Simply Generated Trees, B-series and Wigner Processes

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Abstract

We consider simply generated trees and study multiplicative functions on rooted plane trees. We show that the associated generating functions satisfy differential equations or difference equations. Our approach considers B-series from Butcher’s theory, the generating functions are seen as generalized Runge-Kutta methods.

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Simply Generated Trees, B-Series 
and Wigner Processes

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Abstract

We consider simply generated trees, like rooted plane trees, and consider the problem of computing generating functions of so-called bare functionals, like the tree factorial, using B-series from Butcher’s theory. We exhibit a special class of functionals from probability theory: the associated generating functions can be seen as limiting traces of product of semi-circular elements.

Key words and phrases. B-series, random matrices, rooted plane trees, Runge-Kutta methods, simply generated trees.

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1 Introduction

Let $F_n$ denote the set of rooted plane trees of size n. Simply generated trees are families of trees obtained by assigning weights $\omega(t)$ to the elements $t \in F = \bigcup_n F_n$ using a degree function $\psi(z) = 1 + \sum_{k \geq 1} \psi_k z^k$ (see [20]). Basically, the weight $\omega(t)$ of some $t \in F$ is obtained by multiplying the factors $\psi_{d(v)}$ over the nodes v of t, where $d(v)$ denotes the outdegree of v.

Our main topic is the study of generating functions

\[ Y(z) = \sum_{t \in F} \omega(t)B(t)z^{|t|}, \]

associated with multiplicative functions $B : \mathcal{F} \rightarrow \mathbb{R}$ defined recursively by using a sequence of real numbers $\{B_k\}_{k \in \mathbb{N}^+}$. We call such multiplicative functions bare Green functions: $\sum_{t \in F_n} B(t)\omega(t)$ represents the sum of the
Feynman amplitudes associated to the relevant diagrams of size $n$ in some field theory, and the generating function is then a part of the perturbative expansion of the solution of some equation describing the system (see [3, 6, 8, 15]).

In Section 4 we give an equation satisfied by $Y$ when the weights $B_k$ come from some master function $L(z) = \sum_{m \geq 0} L_m z^m$, with $B_k \equiv L(k)/k$, $\forall k \in \mathbb{N}^+$. We use series indexed by trees, the so-called B-series, as defined in [13, 14], to show in Theorem 1 that $Y$ solves

$$Y' = L(1 + \theta)\Psi(Y),$$

where $\theta$ is the differential operator $\theta = zd/dz$. [1] considers a similar problem for additive tree functionals $s(t)$ defined on varieties of increasing trees, like $s(t) = \ln(B(t))$. Assuming some constraints on the degree function $\Psi(z)$, it is proven that the exponential generating function $S(z) = \sum \omega(t)s(t)z^{|t|}/|t|!$, is given by the formula

$$S(z) = W'(z) \int_0^z (F'(u)/W'(u))du,$$

where $F(u) = \sum_{m \geq 0} \ln(B_m)W_mu^m/m!$ and $W(z) = \sum_{m \geq 0} W_m z^m/m!$ solves $W' = \Psi(W)$. We also consider a central functional called the tree factorial, denoted by $t!$ in the sequel, which is relevant in various fields, like algorithmics [9, 13], stochastics [11, 21], numerical analysis (see for example [5, 14]), and physics [6, 15]. We focus on its negative powers $1/(t!)^{l+1}$, $l \in \mathbb{N}$, which do not admit a master function when $l \geq 1$. [6] solved the case $l = 1$ by using the so-called Butcher’s group (see for example [13, 14]). We provide in Theorem 2 a differential equation for the associated generating function, $\forall l \in \mathbb{N}$.

In Section 5 we define special multiplicative functionals for which the weights $B_k$ are related to the covariance function $r$ of some gaussian process, as $B_k = \beta^2 r(2k - 1)$, for some positive constant $\beta > 0$. We show that the generating function $Y$ is related to the mean normalized trace of products of large symmetric random matrices having independent and identically distributed versions of the process as entries. Theorem 3 gives then a differential equation for the evolution of the trace of a stationary Wigner processes. It follows that most of the examples given in [3, 15] can be expressed in terms of traces of large random matrices. In Section 6 we show how B-series can be useful for studying traces of triangular operators appearing in free probability.
2 Basic notions

A rooted tree \( t \in \mathcal{R} \) is a triple \( t = (r, V, E) \) such that i) \( (V, E) \) is a non-empty directed tree with node set \( V \) and edge set \( E \), ii) all edges are directed away from the root \( r \in V \). The set of rooted trees of order \( n \) is denoted by \( \mathcal{R}_n \), and the set of rooted trees is \( \mathcal{R} = \cup_n \mathcal{R}_n \). A rooted plane tree \( t \in \mathcal{F} \) is a quadruple \( t = (r, V, E, L) \) satisfying i) and ii) and iii) \( L := \{(w : vw \in E) \cup L_v) : v \in V \} \) is a collection of \( |V| \) linear orders. Given \( v \in V \), let \( \text{ch}(v) := \{w : vw \in E\} \) be the set of children of \( v \). \( d(v) := |\text{ch}(v)| \) is the outdegree of \( v \). A rooted planar tree can be seen in the plane with the root in the lowest position, such that the orders \( L_v \) coincide with the left-right order. Next consider the partial ordering \( (V, \leq) \) defined by \( u \leq v \) if and only if \( u \) lies on the path linking \( r \) and \( v \). Given \( v \in V \) and \( t \in \mathcal{R} \) let \( t_v \) be the subtree of \( t \) rooted at \( v \) spanned by the subset \( \{w : v \leq w\} \). A rooted labelled tree is a quadruple \( t = (r, V, E, l) \) satisfying i) and ii), with a labelling \( l : V \setminus \{r\} \rightarrow [1, \cdots, |V|] := \{1, \cdots, |V|\} \) such that \( l(u) < l(v) \) when \( u < v \). The set of rooted labelled trees of order \( n \) is denoted by \( \mathcal{L}_n \). Let \( \mathcal{L} = \cup_n \mathcal{L}_n \). This family is a special variety of increasing trees, as defined in [1, 12].

We next assign weights to the elements of \( \mathcal{F}_n \), the set of rooted planar trees of order \( n \): the resulting family of trees is said to be simply generated (see [21]). Given a sequence \( \psi = \{\psi_k\}_{k \in \mathbb{N}} \) of real numbers with \( \psi_0 = 1 \), define recursively the weight \( \omega(t) \) of \( t \in \mathcal{F} \) as

\[
\omega(t) = \psi_k \prod_{i=1}^{k} \omega(t_i), \quad k = d(r), \quad \omega(t) = \prod_{v \in V} \psi_{d(v)}.
\]

where \( t_1, \cdots, t_k \) are the \( d(r) \) subtrees of \( t \) rooted at \( \text{ch}(r) \). Let \( \psi(z) := 1 + \sum_{k=1}^{\infty} \psi_k z^k \) be the generating function of the weight sequence \( \psi \). Our favourite example is \( \psi(z) = 1/(1 - z) \), with \( \omega(t) = 1, \forall t \) (see [11, 20] for various interesting choices).

We will be concerned with functionals \( B : \mathcal{F} \rightarrow \mathbb{R} \), where \( \mathcal{F} = \cup_n \mathcal{F}_n \), called bare Green functions. This terminology is taken from quantum field theory where bare Green functions occur during the action of the renormalization group (see for example [9], § 4.2 or [10], § 6.1). Let \( \mathcal{B} \) denote the set of bare Green functions. Any element \( B \in \mathcal{B} \) is given through a sequence of functions \( B_k : \mathbb{R} \rightarrow \mathbb{R}, k \in \mathbb{N}^+ \), which are usually Laurent series in some variable \( x \) (see for example [31]). In what follows, we simply write the sequence as \( \{B_k\}_{k \in \mathbb{N}^+} \).

**Definition 1** The bare Green function \( B \in \mathcal{B} \), \( B : \mathcal{F} \rightarrow \mathbb{R} \), associated
with the sequence of functions \( \{B_k\}_{k \in \mathbb{N}_+} \) is defined recursively as

\[
B(t) = B_{|t|} \prod_{i=1}^{k} B(t_i),
\]

where \( t_1, \ldots, t_k \) are the \( d(r) \) subtrees of \( t \) rooted at \( \text{ch}(r) \), and where \( |t| \) denotes the number of nodes of \( t \).

Notice that the value of \( B \) at \( t \in \mathcal{F} \) does not depend on the linear orders and is independent of the labellings. When dealing with rooted trees, we will adopt the notation \( t = B_+(t_1, \ldots, t_k) \) for the operation of grafting the rooted trees \( t_1, \ldots, t_k \), that is by considering the tree \( t \) obtained by the creation of a new node \( r \) (the root) and then joining the roots of \( t_1, \ldots, t_k \) to \( r \). Bare Green functions appeared also in the probabilistic literature in specific situations. The basic example, in algorithmics \([9, 18]\), in numerical analysis (see \([4, 5, 14]\)), in stochastics \([11, 20]\) and in physics (see for example \([6]\)) is the tree factorial, defined by

**Definition 2** Let \( t \in \mathcal{R} \) with \( t = B_+(t_1, \ldots, t_k) \). Then the tree factorial \( t! \) is the functional \( B \in \mathcal{B} \) defined by \( t! = |t| \prod_{i=1}^{k} t_i! \), associated with the sequence \( \{B_k\} \) given by \( B_k \equiv k \).

**Remark 3** It should be pointed out that the functional acting on trees, given as \( s(t) = \ln(B(t)) \), for \( B \in \mathcal{B} \) with \( B_k > 0, \forall k \geq 1 \), is an inductive map or an additive tree functional, as defined in \([17]\). Interestingly, \( B(t) = 1/t! \) is used in \([18]\) to define a probability measure on random search binary trees, and \([9, 11]\) provide precise asymptotics for \( \ln(t!) \).

### 3 Generating functions

We first give some basic results on tree factorials, symmetry factors, and generating functions associated with bare Green functions.

**Definition 4** Let \( t \in \mathcal{R} \). Then \( \alpha(t) \) is the number of rooted labelled trees \( t' \in \mathcal{L} \) of shape \( t \in \mathcal{R} \), where the shape of a labelled tree \( (r, V, E, l) \) is \( (r, V, E) \), \( \kappa(t) \) is the number of rooted plane trees of shape \( t \), and \( \sigma(t) \) is the symmetry factor of the tree, to be defined later. Moreover, let \( \omega_\mathcal{L} \) be the weight function associated with elements of \( \mathcal{L} \), with weights given by \( \psi_k \equiv 1/k! \).

Notice that \( \alpha(t) \) is the Connes-Moscovici weight in quantum field theory (see \([3, 7]\)). The symmetry factor satisfies the recursive definition:

\[
\sigma(\{r\}) = 1,
\]
\[ \sigma(B_{\lambda}(t_{i_1}^{n_1}, \cdots, t_{i_k}^{n_k})) = n_1! \sigma(t_1)^{n_1} \cdots n_k! \sigma(t_k)^{n_k}, \]

where the indices \( n_i \) means that \( t \) is obtained by grafting \( n_1 \) times the tree \( t_1 \), and so on, where we assume that the \( t_i \) are all different as rooted trees.

**Lemma 5** Let \( t \in \mathcal{R} \). Then

\[ \alpha(t) \sigma(t) = \frac{|t|!}{t!}, \quad (1) \]

and

\[ \alpha(t)t! = |t|! \omega_{\mathcal{L}}(t) \kappa(t). \quad (2) \]

**Proof:** (1) is well known (see for example [5]). Suppose that \( t \in \mathcal{R} \) is such that \( t = B_{\lambda}(t_{i_1}^{n_1}, \cdots, t_{i_k}^{n_k}) \), the grafting of \( n_1 \) times the tree \( t_1 \), and so on, where we set that the trees \( t_1, \cdots, t_k \) are different as rooted trees. Then

\[ \kappa(t) = \frac{(n_1 + \cdots + n_k)!}{n_1! \cdots n_k!} \kappa(t_1)^{n_1} \cdots \kappa(t_k)^{n_k}. \]

Using the recursive definition of \( \omega(t) \) and the definition of \( \omega_{\mathcal{L}} \), we have

\[ \omega_{\mathcal{L}}(t) = \frac{1}{(n_1 + \cdots + n_k)!} \omega_{\mathcal{L}}(t_1)^{n_1} \cdots \omega_{\mathcal{L}}(t_k)^{n_k}. \]

Therefore

\[ \frac{1}{\omega_{\mathcal{L}}(t) \kappa(t)} = n_1! \cdots n_k! (\frac{1}{\omega_{\mathcal{L}}(t_1) \kappa(t_1)})^{n_1} \cdots (\frac{1}{\omega_{\mathcal{L}}(t_k) \kappa(t_k)})^{n_k}, \]

and the results follows from the recursive definition of the symmetry factor.

\[ \square \]

Then

\[ \sum_{t \in \mathcal{F}_n} B(t) \omega(t) = \sum_{t \in \mathcal{R}_n} B(t) \omega_{\mathcal{L}}(t) \alpha(t) \frac{t!}{|t|!}, \quad (3) \]

where we have used (2) of Lemma 5.

Consider the generating function

\[ Y(z) = \sum_{n \in \mathbb{N}^+} \sum_{t \in \mathcal{R}_n} \alpha(t) B(t) t! \omega(t) / \omega_{\mathcal{L}}(t). \quad (4) \]

Given \( t \in \mathcal{R} \), the ratio \( \omega / \omega_{\mathcal{L}} \) is associated with the weight sequence \( \tilde{\psi}_k \equiv \psi_k k! \); using the expansion \( \tilde{\psi}(z) = 1 + \sum_{k \geq 1} \psi_k z^k = 1 + \sum_{k \geq 1} (\psi_k / k!) z^k \), we see that \( \tilde{\psi}_k \equiv \psi^{(k)}(0) \). Consider the *elementary differentials* \( \delta \) (see Section 4) defined by
Definition 6

\[ \delta_{\{s\}} = 1, \quad \delta_t = \psi^{(k)}(0) \prod_{i=1}^{k} \delta_{t_i}, \quad \frac{\omega}{\omega_L} = \delta, \]

when \( t = B_+(t_1, \cdots, t_k) \), where * denotes the tree of a single node. For a map \( a : \mathbb{R} \cup \{\emptyset\} \to \mathbb{R} \), a formal power series of the form \( Y(z) = a(\emptyset)y_0 + \sum_{t \in \mathcal{R}} z^{[t]} a(t) \delta_t \alpha(t)/|t|! \) is called a B-series \([13, 14]\).

Remark 7 When \( B(t) = t! \), the series \( Y \) is given by

\[ Y(z) = \sum_{t \in \mathcal{L}} (\omega(t)/\omega_L(t)) z^{[t]}/|t|!. \]

Set \( \phi_k = \psi_k k!, \forall k \), and consider the degree function \( \phi(z) = 1 + \sum_{k \geq 1} (\phi_k / k!) z^k \).

Following \([20]\), \( Y \) solves \( Y' = \phi(Y) \) (see also \([20]\)). We shall see in the next section that it is a natural consequence of B-series expansions of solutions of ordinary differential equations.

4 Runge-Kutta methods for functionals over trees

Consider a dynamical system on \( \mathbb{R} \)

\[ \frac{d}{ds} X(s) = F(X(s)), \quad X(s_0) = X_0, \]

for some smooth \( F : \mathbb{R} \to \mathbb{R} \). The solution of this dynamical system has a B-series expansion of the form

\[ X(s) = X_0 + \sum_{t \in \mathcal{R}} \frac{(s - s_0)^{|t|}}{|t|!} \alpha(t) \delta_t(s_0), \]

where the elementary differentials \( \delta \) is defined recursively by

\[ \delta_{\{s\}} = f(s_0), \quad \delta_{t} = \frac{\partial^k F}{\partial s^{k_1} \partial t_1 \cdots \partial t_k}, \]

when \( t = B_+(t_1, \cdots, t_k) \). These kinds of expansions have been treated in great detail in \([4]\) and \([5]\) and developed independently in combinatorics (see for example \([16, 17]\)). Suppose that \( s_0 = 0 \) for simplicity. Butcher considered what happens with numerical approximations of the exact solution, the Runge-Kutta methods, which are themselves B-series \([13, 14]\); here we focus on specific B-series, which are associated to bare Green functions. Let \( B \in \mathcal{B} \) be such that there exists a power series

\[ L(z) = \sum_{m \geq 0} L_m z^m, \]
with

\[ B_k = \frac{L(k)}{k!}, \forall k \in \mathbb{N}^+. \]

Bare Green functions satisfying (5) are used in practical situations in quantum field theory (see 3, § 4 and 6, § 6.1). Consider Euler’s operator \( \theta = z(d/dz) \), with \( P(\theta)(z^n) = P(n)z^n, \forall n \in \mathbb{N} \), for any polynomial \( P \), and consider the formal operator \( L(\theta + 1) \) acting on monomials as

\[ L(\theta + 1)(z^n) = \sum_{m \geq 0} L_m (\theta + 1)^m (z^n) = \sum_{m \geq 0} L_m (n + 1)^m z^n = L(n + 1)z^n. \]

Given a power series \( Y(z) = \sum_{m \geq 0} a_m z^m \) converging for \( |z| \leq 1 \), we can define \( L(\theta + 1)(Y)(z) := \sum_{m \geq 0} a_m L(m + 1)z^m \), which converges for \( |z| \leq 1 \) when the sequence \( (L(k))_{k \geq 1} \) grows subexponentially. We will not focus on convergence questions here, and work at the formal level. Let \( B \) be a bare Green function with weights \( (B_k)_{k \geq 1} \), such that (5) holds for some power series \( L \). It should be pointed out that 3, 6, 15 deal with the master function \( L \), but do not give explicitly an equation for \( Y \). The next Theorem provides an equation; its proof uses explicitly B-series.

**Theorem 1** The formal power series

\[ Y(z) = \sum_{t \in \mathcal{R}} \frac{z^{|t|}}{|t|!} \alpha(t)! B(t) \delta_t, \]

solves \( Y' = L(1 + \theta) \psi(Y) \).

**Proof:**

\[ \psi(Y(z)) = \sum_{k \geq 0} \frac{\psi^{(k)}(0)}{k!} \sum_{(t_1 \ldots t_k) \in \mathcal{R}^k} z^{\sum_i |t_i|} \prod_{i=1}^k \alpha(t_i) B(t_i) t_i! \delta_t. \]

For given \( (t_1 \ldots t_k) \in \mathcal{R}^k \), set \( t = B_+(t_1, \ldots, t_k) \). Then \( \sum_i |t_i| = |t| - 1 \), \( \psi^{(k)}(0) \delta_{t_1} \cdots \delta_{t_k} = \delta_t \), and \( B(t_1) \cdots B(t_k) = B(t)/B_{|t|} \). The associated term becomes

\[ z^{|t|-1} \frac{B(t)}{B_{|t|}} \delta_t \alpha(t_1) \cdots \alpha(t_k) \frac{t!}{|t_1|! \cdots |t_k|!}. \]

Next, every rooted tree \( t \in \mathcal{R} \) can be decomposed uniquely as \( t = B_+ (t_1^{m_1}, \ldots, t_m^{m_m}) \), meaning that \( t \) is obtained by grafting \( n_1 \) times \( t_1 \) and so on, where the \( t_i \) are different as rooted trees, with \( k = n_1 + \cdots + n_m \). Collecting the terms associated with \( t \), we get the contribution

\[ \frac{z^{|t|-1}}{k!} \frac{B(t)}{B_{|t|}} \delta_t \frac{t!}{|t|} \sum_{(t_1' \cdots t_k') \in \mathcal{R}^k} \alpha(t_1') \cdots \alpha(t_k') \frac{|t_1'|! \cdots |t_k'|!}{|t'_1|! \cdots |t'_k|!}. \]
where * means that the sum is taken over all the collections \((t'_1 \cdots t'_k) \in \mathcal{R}^k\) such that \(t = B_+(t'_1, \ldots, t'_k)\). The above sum reduced then to
\[
\frac{(n_1 + \cdots + n_m)! \alpha(t_1)^{n_1} \cdots \alpha(t_m)^{n_m}}{n_1! \cdots n_m!} \cdot \frac{1}{k! |t_1|! n_1! \cdots |t_m|! n_m!} \cdot \frac{1}{\sigma(t_1)^{n_1} \cdots \sigma(t_m)^{n_m}},
\]
where we use the first identity of Lemma \(\text{A}\). Using the recursive definition of the symmetry factor \(\sigma\), we obtain
\[
\sum_{(t'_1 \cdots t'_k) \in \mathcal{R}^k} \frac{\alpha(t'_1) \cdots \alpha(t'_k)}{k! |t'_1|! \cdots |t'_k|!} = \frac{1}{t_1! n_1! \cdots t_m! n_m! \sigma(B_+(t'_1, \ldots, t'_k))} = \frac{|t|}{t! \sigma(t)} = \frac{|t| \alpha(t)}{|t|!}.
\]
We thus get that the contribution associated with \(t \in \mathcal{R}\) is given by
\[
z^{|t| - 1} B(t) \frac{\alpha(t)!}{B_{|t|}} \frac{\delta_t}{|t|} = \frac{|t| - 1}{B_{|t|}} \frac{z^{|t| - 1} B(t) \alpha(t) \delta_t}{|t|!}.
\]
Therefore
\[
L(\theta + 1) \psi(Y) = \sum_{t \in \mathcal{R}} \frac{B(t)}{B_{|t|}} \frac{\alpha(t)! \delta_t}{|t|} \frac{L(\theta + 1)(z^{|t| - 1})}{|t|!}
\]
\[
= \sum_{t \in \mathcal{R}} \frac{B(t)}{B_{|t|}} \frac{\alpha(t)! \delta_t}{|t|} \frac{L(|t|) z^{|t| - 1}}{|t|!}
\]
\[
= \sum_{t \in \mathcal{R}} \frac{B(t)}{B_{|t|}} \frac{\alpha(t)! \delta_t}{|t|} \frac{|t|!}{|t|!} \frac{B(t) |t|! \delta_t}{|t|!} = \sum_{t \in \mathcal{R}} \frac{z^{|t| - 1}}{|t|!} B(t) t! \delta_t = \frac{dY}{dz}.
\]

Remark 8 As we have observed in Remark \(\text{A}\), \(s(t) = \ln(B(t))\) is an inductive map when the weights \(B_k\) are positive. It turns out that the exponential generating function associated with \(s\) can be given as an integral transform for varieties of increasing trees (see for example Section \(\text{B}\)). This is the topic of \(\text{C}\).

Example 9

When \(L(z) = z\), with \(B_k \equiv 1\), and \(\psi(z) = 1/(1 - z)\), one has \(\sum_{t \in \mathcal{F}_n} B(t) = C_n\), the Catalan number of order \(n\), with \(C_n = \binom{2n}{n}/(n + 1)\). Then \(Y(z) = \)
\[ z \sum_{n \geq 0} z^n C_n \text{ is solution of the differential equation } Y''(z) = L(1 + \theta)(1/(1 - Y(z))), \text{ that is } Y''(z) = \left(z/(1 - Y(z))\right)' \]. The unique solution with \( Y(0) = 0 \) satisfies \( Y(z) = z/(1 - Y(z)) \), or \( Y(z) = (1 - \sqrt{1 - 4z})/2 \), corresponding to a well known result.

\[ \Box \]

[6], § 5.3, considers the case where \( B(t) = (1/t!)^2 \), which is not of the form given in [5]; in this situation, \( B_k = 1/k^2 \), with \( L(z) = 1/z \). The solution is obtained by using the structure of the so-called Butcher’s group of B-series (that is series of the form [6], where the group structure in given in [13, 14]) by tensoring known B-series:

**Example 10**

Consider the bare functional given by \( B_k \equiv 1/k^2 \), with \( B(t) = 1/t^2 \). Following Brouder, the associated B-series, as given in [6], is solution of the second order differential equation

\[ zY'' + Y' = \psi(Y). \]

When \( \psi(z) = \exp(z) \), the solution is given by

\[ Y(z) = -2 \ln(1 - z/2) = \sum_{n=1}^{\infty} \frac{z^n}{n 2^{n-1}}, \]

giving

\[ \sum_{t \in \mathbb{R}_n} \frac{\alpha(t)}{t!} = \frac{(n - 1)!}{2^{n-1}}. \]

\[ \Box \]

We study the general moment problem \( B(t) = (1/t!)^{l+1}, l \in \mathbb{N} \), by working directly on a suitable differential equation as follows: the operator \( L(\theta + 1) \) takes the form \( L(\theta + 1) = 1/(\theta + 1)^l \). Assume that the differential operator \( L(\theta + 1) \) is invertible. Then the formal systems becomes

\[ L(\theta + 1)^{-1} \frac{d}{dz} Y = \psi(Y). \] (7)

Consider again the second moment problem for the tree factorial, with \( B_k \equiv 1/k^2 \) and \( L(k) \equiv 1/k \). Choose \( L \) such that \( L(z) = 1/z \); the inverse operator might be equal to \( L(\theta + 1)^{-1} = \theta + 1 \), and, if this is the case,

\[ (\theta + 1) \frac{d}{dz} Y = \psi(Y), \]

with \((\theta + 1)(d/dz) = z(d^2/dz^2) + (d/dz)\), see Example [10].
More generally, if one considers the moment of order \( l + 1 \in \mathbb{N} \) of the inverse tree factorial, the choice \( L(k) = 1/k^l \) should give \( (\theta + 1)^l \frac{d}{dz} Y(z) = \psi(Y) \). Our result, Theorem 2 below shows that the formalism of inversion is correct in term of power series. This result sheds some light and extends the computations done in [6] for the second moment, and its proof avoids computations in the Butcher’s group.

**Theorem 2** The B-series \( Y(z) \) associated with the moment of order \((l + 1)\) of the inverse tree factorial satisfies the differential equation

\[
(\theta + 1)^l \frac{d}{dz} Y = \psi(Y).
\]

**Proof:** Let

\[
Y(z) = \sum_{t \in \mathcal{R}} \frac{z^{|t|}}{|t|!} \alpha(t) \frac{1}{t!} \delta_t.
\]

Then

\[
\psi(Y(z)) = \sum_{k \geq 0} \frac{\psi^{(k)}(0)}{k!} \sum_{(t_1, \ldots, t_k) \in \mathcal{R}^k} \frac{z^{\sum_i |t_i|}}{|t_1|! \cdots |t_k|!} \frac{\alpha(t_1) \cdots \alpha(t_k)}{(t_1! \cdots t_k!)^l} \delta_{t_1} \cdots \delta_{t_k}.
\]

For given \((t_1 \cdots t_k) \in \mathcal{R}^k\), set \( t = B_+(t_1, \ldots, t_k) \). Then \( \sum_i |t_i| = |t| - 1 \), \( \psi^{(k)}(0) \delta_{t_1} \cdots \delta_{t_k} = \delta_t \), and \((t_1! \cdots t_k!)^l = t^l/|t|^l \). The associated term becomes

\[
\frac{z^{|t|-1} |t|^l}{t^l} \frac{|t| \alpha(t)}{|t|!} \delta_t = \frac{z^{|t|-1} |t|^l}{(|t| - 1)!} \frac{|t|^l \alpha(t)}{t!} \delta_t.
\]

Proceeding as in the proof of Theorem 1, we get that the contribution associated with \( t \in \mathcal{R} \) is given by

\[
\frac{z^{|t|-1} |t|^l}{t!} \frac{|t| |t| \alpha(t)}{|t|!} \delta_t = \frac{z^{|t|-1} |t|^l}{(|t| - 1)!} \frac{|t|^l \alpha(t)}{t!} \delta_t.
\]

On the other hand,

\[
(\theta + 1)^l \frac{d}{dz} Y(z) = (\theta + 1)^l \sum_{t \in \mathcal{R}} \frac{z^{|t|-1}}{(|t| - 1)!} \alpha(t) \frac{1}{t!} \delta_t
\]

\[
= \sum_{t \in \mathcal{R}} \frac{|t|^l z^{|t|-1}}{(|t| - 1)!} \alpha(t) \frac{1}{t!} \delta_t.
\]

\[\square\]

In the next section, we show that traces of certain products of Wigner matrices (see for example [24]) provide natural examples of bare Green functions.
5 Wigner processes

Definition 11 The $N$-dimensional random matrices $\Gamma_N := (\gamma_{i,j})_{1 \leq i,j \leq N}$ are called Wigner matrices of variance $\beta^2$ if the following holds.

- Each $\Gamma_N$ is symmetric, that is, $\gamma_{i,j} = \gamma_{j,i}$.
- For $i \leq j$, the random variables $\gamma_{i,j}$ are independent and centered.
- For $i \neq j$, $\mathbb{E}(\gamma_{i,j}^2) = \beta^2$.
- For any $k \geq 2$, $\mathbb{E}(|\gamma_{i,j}|^k) \leq c_k$, where $c_k$ is independent of $i \leq j$.

Definition 12 The sequence $\Gamma_N(k) := (\gamma_{i,j}(k))_{1 \leq i,j \leq N}$ of $N$-dimensional random matrices, indexed by $k \geq 1$, is called a Wigner process of variance $\beta^2$ and correlation function $r$, $r(k,k) = 1$, $|r(k,m)| \leq 1$ and $r(k,m) = r(m,k)$ if the following holds.

- Each $\Gamma_N(k)$ is a Wigner matrix of variance $\beta^2$ in the sense of definition 11.
- For $i \leq j$, each process $(\gamma_{i,j}(k))_k$ is independent of the others.
- For $i \neq j$, the process $(\gamma_{i,j}(k))_k$ is $r$-correlated, that is, for any $k \geq m$,
  \[ \mathbb{E}(\gamma_{i,j}(k)\gamma_{i,j}(m)) := \beta^2 r(k,m). \]
  \[ (9) \]

A Wigner process is stationary when $r$ is such that $r(k,m) = r(|k-m|)$.

Let $D_N$ be a sequence of random diagonal matrices, with independent and identically distributed entries of law $\mu$, having finite moment $\mu_k = \mu(X^k)$, $k \geq 1$, with $\mu_1 = 1$. Let

\[ Q_N^k := N^{-k/2}D_N \prod_{m=1}^{k} (\Gamma_N(m)D_N), \]

and set
\[ B_N^k(r) = N^{-1}\mathbb{E}(\text{tr}(Q_N^k)). \]

Involutions, Dyck paths and rooted plane trees

For $k \geq 1$, $[k] := \{1,2,\ldots,k\}$, $\mathcal{I}(k)$ is the set of the involutions of $[k]$ with no fixed point, $\mathcal{J}(k)$ is the subset of $\mathcal{I}(k)$ of the involutions $\sigma$ with
no crossing. This means that the configurations \(i < j < \sigma(i) < \sigma(j)\) do not appear in \(\sigma \in \mathcal{J}(k)\). Let \(i \in \text{cr}(\sigma)\) denote the fact that \(i < \sigma(i)\). Let \(\mathcal{D}(2k)\) be the set of the Dyck paths of length \(2k\), that is, of the sequences \(c := (c_n)_{0 \leq n < 2k}\) of nonnegative integers such that \(c_0 = c_{2k} = 0, c_n - c_{n-1} = \pm 1, n \in [2k]\). Thus, exactly \(k\) indices \(n \in [2k]\) correspond to ascending steps \((c_{n-1}, c_n)\), that is, to steps when \(c_n = c_{n-1} + 1\). We denote this by \(n \in \text{asc}(c)\). The other \(k\) indices correspond to descending steps, that is, to steps when \(c_n = c_{n-1} - 1\), and we denote this by \(n \in \text{desc}(c)\). We make use of bijections between \(\mathcal{D}(2k)\) and \(\mathcal{J}(2k)\) \([2]\). If \(c \in \mathcal{D}(2k)\), \(\phi(c) := \sigma \in \mathcal{J}(2k)\) is an involution which maps each element of \(\text{desc}(c)\) to a smaller element of \(\text{asc}(c)\). Thus, \(\text{cr}(\sigma) = \text{asc}(c)\). More specifically, if \(n \in \text{desc}(c)\), \(\sigma(n)\) is the greatest \(m \leq n\) such that \((c_{m-1}, c_m) = (c_n, c_{n-1})\). Finally, the set \(\mathcal{D}(2k)\) is in bijection with \(\mathcal{F}_{k+1}\), the set of rooted plane trees on \(k+1\) nodes, where the bijection is given by the walk on the tree from the right to the left (see for example \([25]\)). Let \(\sigma_t\) denote the involution of \(\mathcal{J}(2(|t| - 1))\) corresponding to \(t \in \mathcal{F}\). Given \(t \in \mathcal{F}_{k+1}\), consider the walk on \(t\) from the right to the left: every edge \((v, w)\) with \(w \in \text{ch}(v)\), is crossed at some instant \(s_v \in [2k]\) as \((v \to w)\) and at a later time \(s_w \in [2k]\) as \((w \to v)\). Clearly, \(s_w = s_v + 2(|t_w| - 1) + 1\), where \(t_w\) is the subtree of \(t\) rooted at node \(w\), that is the subgraph of \(t\) induced by the nodes \(u\) with \(u \geq w\). \(\sigma_t\) is such that \(\sigma_t(s_v) = s_w\) and vice versa.

![Fig 1. Bijections between \(\mathcal{F}_{k+1}, \mathcal{D}(2k)\) and \(\mathcal{J}(2k)\) (\(s_v = 2\) and \(s_w = 5\)).](image)

**Proposition 13** Assume that the covariance \(r\) is such that \(r(l, m) = r(|l-m|)\). Then, the functional \(B^r \in \mathcal{B}\) given by the weights

\[
B^r_k = \beta^2 r(2k-1), \forall k \geq 1,
\]

is such that

\[
\frac{B^r(t)}{B^r_{|t|}} = \prod_{i \in \text{cr}(\sigma_t)} (\beta^2 r(i, \sigma_t(i))).
\]
Proof: Let \( t \in \mathcal{F} \). Let \( s_v < s_w \) be the instants where the oriented edges \((v \to w)\) and \((w \to v)\), \( w \in \text{ch}(v) \), are crossed during the walk on the tree. \( r(s_v, \sigma_l(s_v)) = r(s_w - s_v) = r(2|t_w| - 1) \), and thus \( \beta^2 r(s_v, \sigma_l(s_v)) = B^r_{|t_w|} \).

Finally, \( \prod_{l \in \text{cr}(\sigma_l)} \beta^2 r(i, \sigma_l(i)) = \prod_{w \neq \text{root}} B^r_{|t_w|} = B^r(t) / B^r_{|t|} \), as required.

\[ \square \]

As we have just seen, every Wigner process with covariance \( r \) such that \( r(l, m) = r(|l - m|) \) produces a bare Green function \( B^r \in \mathcal{B} \). The converse is not true, that is, there exists \( B \in \mathcal{B} \) such that \( B \) is not of the form \( B = B^r \) for some covariance function \( r \). Set \( \mathcal{B}^w = \{ B \in \mathcal{B} ; \exists \) a covariance \( r \) with \( B = B^r \} \).

Let \( \psi_\mu \) be the generating function of the weight sequence \( \psi_k = \mu_{k+1} \), and let \( \omega_\mu(t) = t \in \mathcal{F} \) be the associated weight function.

**Theorem 3** Let \( (\Gamma_N(k))_{k \geq 1} \) be a stationary Wigner process of covariance function \( r \) and variance \( \beta^2 \), and let \( D_N \) be a sequence of random diagonal matrices, independent of the Wigner process, with i.i.d. entries \( \lambda_j \) of law \( \mu \), with \( \mu_1 = \mu(\lambda) = 1 \) and finite moments \( \mu_k = \mu(\lambda^k) , \forall k \). Then

\[
B^{N(2)}_{2k}(r) \longrightarrow B_{2k}(r) = \frac{1}{B_{k+1}^r} \sum_{t \in \mathcal{F}_{k+1}} B^r(t) \omega_\mu(t),
\]

and \( B^{N(2)}_{2k+1}(r) \longrightarrow 0 \), \( N \to \infty \). Assume that the covariance is such that there exists a power series \( L^r(z) \) with \( B_k^r = L^r(k)/k \), \( \forall k \). Then the formal power series

\[
Y(z) = \sum_{k \geq 1} z^k B_k^r B_{2(k-1)}(r),
\]

solves

\[
Y' = L^r(\theta + 1) \psi_\mu(Y).
\]

Moreover

\[
\sum_{k \geq 1} z^k B_{2(k-1)}(r) = z \psi_\mu(Y).
\]

(10)

**Example 14**

Let \( B(t) = 1/t! \). If a tree \( t \) has \( n \) nodes and \( n - 1 \) edges, then the requirement \( B_n = 1/n \) is satisfied if \( \beta^2 r(2n - 1) = 1/n \), that is \( r \) must be such that \( \beta^2 r(k) = 2/(k + 1) \), \( k \in 2\mathbb{N} + 1 \). By construction, \( r(0) = 1 \) and therefore \( \beta^2 = 2. 1/(x + 1) \) is positive definite, which implies that \( B(t) = 1/t! \) is element of \( \mathcal{B}^w \). Next, from Theorem 2, the generating function \( Y(z) = \sum_{t \in \mathcal{F}} z^{|t|} B^r(t) \omega_\mu(t) \) is solution of the system \((d/dz)Y(z) = \psi_\mu(Y)\).
Assume that $\mu$ is the point mass $\delta_1$, that is each matrix $D_N$ is the identity matrix of size $N$, with $\psi_\mu(z) = 1/(1 - z)$. The solution of the system is $Y(z) = 1 - \sqrt{1 - 2z} = 2\hat{Y}(z/2)$, where $\hat{Y}$ is the series given in Example 9. On the other hand, Proposition 13 and Theorem 3 show that $Y(z) = \sum_{k\geq 1} \varepsilon^k B_k^2 B_{2(k-1)}(r)$. Therefore the limiting mean normalized trace $B_{2k}(r)$ of the product of correlated random matrices $\prod_{m=1}^{2k} \Gamma_N(m)$ is such that $B_{2k}(r) = E(Z^{2k})/k!$, where $Z$ denotes a normal $N(0,1)$ random variable.

\[\blacksquare\]

**Example 15**

Consider as in Example 9 the special case where $L(z) = z$. The associated inductive parameter (see Remark 8) is the tree size. The covariance $r$ is constant with $r(k) \equiv 1$, and $B_k^2 \equiv 1$. Then the generating function $Y$ is solution of the fixed point equation $Y(z) = z\psi_\mu(Y(z))$ (either by Theorem 11 or by (10)). Notice that in this situation, $\Gamma_N(m) \equiv \Gamma_N(1)$, and thus $B_k^N(r)$ describes the mean normalized moment of the spectral measure of the random matrix $D_N(\Gamma_N(1)D_N)^k$. This example can be extended by considering $L(z) = z\rho^2$, for some $0 < \rho \leq 1$. When $D_N$ is the identity matrix, $Y(z)$ is related to the Rogers-Ramanujan continued fraction [19], and corresponds to the generating function associated with path length, see [11 25].

**Proof of Theorem 3** The first part is a generalization of Theorem 1 of [19]. Set $\tilde{\gamma}_{ij}(m) = \gamma_{ij}(m)\lambda_j$, and $\tilde{\Gamma}_N(m) = \Gamma_N(m)D_N$. The mean normalized trace adds the contributions $E(i) = E(\lambda_0 \tilde{\gamma}_{i_0 i_1} \cdots \tilde{\gamma}_{i_{k-1} i_k})$, for paths $i = (i_0)_{0\leq l \leq k}$, with $i_l \in [N]$ and $i_0 = i_k$. The $\tilde{\gamma}_{ij}$ are centered, so that any edge $(i, j)$ appearing once appears at least twice. Given $i$, define $\varepsilon_l = 1$ and $\varepsilon_l = -1$ otherwise, and consider the walk $c = (c_l)$ defined by $c_l = \sum_{j=1}^l \varepsilon_j$, with $c_k \leq 0$. The support of $i$ is $s(i) = \{i_l; 0 \leq l \leq k\}$, of size $s = |s(i)|$, with $s \leq 1 + k/2$. The contribution $E(i)$ is independent of the labels $i_l$; they are $N(N-1) \cdots (N-s+1)$ labellings giving the same walk $c$, with the same contribution. Thus, the normalization $N^{-(1+k/2)}$ shows that the only walks surviving in the large $N$ limit are those with $s = 1 + k/2$. This shows that $B_k^N(r) \to 0$ when $k$ is odd. Concerning $B_k^N(r)$, $s = 1 + k$ means that every edge occurring in the path occurs exactly twice, in opposite directions. $c$ is a Dyck path of $D(2k)$; let $t \in F$ be the associated rooted plane tree, with involution $\sigma_t$. Using the right to left walk on $t$ and the independence of the random variables, the contribution $E(i)$ of any path leading to $c$ or $t$ is $E(i) = \prod_{m \in \text{cr}(\sigma_t)} E(\gamma(m)\gamma(\sigma_t(m))) E(\prod_v \lambda_v^{d(v)+1})$ where $d(v) = |\text{ch}(v)|$. [14]
From Proposition 13, one obtains
\[ E(i) = (B^r(t)/B^r_{k+1}) \prod_v \mu_{d(v)+1}, \]
with \( B_{2k}(r) = \sum_{t \in \mathcal{F}_{k+1}} (B^r(t)/B^r_{k+1}) \prod_v \mu_{d(v)+1}, \) as required. (10) is a consequence of the multiplicative form of bare Green functions and of Lemma 1.9, chap. III.1 of [14].

These results show that the elements of \( B^w \) appear naturally in the computation of normalized traces of products of large random matrices (see for example [23]). In the next Section we illustrate B-series by considering triangular operators from free probability.

6 On Dykema-Haagerup triangular operator

Let \( B \) be an algebra and \( A \) be a \( B \) bi-module. Let \( \kappa : A \times A \to B \) be a bilinear map. We follow [22] by defining the product \( a_1 \cdot_\kappa a_2 = \kappa(a_1, a_2) \), \( a_1, a_2 \in A \), and setting
\[
i) \quad (ba_1) \cdot_\kappa a_2 = b(a_1 \cdot_\kappa a_2),
\]
\[
ii) \quad (a_1 b) \cdot_\kappa a_2 = a_1 \cdot_\kappa (ba_2),
\]
\[
iii) \quad a_1 \cdot_\kappa (a_2 b) = (a_1 \cdot_\kappa a_2)b.
\]

Let \( \sigma \in \mathcal{J}(2n) \) be an involution of \( [2n] \) without fixed point and without crossing. Given a word \( a = a_1 \cdots a_{2n} \) in \( A \), \( \sigma \) induces parentheses on \( a \), and the proceedings rules permit the evaluation of this parenthesized word. This extends to a map \( \kappa_\sigma \) on \( A^{2n} \). Sniady defines such maps to prove a conjecture of Dykema and Haagerup on generalized circular elements. Let \( (B \subset A, E) \) be an operator valued probability space, that is \( A \) is a unital *-algebra, \( B \subset A \) an unital *-subalgebra and \( E : A \to B \) be a conditional expectation (linear, \( E(1) = 1 \), and \( E(b_1ab_2) = b_1E(a)b_2, \forall b_1, b_2 \in B, a \in A \)).

**Definition 16** \( T \in A \) is a generalized circular element if there is a bilinear map \( \kappa \) satisfying the rules i), ii) and iii) such that
\[
E(b_1T^{s_1}b_2T^{s_2} \cdots b_{2n}T^{s_{2n}}) = \sum_{\sigma \in \mathcal{J}(2n)} \kappa_\sigma(b_1T^{s_1}, \cdots , b_{2n}T^{s_{2n}}),
\]
\[
E(b_1T^{s_1}b_2T^{s_2} \cdots b_{2n+1}T^{s_{2n+1}}) = 0,
\]
\( \forall b_1, \cdots , b_{2n+1} \in B \) and \( \forall s_1, \cdots , s_{2n+1} \in \{1, \ast\} \).

The triangular operator \( T \) of Dykema and Haagerup is obtained from \( B = \mathbb{C}[x] \), the *-algebra of complex polynomials of one variable by setting
\[
[k(T, bT^\ast)](x) = \int_x^1 b(s)ds,
\]
\[ [\kappa(T^*, bT)](x) = \int_0^x b(s)ds, \]

\[ [\kappa(T, bT)](x) = [\kappa(T^*, bT^*)](x) = 0. \]

\( T \) is the limit for the convergence of \(*\)-moments of large upper triangular random matrices \( T_N \) \( \textbf{[10]} \). Define a trace \( \tau \) as \( \text{see } [22] \)

\[ \tau(a) = \tau(E(a)), \quad \tau(b) = \int_0^1 b(s)ds. \]

In what follows, we use P-series (where P stands for partitioned differential systems, see \[13\]). We follow \[6\], and adapt his notations to P-series. Given some function \( \psi \), and two kernels \((a^x(u, v))_{u, v \in [0, 1]}\) and \((a^y(u, v))_{u, v \in [0, 1]}\), consider the iterated integrals \( \phi^x_u \) and \( \phi^y_v \) which are functionals over \( \mathcal{R} \) defined by \( \phi^x_u(\ast) = \phi^y_v(\ast) = 1 \), and, for \( t = B_+(t_1, \cdots, t_k) \),

\[ \phi^x_u(t) = \prod_{i=1}^k \int_0^1 a^x(u, v)\phi^y_v(t_i)dv, \]

\[ \phi^y_v(t) = \prod_{i=1}^k \int_0^1 a^y(u, v)\phi^x_u(t_i)dv. \]

**Lemma 17** Let \( a^x(u, v) = I_{[0, u]}(v) \) and \( a^y(u, v) = I_{[u, 1]}(v) \). Then

\[ \tau(TT^n) = \sum_{t \in \mathcal{P}_{n+1}} \int_0^1 \phi^x_v(t)dv = \sum_{t \in \mathcal{P}_{n+1}} \int_0^1 \phi^y_v(t)dv. \]

**Proof:** The word \( W = (TT^*) \cdots (TT^*) \) is of the generic form with \( b_1 = \cdots = b_{2n} = 1 \) (Definition 16). Let \( t \in \mathcal{P}_{n+1} \) with associated involution \( \sigma_t \) (see Section 6). Let \( s_v \) and \( s_u \) be the instants where the walk on \( t \) crosses the oriented edges \((v \rightarrow w)\) and \((w \rightarrow v)\), with \( w \in \text{ch}(v) \). We colour these edges by giving colour \( '1' \) to \((v \rightarrow w)\) when the symbol in \( W \) located at position \( s_v \) is \( T \), and give the colour \( '*' \) otherwise. Clearly, both edges have different colours, and the elements of the set of edges \( \{(v \rightarrow w); w \in \text{ch}(v)\} \) (the children of \( v \) in \( t \)) have the same colour. The result is then a consequence of the definition of the product with the rules i), ii) and iii).

\[ \square \]

**Remark 18** Iterated integrals are natural objects to consider in the setting of Butcher’s Theory. For example, in the framework of Theorem 17, the iterated integrals \( \phi_u(t) \) defined by \( \phi_u(t) = \prod_{i=1}^k \int_0^u L(\theta + 1)(\phi_v(t_i))dv \), when \( t = B_+(t_1, \cdots, t_k) \), are such that \( \phi_1(t) = B(t) \), \( \forall t \in \mathcal{F} \).
Proposition 19  The P-series

\[ X_u(s) = X_0 + \sum_{t \in \mathbb{R}} \frac{s^{|t|}}{|t|!} \alpha(t) t! \delta t \int_0^1 a^x(u, v) \phi^y_v(t) dv, \]

and

\[ Y_u(s) = Y_1 + \sum_{t \in \mathbb{R}} \frac{s^{|t|}}{|t|!} \alpha(t) t! \delta t \int_0^1 a^y(u, v) \phi^x_v(t) dv, \]

are solutions of the integral system

\[ X_u(s) = X_0 + s \int_0^1 a^x(u, v) \psi(Y_v(s)) dv, \]

\[ Y_u(s) = Y_1 + s \int_0^1 a^y(u, v) \psi(X_v(s)) dv. \]

Proof: This is consequence of Butcher’s general theory (see [4]). To prove it more directly, proceed as in the proof of Theorem 17.

Corollary 20  Let \( X_0 = Y_1 = 0 \). Assume that \( a^x(u, v) = I_{[0, u]}(v) \) and \( a^y(u, v) = I_{[u, 1]}(v) \). Suppose that \( \psi(z) = 1/(1 - z) \). Then

\[ Y_0(s) = \sum_{t \in \mathbb{F}} \frac{s^{|t|}}{|t|!} \alpha(t) t! \delta t \int_0^1 \phi^x_v(t) dv = \sum_{t \in \mathbb{F}} s^{|t|} \tau((TT^*)^{|t|-1}). \]

This result shows that the generating function of the *-moments of the operator \( TT^* \) can be obtained by solving the system given in Proposition 19. We recover in this way a result of [10], Lemmas 8.5 and 8.8.

Lemma 21  In the setting of Corollary 20, the generating function \( Y_0(s) \) solves

\[ G\left( \frac{s}{1 - Y_0(s)} \right) = s, \]

where \( G(z) = z \exp(-z) \), that is, \( L(s) = s/(1 - Y_0(s)) \) and \( G \) are inverse with respect to composition. Moreover \( \tau(TT^*)^n = n^n/(n + 1)! \).

Proof: We solve the integral system by looking for solutions of the form \( X_u(s) = 1 - \exp(\lambda u) \) and \( Y_u(s) = 1 - \exp(\lambda'(u - 1)) \), with \( (d/du)X_u(s) = s/(1 - Y_u(s)) \) and \( (d/du)Y_u(s) = -s/(1 - X_u(s)) \). We deduce that \( \lambda' = -\lambda \) is solution of the equation \( \lambda + s \exp(-\lambda) = 0 \). The formula for the moments of \( TT^* \) is a consequence of Lagrange’s inversion formula.

□
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References


