text{Sym}_m = \text{Str}(\mathbb{R}_+^m)$ for $m \leq 8$ listed in the introduction.

6 Geometric descriptions of the $N_{2,3}^m(\alpha)$’s

When $d = 2$ or $3$ and $a$ is generic, the spaces $N_d^m(a)$ are smooth manifolds, since $SO(d)$ acts freely on the non-lined configurations. The space $N_2^m(a)$ is also a manifold and the map $N_2^m(a) \to \bar{N}_2^m(a)$ is a 2-sheeted covering. The space $\bar{N}_2^m(a)$ lies in $N_3^m(a)$ as the fixed point set for the involution $\tau$ on $N_3^m(a)$ obtained by reflection through a hyperplane. Observe that $\dim N_3^m = 2(m-3)$ while $\dim \dim \bar{N}_2^m = m - 3$. The manifold $\bar{N}_2^m$ plays the role of a real locus of $N_3^m(a)$, the latter being endowed with a natural Kaehler structure for which the involution $\tau$ is antiholomorphic (see [HK2, §9]). It is shown in [HK2, Th. 9.1] that the cohomology rings $H^2(N_3^m(\alpha);\mathbb{Z}_2)$ and $H^*(\bar{N}_2^m(\alpha);\mathbb{Z}_2)$ are isomorphic, by a graded ring isomorphism dividing the degrees by 2.

The above polygon spaces were previously known for $m \leq 5$ (see, for instance, [HK1, §6]). Our classification by genetic code produces the more systematic tables below. Conventional representatives $a_{\text{min}}(\alpha)$ (see 5.2) are used when available.

<table>
<thead>
<tr>
<th>Table III : the 3-gon spaces</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha$</td>
</tr>
<tr>
<td>$\langle \rangle$</td>
</tr>
<tr>
<td>$\langle 3 \rangle$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Table IV : the 4-gon spaces</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha$</td>
</tr>
<tr>
<td>$\langle \rangle$</td>
</tr>
<tr>
<td>$\langle 4 \rangle$</td>
</tr>
<tr>
<td>$\langle 41 \rangle$</td>
</tr>
</tbody>
</table>
Table V: the 5-gon spaces

<table>
<thead>
<tr>
<th></th>
<th>a_{min}(a)</th>
<th>\mathcal{N}^5_3(\alpha)</th>
<th>\mathcal{N}^5_2(\alpha)</th>
<th>\mathcal{N}^5_2(\alpha)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>(0, 0, 0, 0, 1)</td>
<td>∅</td>
<td>∅</td>
<td>∅</td>
</tr>
<tr>
<td>2</td>
<td>(1, 1, 1, 1, 3)</td>
<td>\mathbb{C}P^2</td>
<td>\mathbb{R}P^2</td>
<td>S^2</td>
</tr>
<tr>
<td>3</td>
<td>(0, 1, 1, 1, 2)</td>
<td>\mathbb{C}P^2 # \mathbb{C}P^2</td>
<td>\Sigma_1</td>
<td>T^2</td>
</tr>
<tr>
<td>4</td>
<td>(1, 1, 2, 2, 3)</td>
<td>(S^2 \times S^2) # \mathbb{C}P^2</td>
<td>\Sigma_2</td>
<td>\Sigma^{or}</td>
</tr>
<tr>
<td>5</td>
<td>(0, 0, 1, 1, 1)</td>
<td>S^2 \times S^2</td>
<td>T^2</td>
<td>T^2 # T^2</td>
</tr>
<tr>
<td>6</td>
<td>(1, 1, 1, 2, 2)</td>
<td>\mathbb{C}P^2 # 3\mathbb{C}P^2</td>
<td>\Sigma_3</td>
<td>\Sigma^{or}</td>
</tr>
<tr>
<td>7</td>
<td>(1, 1, 1, 1, 1)</td>
<td>\mathbb{C}P^2 # 4\mathbb{C}P^2</td>
<td>\Sigma_4</td>
<td>\Sigma^{or}</td>
</tr>
</tbody>
</table>

Our method produces a classification of the spaces \mathcal{N}^m_{\mathbb{R}^n}(\alpha) for \(m \leq 9\), \(\alpha\) a chamber, and \(n \geq 2\). Table VI of Section 8 gives the list of hexagon spaces. The tables for generic \(m\)-gon spaces when \(m = 7, 8, 9\) are too big to be included in this paper. They can be consulted on the WEB page [HRWEB].

We shall now give procedures describing \(\mathcal{N}^m_E(\beta^+)\) in terms of \(\mathcal{N}^{m-1}_E(\beta)\) when \(\beta\) is generic and \(\dim E = 2, 3\). A \(m\)-tuple \((\rho_1, \ldots, \rho_m) \in \mathcal{K}(E^m)\) is called a vertical configuration if \(\rho_m = (0, \ldots, 0, -|\rho_m|)\).

**Proposition 6.1** If \(\beta \in \text{Ch}(\mathbb{R}^{m-1})\), then \(\mathcal{N}^m_2(\beta^+)\) is diffeomorphic to \(\mathcal{N}^{m-1}_2(\beta) \times S^1\).

**Proof:** Let \((b_2, \ldots, b_m) \in \beta\) and let \(\varepsilon > 0\) small enough so that \(a := (\varepsilon, b_2, \ldots, b_m) \in \beta^+\). A class in \(\mathcal{N}^m_2(\alpha)\) has a unique representative \(\rho = (\rho_1, \ldots, \rho_m)\) which is a vertical configuration. As \(b\) is generic, if \(\varepsilon\) is small enough, then \((b_2, \ldots, b_m') \in \beta\) when \(|b_m' - b_m| < \varepsilon\). The \((m - 1)\)-tuple \((\rho_2, \ldots, \rho_m + \rho_1)\) thus represents an element \(\rho' \in \mathcal{N}^{m-1}_2(\beta)\) and the correspondence \(\rho \mapsto (\rho', \rho_1)\) produces a diffeomorphism from \(\mathcal{N}^m_2(\beta^+)\) to \(\mathcal{N}^{m-1}_2(\beta) \times S^1\). \[\]

Let \(\rho = (\rho_1, \ldots, \rho_m) \in (\mathbb{R}^3)^m\). Let \(\rho_m^\perp\) be the orthogonal complement of \(\rho_m\), oriented by the vector \(\rho\). Let \(\pi: \mathbb{R}^3 \to \mathbb{R}^2\) be the composition of the orthogonal projection \(\mathbb{R}^3 \to \rho_m^\perp\) with some chosen isometry \(\rho_m^\perp \overset{\approx}{\to} \mathbb{R}^2\) preserving the orientation. If \(a \in (\mathbb{R}_{>0})^m\) is generic, then \(\pi(\rho_1), \pi(\rho_1 + \rho_2), \ldots, \pi(\rho_1 + \cdots + \rho_{m-2})\) are not all zero. This defines a smooth map

\[
r: \mathcal{N}^m_3(\alpha) \to (\mathbb{R}^2)^{m-2} - \{0\}/SO(2)
\]  

where \(\alpha = \text{Ch}(a)\). The right hand member of Equation (5.1) is homotopy equivalent to \(\mathbb{C}P^{m-3}\). The map \(r\) thus determines a cohomology class \(R \in H^2(\mathcal{N}^m_3(\alpha); \mathbb{Z})\) which is the characteristic class of some principal circle bundle \(E(\alpha) \to \mathcal{N}^m_3(\alpha)\). The class \(R\) was introduced in [HR3, §6 and 7] and will appear again in Section 8 below.
Lemma 6.2 (compare [HK2, Prop. 7.3]) The total space $\mathcal{E}(\alpha)$ is $S^1$-equivariantly diffeomorphic to the spaces of representatives of $\mathcal{N}_3^m(a)$, $(\text{Ch}(a) = \alpha)$ which are vertical configurations.

Proof: Let $\mathcal{E}'(\alpha) \subset (\mathbb{R}^3)^m$ be the space described in the statement. Any element of $\mathcal{N}_3^m(a)$ has at least one representative which is a vertical configuration and any two of those are in the same orbit under the orthogonal action of $S^1 = SO(2)$ fixing the vertical axis. As $a$ is generic, the quotient map $\mathcal{E}'(\alpha) \to \mathcal{N}_3^m(a)$ is then a principal circle bundle. If $\pi : \mathbb{R}^3 \to \mathbb{R}^2$ denotes the projection onto the first two coordinates, the correspondence $\rho \mapsto (\pi(\rho_1), \pi(\rho_1 + \rho_2), \ldots, \pi(\rho_1 + \cdots + \rho_{m-2}))$ defines a smooth $S^1$-equivariant map $\tilde{r} : \mathcal{E}'(\alpha) \to (\mathbb{R}^2)^{m-2}$ which covers the map $r$. This proves that the characteristic class of $\mathcal{E}'(\alpha) \to \mathcal{N}_3^m(a)$ is $R$. \[\]

Example 6.3 The chamber $\alpha = \langle m \rangle$ of $\mathbb{R}^m_+$ has its minimal representative $a = a_{\min}(\alpha) = (1, \ldots, 1, m - 2)$. As, in a vertical configuration $\rho$ of $[\rho] \in \mathcal{N}_3^m(a)$, one has $\sum_{i=1}^{m-1} \rho_i = (0, \ldots, 0, m - 2)$, the sequence of the third coordinate of $\rho_1, \rho_1 + \rho_2, \ldots$, must be strictly increasing. This implies that the map $\tilde{r}$ of the proof of Lemma 6.2 is a smooth $S^1$-equivariant embedding. It induces a diffeomorphism from $\mathcal{N}_3^m(\langle m \rangle)$ onto $\mathbb{C}P^{m-3}$ and an identification of the bundle $\mathcal{E}(\langle m \rangle) \to \mathcal{N}_3^m(\langle m \rangle)$ with the Hopf bundle. (see also [Ha, Remark 4.2]).

Let $\mathcal{D}(\alpha)$ be the total space of the $D^2$-bundle associated to $\mathcal{E}(\alpha) \to \mathcal{N}_3^m(\alpha)$.

Proposition 6.4 If $\beta \in \text{Ch}(\mathbb{R}^m_+)$, then $\mathcal{N}_3^m(\beta^+)$ is diffeomorphic to the double of $\mathcal{D}(\beta)$.

Proof: Let $(b_2, \ldots, b_m) \in \beta$ and let $\varepsilon > 0$ be small enough so that $a := (\varepsilon, b_2, \ldots, b_m) \in \beta^+$. A class in $\mathcal{N}_3^m(a)$ has a representative $\rho = (\rho_1, \ldots, \rho_m)$ which is a vertical configuration and with $\rho_1 = (\varepsilon \cos \theta, 0, \varepsilon \sin \theta)$. Let $\mathcal{E}(a)$ be the space of such representatives. The map sending $\rho$ to $\theta$ is a smooth map $\theta : \mathcal{E}(a) \to [0, \pi]$.

If $\rho \in \mathcal{E}(a)$, then $\rho_m + \rho_1$ is close to $\rho_m$. This defines a smooth map $P : \mathcal{E}(a) \to SO(3)$, sending $\rho$ to $P_\rho$, characterized by $P_\rho(\rho_m + \rho_1) = (0, 0, -|\rho_m + \rho_1|)$ and $P_\rho = \text{id}$ if $\theta(\rho) = 0, \pi$. The smooth map $\tilde{F} : \mathcal{E}(a) \to [0, \pi] \times \mathcal{E}(\beta)$ given by

$$\tilde{F}(\rho) := \left( \theta(\rho), (P_\rho(\rho_2), \ldots, P_\rho(\rho_{m-1}), P_\rho(\rho_m + \rho_1)) \right)$$

is a diffeomorphism. It induces a diffeomorphism

$$F : \mathcal{N}_3^m(\alpha) \cong \mathcal{N}_3^m(a) \longrightarrow [0, \pi] \times \mathcal{E}(\beta) \bigg/ \sim$$

where $\sim$ is the equivalence relation given by $(0, \eta) \sim (0, g \cdot \eta)$ and $(\pi, \eta) \sim (\pi, g \cdot \eta)$ for all $g \in SO(2)$. The right member of (6.4) is diffeomorphic to the double of $\mathcal{D}(\beta)$ which proves the proposition. \[\]
If, in (1.1), one replaces $\mathbb{R}^3$ by $\mathbb{R}^2$, one gets a map

$$r : N^m_2(\alpha) \to \mathbb{R}^{m-1} - \{0\}/\{\pm 1\} \simeq \mathbb{RP}^{m-2}$$

This produces a cohomology class $R \in H^1(N^m_2(\alpha); \mathbb{Z}_2)$ which is the Stiefel-Whitney class of the double covering $N^m_2(\alpha) \to N^m_2(\alpha)$. Lemma 6.2 holds true and Example 6.3 becomes:

**Example 6.5** For the chamber $\langle m \rangle$, realized by $a = (1, \ldots, 1, m - 2)$, the map $r$ is homotopic to a diffeomorphism from $N^m_2(\langle m \rangle)$ onto $\mathbb{RP}^{m-3}$ and gives an identification of the double covering $N^m_2(\langle m \rangle) \to N^m_2(\langle m \rangle)$ with $S^{m-3} \to \mathbb{RP}^{m-3}$.

If $\mathcal{D}(\alpha)$ denotes the total space of the $D_1$-bundle associated to the double covering $N^m_2(\alpha) \to N^m_2(\alpha)$, one proves, as in Proposition 6.4, that

**Proposition 6.6** If $\beta \in \text{Ch}(\mathbb{R}^{m-1})$, then $N^m_2(\beta^+)$ is diffeomorphic to the double of $\mathcal{D}(\beta)$. \(\square\)

**Example 6.7** The chamber $\langle\{m, 1\}\rangle = \langle m - 1 \rangle^+$ is represented by $a = (1/2, 1, \ldots, 1, m - 3)$. As seen in Example 6.3 one has that $N^m_3(\langle m - 1 \rangle)$ is diffeomorphic to $\mathbb{CP}^{m-4}$ and $\mathcal{D}(\langle m - 1 \rangle)$ is the disk bundle associated to the Hopf bundle. Therefore $N^m_3(\langle\{m, 1\}\rangle)$ is diffeomorphic to $\mathbb{CP}^{m-3} \# \mathbb{CP}^{m-3}$. For planar polygons, one has $N^m_2(\langle m - 1 \rangle)$ is diffeomorphic to $\mathbb{RP}^{m-4}$ and $\mathcal{E}(\langle m \rangle) \to N^m_2(\langle m \rangle)$ is the double covering. Therefore, $N^m_2(\langle\{m, 1\}\rangle)$ is diffeomorphic to $\mathbb{RP}^{m-3} \# \mathbb{RP}^{m-3}$ (of course, $\mathbb{RP}^{m-3} = \mathbb{RP}^{m-3}$ when $m$ is even).

**6.8 Case where $\beta$ is non generic.** When $\alpha = \beta^+$ with $\beta$ non generic, some partial information about $N^m_3(\alpha)$ can still be gathered. We proceed as in the proof of Proposition 6.4, with the same notations. If $\varepsilon$ is small enough, the $(m - 1)$-tuple $b_\delta := (b_2, \ldots, b_m + \delta)$ is generic when $0 < |\delta| \leq \varepsilon$; set $\beta_{\pm \varepsilon} := \text{Ch}(b_{\pm \varepsilon})$. The manifold $\mathcal{E}(a)$ is now a cobordism between between $\mathcal{E}(\beta_\varepsilon)$ and $\mathcal{E}(\beta_{-\varepsilon})$. By [Ha, Thm 3.2], the map $-\theta : \mathcal{E}(a) \to [-\pi, 0]$ is a Morse function. It has only one critical value, the angle for which the diagonal length $|\rho_m + \rho_1|$ is equal to $b_m$. The preimage of this critical value is diffeomorphic to $N^{m-1}_3(\beta)$ and the (isolated) critical points are the lined configurations. There is one for each almost short subset $I^\varepsilon \in S_{m-1}(\beta)$ and its index is equal to $2|I|$ or $|I|$ for planar polygons. As in Equation (6.4), the space $N^m_3(\alpha)$ is diffeomorphic to the quotient of the cobordism $\mathcal{E}(a)$ by the following identifications on its two ends: $\eta \sim g \cdot \eta$ for all $g \in SO(2)$, when $\theta(\eta) = 0, \pi$ (for planar polygons, $g \in O(1)$).

As an application of the results of this section, we will describe all the hexagon spaces in Section 8.
6.9 Use of toric manifolds. Recall that a symplectic manifold $M^{2n}$ is called toric if it is endowed with a Hamiltonian action of a torus $T$ of dimension $n$ (the maximal possible dimension for a Hamiltonian torus action). The moment map $\mu: M \to \text{Lie}(T)^* \simeq \mathbb{R}^n$ has for image a convex polytope, the moment polytope, which determines $M$ up to $T$-equivariant symplectomorphism (see [Gu]).

The spatial polygon space $N_3^m(a)$ with its symplectic structure (see 2.6) may admit Hamiltonian torus actions by so the so called bending flows (see [K], [KM], [HT]), which we recall now. For $I \in m$, define $f_I: N_3^m(a) \to \mathbb{R}$ by $f_I(\rho) := \left| \sum_{i \in I} \rho_i \right|$. If $f_I$ does not vanish, it is a smooth map which generates a Hamiltonian circle action on $N_3^m(a)$. This action rotates at constant speed the set of vectors $\{\rho_i \mid i \in I\}$ around the axis $\sum_{i \in I} \rho_i$ (see [K], §2.1, [KM], Corollary 3.9). The non-vanishing of $f_I$ is equivalent to $I$ being lopsided, that is there exists $i \in I$ with $a_i > \sum_{j \in I \setminus \{i\}} a_j$ (see [HT]).

Suppose that $I \subset \mathcal{P}(m)$ is formed of lopsided subsets satisfying the following “absorption condition”: if $I, J \in I$ with $I \neq J$, then either $I \cap J = \emptyset$ or one is contained in the other. Then, the Hamiltonian flow of the $f_i$’s of $I \in I$ commute and generate a Hamiltonian action of a torus $T_I$. (see [K], §2.1, [HT], Lemma 2.1). Thus, when $\dim T_I = m - 3$, the manifold $N_3^m(a)$ is a toric manifold which is determined by the moment polytope for the $T_I$-action.

For example, when $m = 5$, each chamber $\alpha \in \text{Ch}(\mathbb{R}^5)$ has a representative $a \in \alpha$ with $a_1 \neq a_2$ and $a_3 \neq a_4$. Therefore, $N_3^5(a)$ admits a Hamiltonian action of the 2-dimensional torus $T_I$ for $I = \{\{1, 2\}, \{34\}\}$. This shows that the diffeomorphism type of $N_3^5(a)$ is that of a toric manifold. The determination of all the 2-dimensional moment polytopes was the principle of the classification of the 5-gon spaces given in [HK], §6.1.

The same holds for $m = 6$ since each chamber $\alpha \in \text{Ch}(\mathbb{R}^6)$ has a representative $a \in \alpha$ with $a_1 \neq a_2$, $a_3 \neq a_4$ and $a_5 \neq a_6$. Therefore, all $N_3^6(\alpha)$ are diffeomorphic to toric manifolds. The 3-dimensional moment polytopes can still be visualized but with more difficulties.

The above two cases generalizes in the following

**Proposition 6.10** Let $\alpha \in \text{Ch}(\mathbb{R}^m)$. Suppose that there exists $a \in \alpha \cap \mathbb{Z}^m$ with $a_m \geq \sum_{i=1}^{m-5} a_i$. Then the diffeomorphism type of $N_3^m(\alpha)$ is that of a toric manifold.

**Proof:** One can find $a' \in \alpha$ arbitrarily close to $a$ so that $a_{m-4} \neq a_{m-3}$, $a_{m-2} \neq a_{m-1}$ and $a_m > \sum_{i=1}^{m-5} a_i$. Therefore the family of lopsided sets

$\{m, 1\}, \{m, 2, 1\}, \ldots, \{m, m-5, m-4, \ldots, 1\}, \{m-4, m-3\}, \{m-2, m-1\}$

satisfy the absorption condition. Their bending flows generate a Hamiltonian action of a torus of dimension $m - 3$, which proves the proposition. □
Examples: Consulting the table of the 135 seven-gons (see [HRW]), we see that there are only three \( \alpha \in \text{Ch}(\mathbb{R}^7) \) for which \( a = a_{\min}(\alpha) \) does not satisfy the hypothesis of Proposition 6.10, that is, here, \( a_7 \geq a_1 + a_2 \). These are

\[
\begin{array}{cc}
\alpha & a_{\min}(\alpha) \\
\langle 754, 762 \rangle & (3,3,4,4,5,5) \\
\langle 764 \rangle & (2,2,2,3,3,3) \\
\langle 765 \rangle & (1,1,1,1,1,1)
\end{array}
\]

Thus, all the other 133 heptagon spaces are diffeomorphic to toric manifolds. We do not know whether the above three heptagon spaces are diffeomorphic to toric manifolds.

The same experiment with \( m = 8 \) or \( 9 \) gives the following results: 217 elements of \( \text{Ch}(\mathbb{R}^8) \) (out of 2400) and 56550 elements of \( \text{Ch}(\mathbb{R}^9) \) (out of 175428) do not satisfy the hypothesis of 6.10.

7 Cohomology invariants of \( \mathcal{N}^m_3(\alpha) \)

Let \( \alpha \) be a chamber of \( (\mathbb{R}_{>0})^m \). In [HK2], presentations of the cohomology rings \( H^*(\mathcal{N}^m_3(\alpha); \mathbb{Z}) \) and \( H^*(\mathcal{N}^m_2(\alpha); \mathbb{F}_2) \) were obtained in terms of \( \alpha \). Our algorithms allowed us, with the help of a computer, to find enough information about these rings to prove that, for \( 5 \leq m \leq 7 \), \( \alpha = \alpha' \) if and only if the mod 2 cohomology rings of \( \mathcal{N}^m_3(\alpha) \) and of \( \mathcal{N}^m_3(\alpha') \) are isomorphic.

We start by the Poincaré polynomial. Recall that \( \mathcal{N}^m_3(\alpha) \) has a cellular decomposition with only even-dimensional cells [HK2, §4], so its Poincaré polynomial is the same for any field \( \mathbb{F} \):

\[
P(t) = \sum_{i=0}^{2(m-3)} \dim_{\mathbb{F}} H^i(\mathcal{N}^m_3(\alpha); \mathbb{F}) t^i
\]

and has only terms of even degree. Moreover, the polynomial \( P(\sqrt{t}) \) is the Poincaré polynomial of \( \mathcal{N}^m_2(\alpha) \) for the coefficient field with two elements \( \mathbb{F}_2 \) [HK2, §9]. The first formula for computing \( P(t) \) in terms of \( \alpha \) was found by A. Kliachko [K, Th. 2.2.4]. We will use the more economical formula, using only elements of \( S_m(\alpha) \), obtained in [HK2, Cor. 4.3]. With our notation, this is:

Proposition 7.1 Let \( \alpha \in \text{Ch}((\mathbb{R}_{>0})^m) \). The Poincaré polynomial of \( \mathcal{N}^m_3(\alpha) \) is

\[
P(t) = \frac{1}{1-t^2} \sum_{J \in S_m(\alpha)} (t^{2(|J|)} - t^{2(m-1-|J|)}).
\]
Remark 7.2 The difference between the formula in Proposition 7.1 and that of [HK2, Cor. 4.3] comes from that, there, the notation $S_m$ is used for the set of $I \in \mathcal{P}(m-1)$ such that $J \cup \{m\} \in S$. Recall that, here, $S_m = S \cap \mathcal{P}_m(m)$. So, each occurrence of $|J|$ in [HK2, Cor. 4.3] is replaced here by $|J| - 1$.

For $\beta \subset \mathcal{P}(m)$, denote by $NS_\beta(\beta)$ the number of sets $I \in \beta$ with $|I| = i + 1$. The formula of Proposition 7.1 gives the Betti numbers $b_{2i} := b_{2i}(\alpha) := \dim_F H^i(N_3^m(\alpha); F)$ as the solution of the system of equations

$$b_{2i} - b_{2i-2} = NS_i(S_m(\alpha)) - NS_{m-2-i}(S_m(\alpha)), \quad (7.3)$$

starting with $b_{2i} = 0$ if $i < 0$. For instance, if $\alpha = \langle 54 \rangle$, realized by $(1, 1, 1, 1, 1)$, one has

$$S_5(\alpha) = \{5, 51, 52, 53, 54\}$$

thus $NS_0(S_5(\alpha)) = 1$, $NS_1(S_5(\alpha)) = 4$ and the other $NS_i(S_5(\alpha))$ vanish. This gives $b_0 = 1$, $b_2 = 5$ and $b_4 = 1$, which are indeed the Betti numbers of $N_3^5(\langle 54 \rangle) = \mathbb{CP}^2 \# 4\mathbb{CP}^2$ [HK2, Example 10.4].

As our computer algorithm had to list all the sets of $S_m(\alpha)$ (for instance, for the realization), the numbers $NS_i(S_m(\alpha))$ are available and so are the $b_{2i}$’s.

We now recall the presentation of $H^*(N_3^m(\alpha); \mathbb{Z})$ obtained in [HK2, Thm 6.4]. Taking care of Remark 7.2, this gives:

**Proposition 7.4** Let $\alpha \in \text{Ch}((\mathbb{R}_{>0})^m)$. The cohomology ring of the polygon space $N_3^m(\alpha)$ with coefficient in a ring $\Lambda$ is

$$H^*(N_3^m(\alpha); \Lambda) = \Lambda[R, V_1, \ldots, V_{m-1}]/\mathcal{I}(\alpha)$$

where $R$ and $V_i$ are of degree 2 and $\mathcal{I}(\alpha)$ is the ideal of $\Lambda[R, V_1, \ldots, V_{m-1}]$ generated by the three families

1. (R1) $V_i^2 + RV_i$ for $i = 1, \ldots, m-1$
2. (R2) $\prod_{i \in L} V_i$ for all $L \in \mathcal{P}(m-1)$ with $L \cup \{m\} \notin S(\alpha)$
3. (R3) $\sum_{S \subset L, \sum i(S) \in \alpha} (\prod_{i \in S} V_i)R^{|L-S|-1}$ for all $L \in \mathcal{P}(m-1)$ with $L \notin S(\alpha)$

We shall use this presentation, first to compute a homotopy invariant $r_\cup(\alpha) \in \mathbb{N}$, defined as the rank of the linear map $x \mapsto x \cup x$ from $H^2(N_3^m(\alpha); \mathbb{F}_2)$ to $H^4(N_3^m(\alpha); \mathbb{F}_2)$ (recall that $x \mapsto x^2$ is a linear map in an algebra over $\mathbb{F}_2$). Observe that $r_\cup(\alpha)$ is also the rank of the same map from $H^1(N_2^m(\alpha); \mathbb{F}_2)$ to $H^2(N_2^m(\alpha); \mathbb{F}_2)$.

**Proposition 7.5** For all $\alpha$ a chamber of $\mathbb{R}_m^m$, one has

$$r_\cup(\alpha) = 1 + NS_1(S_m(\alpha)) - NS_{m-3}(S_m(\alpha)) - NS_{m-4}(S_m(\alpha)). \quad (7.5)$$
PROOF: One has $1 + \text{NS}_1(S_m(\alpha)) - \text{NS}_{m-3}(S_m(\alpha)) = b_2$ by equations \[7.3\]. Let us first consider the case $\text{NS}_{m-3}(S_m(\alpha)) \neq 0$. This means that $S_m(\alpha)$ contains a set with $m - 2$ elements and thus contains the smallest of those for the order $\leftrightarrow$, which is $I := \{m, m - 3, m - 4, \ldots, 1\}$. Then $\alpha = \langle I \rangle$. Indeed, if $\alpha \neq \langle I \rangle$, then $\alpha$ would contain $J := \{m, m - 2\}$, which is impossible since $J \not\subseteq I$. Therefore,

$$\text{NS}_1(S_m(\alpha)) = m - 3 \quad , \quad \text{NS}_{m-4}(S_m(\alpha)) = m - 3 \quad , \quad \text{NS}_{m-3}(S_m(\alpha)) = 1$$

and the right hand member of (7.3) is equal to zero. On the other hand, Relator (R3) of Proposition \[7.4\], with $L = \{m - 2, m - 1\}$ and $S = \emptyset$ gives the equality $R = 0$. By Relators (R1), all squares vanish and $r_\cup = 0$. Formula (7.3) is then proven in the case $\text{NS}_{m-3}(S_m(\alpha)) \neq 0$. Observe that $\mathcal{N}_3^m((I))$ is diffeomorphic to a product of $m - 3$ copies of the sphere $S^2$ [HK2, Example 10.2].

Assume then that $\text{NS}_{m-3}(S_m(\alpha)) = 0$. By Proposition \[7.4\], the vector space $H^2(\mathcal{N}_3^m(\alpha); \mathbb{F}_2) = H^2(\mathcal{N}_3^m(\alpha); \mathbb{Z}) \otimes \mathbb{F}_2$ has the basis $R, V_1, \ldots, V_p$ for $p = \text{NS}_1(S_m(\alpha))$. Indeed, $V_i = 0$ for $i > p$ by Relator (R2) with $L = \{i\}$. The image of $x \mapsto x^2$ is generated by $R^2, V_1^2, \ldots, V_p^2$. The relations between these generators come from relators (R3) of Proposition \[7.4\] with $|L| = 3$. For such an $L = \{i, j, k\}$ (denoted by $ijk$), the relation is

$$R^2 + RV_i + RV_j + RV_k + V_iV_j + V_iV_k + V_jV_k = 0. \quad (7.6)$$

The three last terms of the left hand member vanish by Relator (R2). Indeed, since $ijk \notin \alpha$ and $ijk \not\leftrightarrow ijm$, then $ijm \notin \alpha$. By Relator (R1), Equation (7.6) becomes

$$R^2 - V_i^2 - V_j^2 - V_k^2 = 0. \quad (7.7)$$

To establish Proposition \[7.3\], it is enough to prove that, for distinct $i, j, k$, Equations (7.7) are independent. But, by Proposition \[7.4\], the vector space $H^4(\mathcal{N}_3^m(\alpha); \mathbb{F}_2)$ is generated by the $b_1$ elements $R^2, V_1^2, \ldots, V_p^2$ together with the NS$_2(S_m(\alpha))$ non-vanishing products $V_iV_j$ ($ijm \in \alpha$) and these generators are just subject to Equations (7.7). This implies that $b_2 \geq b_1 + \text{NS}_2(S_m(\alpha)) - \text{NS}_{m-4}(S_m(\alpha))$. By Equations (7.3), this inequality is an equality, showing that Equations (7.7) are independent.

The idea of our last cohomology invariant Was given to us by R. Bacher. Define $\mathcal{I}_k(\alpha)$ to be the ideal of $\mathbb{F}[R, V_1, \ldots, V_{m-1}]$ generated by the elements of $\mathcal{I}(\alpha)$ which are polynomials of degree $\leq k$ in the variables $R$ and $V_i$’s (then giving elements of degree $\leq 2k$ in $H^*(\mathcal{N}_3^m(\alpha); \mathbb{F})$). Define $\text{Sol}_k(\alpha; \mathbb{F}) \subset \mathbb{F}P^{m-1}$ to be the projective variety defined by the equations $W = 0$ for all $W \in \mathcal{I}_2(\alpha)$.

Proposition 7.8 a) Let $\alpha, \alpha'$ be chambers of $\mathbb{R}^m_\geq$. Any graded ring isomorphism from $H^*(\mathcal{N}_3^m(\alpha); \mathbb{F})$ onto $H^*(\mathcal{N}_3^m(\alpha'); \mathbb{F})$ induces, for $k \geq 1$, a bijection from $\text{Sol}_k(\alpha'; \mathbb{F})$ to $\text{Sol}_k(\alpha; \mathbb{F})$.  

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b) Suppose that \( k \geq 2 \). Then, any element \( \zeta \in \text{Sol}_k(\alpha; \mathbb{F}) \) has a unique representative of the form \((-1, v_1, \ldots, v_{m-1}) \in \mathbb{F}^m \) with \( v_i = 0 \) or 1. In particular, the set \( \text{Sol}_k(\alpha; \mathbb{F}) \) is finite.

c) The finite set \( \text{Sol}_2(\alpha; \mathbb{F}) \) does not depend on the field \( \mathbb{F} \).

**Proof:**

**Proof of a):** Let \( q : H^*(\mathcal{N}_3^m(\alpha); \mathbb{F}) \to H^*(\mathcal{N}_3^m(\alpha'); \mathbb{F}) \) be a graded ring homomorphism. By Proposition 7.4, the homomorphism \( q \) is covered by a graded ring homomorphism \( \tilde{q} : \mathbb{F}[R, V_1, \ldots, V_{m-1}] \to \mathbb{F}[R', V'_1, \ldots, V'_{m-1}] \) which sends \( \mathcal{I}_k(\alpha) \) into \( \mathcal{I}_k(\alpha') \) for all \( k \). Such a lifting \( \tilde{q} \) is well defined up to a homomorphism with image in \( \mathcal{I}_1(\alpha') \), therefore \( q \) functorially induces homomorphisms \( \tilde{q} : \mathbb{F}[R, V_1, \ldots, V_{m-1}]/\mathcal{I}_k(\alpha) \to \mathbb{F}[R', V'_1, \ldots, V'_{m-1}]/\mathcal{I}_k(\alpha') \) for all \( k \geq 1 \). Observe that \( \text{Sol}_k(\alpha; \mathbb{F}) \) can be identified with the projectivization of the vector space of ring homomorphisms from \( \mathbb{F}[R, V_1, \ldots, V_{m-1}]/\mathcal{I}_k(\alpha) \) to \( \mathbb{F} \). Therefore, the homomorphism \( q \) will functorially induce maps \( \tilde{q} : \text{Sol}_k(\alpha'; \mathbb{F}) \to \text{Sol}_k(\alpha; \mathbb{F}) \) for all \( k \geq 1 \). By this functoriality, if \( q \) is an isomorphism, then \( \tilde{q} \) is a bijection, which proves a).

**Proof of b):** Let \( z = (r, v_1, \ldots, v_{m-1}) \in \mathbb{F}^m - \{0\} \) represent an element \( \zeta \in \text{Sol}_k(\alpha; \mathbb{F}) \). Then, \( r \neq 0 \), since, otherwise relators (R1) of Proposition 7.4 would give equations \( v_i^2 = -rv_i \), implying that \( z = 0 \). Then, \( [z] \) has a unique representative with \( r = -1 \) and the equations \( v_i^2 = -rv_i \) imply that \( v_i \in \{0, 1\} \).

**Proof of c):** The equations defining \( \text{Sol}_2(\alpha; \mathbb{F}) \), coming from Relators (R1)-(R3) of 7.4, are, for \( i, j, k = 1, \ldots, m - 1 \):

(i) \( v_i^2 = -rv_i \). Having normalized \( r = -1 \), these are equivalent to \( v_i = 0, 1 \).

(ii) \( v_i = 0 \), for \( \{i, m\} \notin \alpha \) and \( v_iv_j = 0 \) for \( \{i, j, m\} \notin \alpha \).

(iii) Equations (7.7) which, after (i), become \( v_i^2 + v_j^2 + v_k^2 = 1 \), for \( \{i, j, k\} \notin \alpha \).

The solutions \( v_i = 0, 1 \) of Equations (ii) are clearly independent of the ground field \( \mathbb{F} \). The solutions \( v_i = 0, 1 \) of an equation like \( v_i^2 + v_j^2 + v_k^2 = 1 \) seem, a priori, to depend on the characteristic of \( \mathbb{F} \). But, as seen just before Equation (7.7), such an equation occurs only if \( r_ir_j = rjr_k = rkr_i = 0 \). Thus, Equation (iii) is equivalent to the fact that exactly one of the \( v_i, v_j, v_k \) is equal to one, a condition independent of \( \mathbb{F} \).

**Definition 7.9** We set \( s(\alpha) \) to be the number of elements of \( \text{Sol}_2(\alpha; \mathbb{F}) \). This does not depend on the field \( \mathbb{F} \) by Proposition 7.8.

It is not difficult to compute \( s(\alpha) \) with or without the help of a computer. We select the elements of \( W(\mathcal{R}, V_1, \ldots, V_{m-1}) \in \mathcal{I}_2(\alpha) \) and count how many of them vanish when \( R = -1 \) and \( V_i \in \{0, 1\} \).

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7.10 Proof of Proposition B: The following implies Proposition B of the introduction:

**Proposition 7.11** Let $5 \leq m \leq 7$ and let $\alpha, \alpha' \in \text{Ch}(\mathbb{R}^m)$. Then $\alpha = \alpha'$ if and only if $\mathcal{N}_3^m(\alpha)$ and $\mathcal{N}_3^m(\alpha')$ have the same Betti numbers, $r(\alpha) = r(\alpha')$ and $s(\alpha) = s(\alpha')$.

**Proof:** For $m = 5$, by the list of Table V, The only case where two 5-gon spaces have the same Betti numbers are $\mathcal{N}_5^3(\langle 52 \rangle) \approx \mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$ and $\mathcal{N}_5^3(\langle 521 \rangle) \approx S^2 \times S^2$. But $r(\langle 52 \rangle) = 1$ while $r(\langle 521 \rangle) = 0$. (Taking $\mathbb{F}_2$ coefficients is important here: for instance, these two spaces have isomorphic cohomology ring with real coefficients.)

For $m = 6$, the list of Table VI in Section 8 has been sorted by lexicographic order of the triple $b_2(\alpha), r(\alpha), s(\alpha)$. One thus can check that no such triples occur twice. The same holds for $m = 7$ with $b_2(\alpha), b_4(\alpha), r(\alpha), s(\alpha)$ (table in [HRW]).

**Remark 7.12** By [HK2, §9], the cohomology ring $H^*(\overline{\mathcal{N}}_2^m(\alpha); \mathbb{F}_2)$ admits the presentation of Proposition 7.4, with $R$ and the $V_i$'s of degree 1. Therefore, the above invariants are mod 2 cohomology invariants of the spaces $\overline{\mathcal{N}}_2^m(\alpha); \mathbb{F}_2$ and Proposition 7.11 holds true for these spaces.

**Problem 7.13** When a virtual genetic code $\gamma \in G_m$ is not realizable (for instance when $m = 9$), it gives rise as well to a non-trivial graded ring. Is this ring the cohomology ring of a space? Does it satisfy Poincaré duality? Is this ring the cohomology ring of a manifold?

8 The hexagon spaces

As for Tables III-V of Section 3 the first column of Table VI below contains the list of the 21 genetic codes of type 6, all realized by a chamber $\alpha$ whose minimal realization $a_{\text{min}}(\alpha)$, using conventional representatives (see 5.2), is written in the second column. The next three columns give the cohomology invariants of $\mathcal{N}_3^6(\alpha)$ or $\overline{\mathcal{N}}_2^6(\alpha)$ which are defined in Section 4, with the abbreviations

$$ b := \dim_{\mathbb{F}_2} H^2(\mathcal{N}_3^6(\alpha); \mathbb{F}) = \dim_{\mathbb{F}_2} H^1(\overline{\mathcal{N}}_2^6(\alpha); \mathbb{F}_2) \ , \ r = r(\alpha) \ , \ s = s(\alpha). $$

By Poincaré duality, the number $b$ determines the Poincaré polynomial $P(t)$ of $\mathcal{N}_3^6(\alpha)$ which is

$$ P(t) = 1 + bt^2 + bt^4 + t^6 $$

(for $\overline{\mathcal{N}}_2^6(\alpha)$, this would be $P(t) = 1 + bt^2 + t^4$). The 6-gon spaces have been listed by the lexicographic order of the triples $(b, r, s)$, showing that the homotopy
type of the hexagon spaces in $\mathbb{R}^2$ or $\mathbb{R}^3$ are distinguished by these cohomology invariants.

The last two columns contain some geometric descriptions of $\mathcal{N}_2^6(\alpha)$ and $\mathcal{N}_2^6(\alpha)$ obtained by the methods discussed in Section 6. In the last column, we see the hexagon spaces coming from the 7 generic pentagons by adding a tiny vector; their descriptions uses Proposition 6.1. Lines 2 and 3, illustrate Examples 6.3 and 6.7. Line 4 is a special case of [HK2, Example 10.2] (see also the proof of Proposition 7.3). The other descriptions come from the method of 6.8. Using the notations of 6.8, the 3-manifold $\tilde{E}(a)$ is a cobordism between the orientable surfaces $E(\beta_\varepsilon)$ and $E(\beta_{-\varepsilon})$. When they are connected (all cases except Line 7), we denote their genus respectively by $g_+$ and $g_-$. The map $-\theta : \tilde{E}(a) \to [-\pi, 0]$ is a Morse function with $n_1$ critical points of index 1 and $n_2$ critical points of index 2. This situation is indicated in Table VI by the writing $[g_+; n_1, n_2; g_-]$ (observe that $g_- = g_+ + n_1 - n_2$). The orientable 3-manifold $\mathcal{N}_2^6(\alpha)$ is diffeomorphic to the quotient of the cobordism $\tilde{E}(a)$ by, on each end, a free involution reversing the orientation.

In other words, the notation $[g_+; n_1, n_2; g_-]$ tells us that the orientable 3-manifold $\mathcal{N}_2^6$ is obtained in the following way. For $\Sigma$ a surface, denote by $\bar{D}(\Sigma)$ the mapping cylinder of the orientation covering of $\Sigma$. Let $W_+ = \bar{D}(\Sigma_{g_+})$ union with $n_1$ 1-handles and $W_- = \bar{D}(\Sigma_{g_-})$ union with $n_2$ 1-handles. If we require that $W_\pm$ are orientable, they are well defined since there is only one way, up to diffeomorphism isotopic to the identity, to attach 1-handles to $\Sigma_{g_\pm} \times [0, 1]$ in order to obtain an orientable manifold. Thus, $\mathcal{N}_2^6$ is obtained by gluing $W_+$ to $W_-$ by a diffeomorphism of their boundary. In the case where $n_1 = n_2 = 0$, one has $W_+ = W_- = \bar{D}(\Sigma_2)$ and Proposition 6.6 says that the gluing diffeomorphism is the identity.

We were not able to identify this gluing diffeomorphism in the other cases, so, a priori, the numbers $[g_+; n_1, n_2; g_-]$ do not determine the homeomorphism type of $\mathcal{N}_2^6$.

In the case $\alpha = \langle 632 \rangle$ (line 7 of the table), $\mathcal{N}_2^6(\beta_\varepsilon) = T^2$ (the only case where it is orientable). Therefore $W_+ = T^2 \times [-1, 1]$ and $W_- = \bar{D}(\Sigma_2)$. 

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>$a_{\min}(\alpha)$</th>
<th>$b$</th>
<th>$r_U$</th>
<th>$s$</th>
<th>$\tilde{N}_2^6(\alpha)$</th>
<th>$N_2^6(\alpha)$</th>
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<td>$S^3$</td>
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<td>$\langle 61 \rangle$</td>
<td>0,1,1,1,1,3</td>
<td>2</td>
<td>2</td>
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<td>$\langle 6321 \rangle$</td>
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<td>$T^3$</td>
<td>$T^3 \sqcup T^3$</td>
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<td>$[2; 0, 0; 2]$</td>
<td>$\Sigma_2^{or} \times S^1$</td>
</tr>
</tbody>
</table>
8.1 Problem. Describe more precisely the 3-dimensional manifolds $\mathcal{N}_2^6$, for instance in terms of the Kirby calculus.

References


[HRWeb] [http://www.unige.ch/math/folks/hausmann/polygones](http://www.unige.ch/math/folks/hausmann/polygones) Web page organized by the authors.


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