Cluster X-varieties for dual Poisson-Lie groups II

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Abstract

In the prequel of this paper, we have associated a family of cluster X-varieties to the dual Poisson-Lie group $(G^*, \pi_i^*)$ of $(G, \pi_G)$ when $(G, \pi_G)$ is a complex semi-simple Lie group of adjoint type, given with the standard Poisson structure $\pi_G$ and $\pi_i^*$ is the "dual" Poisson structure defined by the Semenov-Tian-Shansky Poisson bracket on $G$. We describe here the cluster combinatorics involved into the Artin group action on $G^*$ given by the De-Concini-Kac-Procesi Poisson automorphisms.

Reference

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Abstract. In the prequel of this paper, we have associated a family of cluster $\mathcal{X}$-varieties to the dual Poisson-Lie group $G^* \subset (G, \pi_\ast)$ of $(G, \pi_G)$ when $(G, \pi_G)$ is a complex semi-simple Lie group of adjoint type, given with the standard Poisson structure $\pi_G$ and $\pi_\ast$ is the "dual" Poisson structure defined by the Semenov-Tian-Shansky Poisson bracket on $G$. We describe here the cluster combinatorics involved into the Artin group action on $G^*$ given by the De-Concini-Kac-Procesi Poisson automorphisms.

1. Introduction

A cluster $\mathcal{X}$-variety is a Poisson variety obtained by gluing a set of algebraic tori along some specific bi-rational isomorphisms called ($\mathcal{X}$-)mutations, which are strongly related to the mutations of the well-known cluster algebra of Fomin and Zelevinsky introduced in [FZ02]. Each torus is given a log-canonical Poisson structure, that is a set of coordinates $x_i$ and a skew-symmetric matrix $\hat{\varepsilon}$, with generic integer values, such that the equality $\{x_i, x_j\} = \hat{\varepsilon}_{ij}x_ix_j$ is satisfied. Because mutations are Poisson maps relative to these log-canonical Poisson structures, cluster $\mathcal{X}$-varieties are naturally given a kind of Darboux coordinates. Therefore, in the same way that the cluster algebra machinery can be used to described coordinates ring of affine varieties related to semisimple Lie groups [BFZ05], [S06], we can use cluster $\mathcal{X}$-varieties to study their Poisson geometry [FG06b], [GSV03], [B].

When $G$ is a real split semisimple Lie group, with trivial center, given with the Sklyanin Poisson structure associated with the standard $r$-matrix of the Belavin-Drinfeld classification (this makes $G$ a Poisson Lie group), Fock and Goncharov have, in the paper [FG06a], constructed canonical Poisson birational maps of cluster $\mathcal{X}$-varieties into $G$ (one map for each seed $\mathcal{X}$-torus associated with a double reduced word of the Weyl group $W$ of $G$); this construction provides for $G$ a natural set of rational canonical coordinates. Canonical maps associated with different double reduced words are given by a composition of mutations simply related to the composition of generalized $d$-moves linking double reduced words.

Now, let us recall that any Poisson-Lie group structure on any Lie group $G$ provides its Lie algebra $\mathfrak{g}$ with a bialgebra structure, and thus its dual $\mathfrak{g}^*$ acquires a bialgebra structure as well. Hence Poisson-Lie groups always come by pairs, and the Lie group associated to $\mathfrak{g}^*$ is provided a Poisson-Lie group structure and is called the dual of $G$. In particular, the dual of a complex semisimple Poisson-Lie group $(G, \pi_G)$ of adjoint type, still equipped with the Sklyanin Poisson structure associated with the standard $r$-matrix, can be identified with a subgroup in the direct product of two opposite Borel subgroups of $G$. This group may also be mapped onto a dense open subvariety in $G$. The induced Poisson structure $\pi_\ast$ then extends smoothly to the entire group $G$ and is given by a simple explicit formula [STS85]. In [DCKP92], De Concini, Kac and Procesi described an Artin group action on
$G^* \subset (G, \pi_*)$ by Poisson transformations. It turns out that this action is the semiclassical analog of the Lusztig automorphisms on the universal quantized enveloping algebra $U_q(\mathfrak{g})$ of the Lie algebra $\mathfrak{g}$.

In the prequel to this paper \cite{B}, we have associated a family of cluster $\mathcal{X}$-varieties \{$\mathcal{X}_w \mid w \in W$\} to the dual Poisson-Lie group $G^* \subset (G, \pi_*)$. The underlying combinatorics is based on a factorization of the Fomin-Zelevinsky twist maps into mutations and other new Poisson birational isomorphisms on seed $\mathcal{X}$-tori called tropical mutations, associated to an enrichment of the combinatorics on double words of the Weyl group $W$ of $G$. A double word $i$ of $W$ being a word of $W \times W$, this enrichment is in fact based on a switch $i \mapsto \overline{i}$ of the two copies of $W$, acting on the first or the last letter of a double word $i$. Such new moves are called $\tau$-moves, and tropical mutations are then Poisson birational isomorphisms between the related seed $\mathcal{X}$-tori. Finally, twisted evaluations, strongly influenced by the morphisms of Evens and Lu \cite{EL07}, relate the cluster $\mathcal{X}$-varieties $\mathcal{X}_w$ to $(G, \pi_*)$.

In the present paper, we use the previous combinatorics to get an action of the Artin group associated to the root system of the Lie algebra $\mathfrak{g}$ by Poisson automorphisms on any seed $\mathcal{X}$-torus of any cluster $\mathcal{X}$-variety $\mathcal{X}_w$. Composing these automorphisms with some of the twisted evaluations of \cite{B}, we rediscover, as a particular case, the Artin group action on $G^* \subset (G, \pi_*)$ generated by the De-Concini-Kac-Procesi Poisson automorphisms.

Here is the organization of the paper: Section \ref{sec:prelim} collects preliminaries, Section \ref{sec:cluster} recalls how to associate cluster $\mathcal{X}$-varieties to the Poisson manifolds $(G, \pi_G)$ and $(G, \pi_*)$, Section \ref{sec:Construction} deals with the cluster combinatorics of the Artin group action on $G^* \subset (G, \pi_*)$ leading to the De-Concini-Kac-Procesi Poisson automorphisms, and Section \ref{sec:twisted} details our construction when $G$ is of type $A_1$.

2. Preliminaries

2.1. Lie setting. Let $\mathfrak{g}$ be a complex semi-simple Lie algebra of rank $\ell$, $A$ its Cartan matrix, and $G$ its connected Lie group of adjoint type. Fix a Borel subgroup $B \subset G$, let $B_-$ be the opposite Borel subgroup, $H = B \cap B_-$ the associated Cartan subgroup and $\mathfrak{h} \subset \mathfrak{g}$ the corresponding Cartan subalgebra. In the following, we will denote $[1, \ell] = \{1, \ldots, \ell\}$. Let $\alpha_1, \ldots, \alpha_\ell$ be the simple roots of $\mathfrak{g}$, and let $\omega_1, \omega_2, \ldots, \omega_\ell \in \mathfrak{h}^*$ be the corresponding fundamental weights. For every $i \in [1, \ell]$, let $(e_i, f_i, h_i)$ be the Chevalley generators of $\mathfrak{g}$; they generate a Lie subalgebra $\mathfrak{g}_{\alpha_i}$ of $\mathfrak{g}$. In particular, we have $\omega_j(h_k) = \delta_{jk}$ for every $j, k \in [1, \ell]$. Let us recall that the weight lattice $\Lambda$ is the set of all weights $\gamma \in \mathfrak{h}^*$ such that $\gamma(h_i) \in \mathbb{Z}$ for all $i$. Every weight $\gamma \in \Lambda$ gives rise to a multiplicative character $a \mapsto a^\gamma$ of the Cartan subgroup $H$, given by $\exp(h)^\gamma = e^{\gamma(h)}$, with $h \in \mathfrak{h}$. We introduce a new basis on $\mathfrak{h}$ putting

$$h^i := \sum (A^{-1})_{ij} h_j.$$  

Let $D = \text{diag}(d_1, \ldots, d_\ell)$ be the diagonal matrix symmetrizing the Cartan matrix; we put $a_{ij} = d_i a_{ij} = a_{ij} d_j$. For every $x \in \mathbb{C}$ and $i \in [1, \ell]$, we define the group elements $E_i = \exp(e_i), F_i = \exp(f_i)$ and $H_i(x) = \exp(\log(x) h_i)$ related to the generators $e_i, f_i$ and $h_i$ of $\mathfrak{g}$. Because of the relation (2.1), the canonical inclusions $\varphi_i : SL(2, \mathbb{C}) \hookrightarrow G$ satisfy

$$\varphi_i \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} = H_i(x) E_i H_i(x^{-1}), \quad \varphi_i \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} = H_i(x^{-1}) F_i H_i(x).$$

We denote by $W$ the Weyl group of $G$. As an abstract group, $W$ is a finite Coxeter group of rank $\ell$ generated by the set of simple reflections $S = \{s_1, \ldots, s_\ell\}$; it acts on $\mathfrak{h}^*$,
and the Cartan subgroup $H$ by

$$s_i(\gamma) = \gamma - \gamma(\alpha_i^\vee)\alpha_i, \quad s_i(h) = h - \alpha_i(h)\alpha_i^\vee$$

and $a^{w(\gamma)} = (\hat{w}^{-1}a\hat{w})^\gamma$

for every $\gamma \in \mathfrak{h}^*$, $h \in \mathfrak{h}$, $w \in W$ and $a \in H$. Recall now that a reduced word for $w \in W$ is an expression for $w$ in the generators belonging to $S$, which is minimal in length among all such expressions for $w$. Let us denote $\ell(w)$ this minimal length and $R(w)$ the set of all reduced words associated to $w$. As usual, the notation $w_0$ will refer to the longest element of $W$. Let us also recall that $W$ can also be seen as the subgroup $\text{Norm}_G(H)/H$ of $G$. Thus, to every simple reflection $s_i \in W$ we associate the representative

$$\hat{s}_i = \phi_i \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$ 

We can choose representatives in $G$ for every element of $W$ by setting $\hat{w}_1\hat{w}_2 = \hat{w}_1\hat{w}_2$ for every $w_1, w_2 \in W$ as long as $\ell(w_1) + \ell(w_2) = \ell(w_1w_2)$.

2.2. Poisson setting. Let us recall that the so-called standard classical $r$-matrix is given by $r = \sum e_\alpha \wedge f_\alpha \in \mathfrak{g} \wedge \mathfrak{g}$ (the summation is done over all positive roots $\alpha$). Let $\langle , \rangle$ be the Killing form on $\mathfrak{g}$. For every $x \in \mathfrak{g}$, $X \in G$ and $f \in \mathcal{F}(G)$ the left and right gradients are defined respectively by

$$\langle \nabla f(X), x \rangle = \frac{d}{dt} \bigg|_{t=0} f(e^{tx}X) \quad \text{and} \quad \langle \nabla' f(X), x \rangle = \frac{d}{dt} \bigg|_{t=0} f(Xe^{tx}).$$

If $f, g \in \mathcal{F}(G)$, let $\pi_G$ be the following Poisson structure on $G$ given by the Sklyanin bracket which transforms $G$ into a Poisson-Lie group.

$$\{f, g\}_G = \frac{1}{2}(\langle \nabla f \otimes \nabla g, r \rangle - \langle \nabla f \otimes \nabla' g, r \rangle).$$

Denote $r_\pm = r \pm t$, where $t$ is the Casimir element of $\mathfrak{g}$:

$$t = \frac{1}{2} \sum_{i=1}^\ell (h_i \otimes h_i^\dagger + e_i \otimes f_i + f_i \otimes e_i) \in \mathfrak{g} \otimes \mathfrak{g}. $$

**Proposition 2.1** ([STS85]). Let us equip $G$ with the Poisson structure $\pi_*$ given by

$$\{f, g\}_* = \frac{1}{2}(\langle \nabla f \otimes \nabla g, r \rangle + \langle \nabla f \otimes \nabla' g, r \rangle) - \langle \nabla f \otimes \nabla' g, r_+ \rangle - \langle \nabla f \otimes \nabla g, r_- \rangle. $$

There exists a canonical map of the dual group $G^*$ onto the dense open cell $BB_- \subset G$ which is a covering of Poisson manifolds.

To any $u, v \in W$ we associate the double Bruhat cell $G^{u,v} = B\hat{u}B \cap B_-\hat{v}B_-$; we have

$$G = \bigcup_{u,v \in W} G^{u,v}. $$

**Proposition 2.2** ([KZ02]). For every $u, v \in W$, the double Bruhat cells $G^{u,v}$ are the $H$-orbits, by the right (or left) action, of the symplectic leaves of $(G, \pi_G)$.

Now, following [EL07], let us now give a Poisson stratification for $(G, \pi_*)$. Two elements $g_1, g_2 \in G$ are said to be in the same Steinberg fiber if $f(g_1) = f(g_2)$ for every regular function $f$ on $G$ that is invariant under conjugation. For $t \in H$, let $F_t$ be the Steinberg fiber containing $t$. By the Jordan decomposition of elements in $G$, every Steinberg fiber is of the form $F_t$ for some $t \in H$. Moreover, the equality $F_t = F_i$ is satisfied if and only
if there exists \( w \in W \) such that \( t' = w(t) \), where \( W \) acts on \( H \) by the formula (2.2). The group \( G \) has therefore the decompositions

\[
G = \bigcup_{t \in H, w \in W} F_{t,w} = \bigcup_{t \in H \setminus W, w \in W} F_{t,w} \quad \text{where} \quad F_{t,w} := BwB^{-} \cap F_{t}.
\]

**Proposition 2.3.** [EL07] Proposition 3.3] Every \( F_{t,w} \) is a finite union of \( H \)-orbits, with respect to the conjugation action, of the symplectic leaves of \((G, \pi_{*})\).

2.3. Cluster \( \mathcal{X} \)-variety setting. We recall here the definitions introduced by Fock and Goncharov underlying cluster \( \mathcal{X} \)-varieties (see, for example, [FG07a] for more details). A **seed** \( \textbf{I} \) is a quadruple \((I, I_{0}, \varepsilon, d)\) where

- \( I \) is a finite set;
- \( I_{0} \subset I \);
- \( \varepsilon \) is a matrix \( \varepsilon_{ij}, i, j \in I \), such that \( \varepsilon_{ij} \in \mathbb{Z} \), unless \( i, j \in I_{0} \);
- \( d = \{d_{i}\}, i \in I \), is a subset of positive integers such that the matrix \( \hat{\varepsilon}_{ij} = \varepsilon_{ij}d_{j} \) is skew-symmetric.

For every real number \( x \in \mathbb{R} \), let us denote \([x]_{+} = \max(x, 0)\) and

\[
\text{sgn}(x) = \begin{cases} 
-1 & \text{if } x < 0 ; \\
0 & \text{if } x = 0 ; \\
1 & \text{if } x > 0 .
\end{cases}
\]

Let \( \textbf{I} = (I, I_{0}, \varepsilon, d), \textbf{I}' = (I', I'_{0}, \varepsilon', d') \) be two seeds, and \( k \in I \setminus I_{0} \). A **mutation in the direction** \( k \) is a map \( \mu_{k} : I \rightarrow I' \) satisfying the following conditions:

- \( \mu_{k}(I_{0}) = I'_{0} ; \)
- \( d'_{\mu_{k}(i)} = d_{i} ; \)
- \( \varepsilon_{\mu_{k}(i)\mu_{k}(j)} = \begin{cases} 
-\varepsilon_{ij} & \text{if } i = k \text{ or } j = k ; \\
\varepsilon_{ij} + \text{sgn}(\varepsilon_{ik})[\varepsilon_{ik}\varepsilon_{kj}]_{+} & \text{if } i, j \neq k .
\end{cases} \)

A **symmetry** on a seed \( \textbf{I} = (I, I_{0}, \varepsilon, d) \) is an automorphism \( \sigma \) of the set \( I \) preserving the subset \( I_{0} \), the matrix \( \varepsilon \), and the numbers \( d_{i} \). That is to say:

- \( \sigma(I_{0}) = I_{0} ; \)
- \( d_{\sigma(i)} = d_{i} ; \)
- \( \varepsilon_{\sigma(i)\sigma(j)} = \varepsilon_{ij} . \)

Let \(|I|\) be the cardinal of every finite set \( I \) and \( \mathbb{C}_{\neq 0} \) be the set of non-zero complex numbers. The **seed \( \mathcal{X} \)-torus** \( \mathcal{X}_{\textbf{I}} \) associated to a seed \( \textbf{I} \) is the torus \((\mathbb{C}_{\neq 0})^{|I|}\) with the Poisson bracket

\[
\{x_{i}, x_{j}\} = \hat{\varepsilon}_{ij}x_{i}x_{j} ,
\]

where \( \{x_{i}|i \in I\} \) are the standard coordinates on the factors. Symmetries and mutations on seeds induce involutive maps between the corresponding seed \( \mathcal{X} \)-tori, which are denoted by the same symbols \( \mu_{k} \) and \( \sigma \), and given by

- \( \sigma^{*}x_{\sigma(i)} = x_{i} \)
- \( \mu_{k}^{*}x_{\mu_{k}(i)} = \begin{cases} 
x_{i}^{-1} & \text{if } i = k ; \\
x_{i}^{[\varepsilon_{ik}]_{+}}(1 + x_{k}^{[\varepsilon_{ik}]_{+}}) & \text{if } i \neq k .
\end{cases} \)

Finally, a **cluster transformation** linking two seeds (and two seed \( \mathcal{X} \)-tori) is a composition of symmetries and mutations, and the **cluster \( \mathcal{X} \)-variety** \( \mathcal{X}_{\textbf{I}} \) associated to a seed \( \textbf{I} \) is obtained by taking every seed \( \mathcal{X} \)-tori obtained from \( \mathcal{X}_{\textbf{I}} \) by cluster transformations, and gluing them via the previous bi-rational isomorphisms.
3. Cluster $\mathcal{X}$-varieties related to $(G, \pi_G)$ and $(G, \pi_\epsilon)$

3.1. Cluster $\mathcal{X}$-varieties related to $(G, \pi_G)$. We briefly recall the construction of 
[FG06b]. A (reduced) word of $W \times W$ is called a double (reduced) word. To avoid confusions, we denote $\overline{1}, \ldots, \overline{\ell}$ the indices of the reflections associated to the first copy of the Weyl group $W$ and $1, \ldots, \ell$ the indices of the reflections associated to the second copy of $W$. A double reduced word of $(u, v)$ is a then shuffle of a reduced word of $u$, written in the alphabet $[\overline{1}, \overline{\ell}]$, and of a reduced word of $v$, written in the alphabet $[1, \ell]$. We denote $R(u, v)$ the set of all double reduced words of $(u, v)$, and $1 \in R(1, 1)$ the unity of $W \times W$.

For every $u, v \in W$, $i \in R(u, v)$ and $j \in [1, \ell]$, let $N^j(i)$ be the number of times the letter $j$ or $\overline{j}$ appear in $i$. Let $I(i)$ (resp. $I^\overline{i}(i)$ and $I_0(i)$) be the set of all ordered pairs $(j, \ell)$ such that $j \in [1, \ell]$, and $0 \leq k \leq N^j(i)$ (resp. $k = N^j(i)$ and $k \in \{0, N^j(i)\}$). Let us start to describe the trivial seed $(I(1), I_0(1), \varepsilon(1), d(1))$. It is defined by the zero square matrix $\varepsilon(1)$ of size $\ell$ and $\ell$-vectors $d(1)(\ell) = d_j$.

Now, let us describe the elementary seeds $(I(i), I_0(i), \varepsilon(i), d(i))$ and $(I(\overline{i}), I_0(\overline{i}), \varepsilon(\overline{i}), d(\overline{i}))$ for every $i \in [1, \ell]$ and $\overline{i} \in \{i, \overline{i}\}$. Let $\varepsilon(i), \varepsilon(\overline{i})$ be the square matrices of size $\ell + 1$, with entries labeled by the elements of $I(i)$ and given by

$$
\varepsilon(i)_{(\ell)(\ell)} = \frac{a_{ij}}{2} = -\varepsilon(i)_{(0)(\ell)}, \quad \varepsilon(\overline{i})_{(\ell)(\ell)} = -\frac{a_{j\overline{i}}}{2} = -\varepsilon(\overline{i})_{(\ell)(0)},
$$

and zero otherwise. In a similar way, let $d(i)$ and $d(\overline{i})$ be the $(\ell + 1)$-vectors with components labeled by the elements of $I(i)$, $d(i)(\ell) = d_j = d(\overline{i})(\ell)$.

For every $i \in \{1, \overline{1}, \overline{\overline{1}}\}$ we then denote $\mathcal{X}_i$ the seed $\mathcal{X}$-torus associated to the seed $(I(i), I_0(i), \varepsilon(i), d(i))$ and $ev_1 : \mathcal{X}_i \to G$ the related evaluation map

$$
ev_1 : \mathbb{C}^\ell_{\neq 0} \to G : (x_{(0)}, \ldots, x_{(\ell)}) \mapsto \prod_j H^2(x_{(0)}),$$

$$
ev_i : \mathbb{C}^{\ell+1}_{\neq 0} \to G : (x_{(0)}, \ldots, x_{(\ell)}, x_{(\overline{0})}, x_{(\overline{1})}) \mapsto \prod_j H^2(x_{(0)})F^iH^1(x_{(\overline{i})}),$$

$$
ev_{\overline{i}} : \mathbb{C}^{\ell+1}_{\neq 0} \to G : (x_{(0)}, \ldots, x_{(\ell)}, x_{(\overline{0})}, x_{(\overline{1})}) \mapsto \prod_j H^2(x_{(0)})F^\overline{i}H^1(x_{(\overline{i})}).$$

To associate seeds to longer words we proceed inductively. We associate to a product $ij$ of double words $i$ and $j$ an amalgamated seed $(I(ij), I_0(ij), \varepsilon(ij), d(ij))$ in the following way. The elements of the set $d(ij)$ are defined by $d(ij)(\ell) = d_j$, and the matrix $\varepsilon(ij)$ is given by

$$
\varepsilon(ij)_{(\ell)(\ell)} = \begin{cases} 
\varepsilon(i)_{(\ell)(\ell)} & \text{if } k < N^i(i) \text{ and } l < N^j(i); \\
\varepsilon(i)_{(\ell)(\ell)} + \varepsilon(j)_{(0)(\ell)} & \text{if } k = N^i(i) \text{ and } l = N^j(i); \\
\varepsilon(j)_{(\ell)(0)} & \text{if } k > N^i(i) \text{ and } l > N^j(i); \\
0 & \text{otherwise}. 
\end{cases}
$$
This operation induces a homomorphism \( m : X_1 \times X_j \to X_{ij} \) between the corresponding seed \( X \)-tori called \textit{amalgamation} and given by

\[
m^* z_{(k)} = \begin{cases} 
  x_{(i)} & \text{if } 0 \leq k < N^i(i); \\
  x_{(i)} y_{(j)} & \text{if } k = N^i(i); \\
  y_{(k_{-1} N^i(i))} & \text{if } N^i(i) < k \leq N^i(i) + N^i(j),
\end{cases}
\]

where \( x_i, y_j \) and \( z_k \) denote respectively the coordinates functions on \( X_1, X_j, \) and \( X_{ij} \). Notice that this amalgamated product is associative. Now, let \( i = i_1 \ldots i_k \) be a double word, \( X_i \) be the seed \( X \)-torus given by the associated amalgamation \( m : X_{i_1} \times \cdots \times X_{i_k} \to X_i \), and \( z \in X_i \) be an amalgamation \( m(x_1, \ldots, x_k) \) of \( x_1 \in X_{i_1}, \ldots, x_k \in X_{i_k} \). We define the evaluation map

\[
ev_i : X_i \to G : z \mapsto \ev_{i_1}(x_1) \ldots \ev_{i_k}(x_k).
\]

**Theorem 3.1 ([FG06b]).** For any \( u, v \in W \) and \( i \in R(u, v) \) the map \( \ev_i : X_i \to (G^{u, v}, \pi_G) \) is a Poisson birational isomorphism onto a Zariski open set of the double Bruhat cell \( G^{u, v} \).

Let \( i \) be a reduced word. Following [BZ01], we call a \textit{d-move} a transformation of \( i \) that replaces \( d \) consecutive entries \( i, j, i, j, \ldots \) by \( j, i, j, i, \ldots \), for some \( i \) and \( j \) such that \( d \) is the order of \( s_i s_j \), that is: if \( a_{ij} a_{ji} = 0 \) (resp. 1, 2, 3), then \( d = 2 \) (resp. 3, 4, 6). By the Tits theorem, every two reduced words which represent the same element of a Coxeter group are related by a sequence of \( d \)-moves. Next, let us say that a letter \( i \) is \textit{positive} if \( i \in [1, \ell] \) and \textit{negative} if \( i \in [\ell + 1, \ell + \ell] \). Considering the group \( W \times W \), we conclude that every two double reduced words \( i, j \in R(u, v) \) can be obtained from each other by a sequence of \textit{generalized \( d \)-moves}, i.e. \textit{positive} \( d \)-moves for the alphabet \([1, \ell]\), \textit{negative} \( d \)-moves for the alphabet \([\ell + 1, \ell + \ell]\), or \textit{mixed} \( 2 \)-moves that interchange two consecutive indices of opposite signs. Let us say that a double reduced word of length \( d \) is \textit{minimal} if we can perform a generalized \( d \)-move on it. To any two minimal double words \( i, j \) related by a generalized \( d \)-move \( \delta : i \mapsto j \), we associate a cluster transformation \( \mu_{i \to j} : X_i \to X_j \) in the following way:

\[
\mu_{i \to j} = \begin{cases} 
  \mu_{(1)} & \text{if } \delta \text{ is a move } i \leftrightarrow j \text{ or a } 3 \text{-move}; \\
  \mu_{(1)} \mu_{(j)} & \text{if } \delta \text{ is a } 4 \text{-move}; \\
  \mu_{(2)} \mu_{(1)} \mu_{(j)} & \text{if } \delta \text{ is a } 6 \text{-move}; \\
  \text{the identity map} & \text{otherwise}.
\end{cases}
\]

Since mutations commute with amalgamation, we may extend these definitions to any two double words \( i, j \in R(u, v) \) related by a generalized \( d \)-move. Finally, if \( i, j \) are double words linked by a sequence of generalized \( d \)-moves and \( i \to i_1 \to \cdots \to i_{n-1} \to j \) is the associated chain of elements, we define the cluster transformation \( \mu_{i \to j} \) as the composition \( \mu_{i_{n-1} \to i} \circ \cdots \circ \mu_{i_{1} \to i} \). A cluster \( X \)-variety is associated to every double Bruhat cell via the following theorem.

**Theorem 3.2 ([FG06b]).** For any \( u, v \in W \) and \( i, j \in R(u, v) \), we have \( \ev_i = \ev_j \circ \mu_{i \to j} \).

### 3.2. Cluster \( X \)-varieties related to \( (G, \pi_+) \)

We sum-up here some definitions and results of [B]. To state an analog of Theorem 3.1 and Theorem 3.2 for \( (G, \pi_+) \), we need new evaluation maps and an extension of the combinatorics. Let us first recall that the fundamental weights \( \omega_i \in h^* \) are permuted by the transformation \((-w_0)\). We denote \( i \mapsto \omega_i \) the induced permutation of the indices of these weights, that is \( \omega_i^* = -w_0(\omega_i) \). Let \( w \in W \) and \( s_{i_1} \ldots s_{i_n} \) be a reduced decomposition of \( w \), then \( w^* \in W \) is the element given...
by \( w^* = s_{i_1} \cdots s_{i_k} \). (The Tits theorem implies that this definition doesn’t depend on the choice of the reduced expression for \( w \).)

For every \( w, w' \in W \), we denote \( w \rightarrow w' \) if and only if we can find \( i \in [1, \ell] \) such that \( w = s_iw' \) and \( \ell(w) = \ell(w') + 1 \), and denote \( \leq \) the right weak order on \( W \), i.e. \( w' \leq w \) if there exists a chain \( w \rightarrow \cdots \rightarrow w' \).

**Definition 3.3.** Let \( w_1 \leq v, w_2 \in W \). A \((w_1, w_2)_\circ\)-word \( i \) is a double word linked to a product \( i_1, i_2 \), with \( i_2 \in R(w_1^{-1}, v)w_2^{-1} \) and \( i_2 \in R(w_1^{-1}, v)w_2^{-1} \), by a sequence of mixed 2-moves. The product \( i_1i_2 \) is called a trivalent \((w_1, w_2)_\circ\)-word. In particular, the set \( R(v_w^{-1}, w_0) \) is the set of \((v, e)_\circ\)-words. Let \( W(w_1, w_2)_\circ \) denote the set of \((w_1, w_2)_\circ\)-words and \( D(v) \) be the union over \( w_1 \leq v, w_2 \in W \) of all the sets \( W(w_1, w_2)_\circ \). In particular, we have \( R(v_w^{-1}, w_0) \subset D(v) \).

The set \( D(v) \) are going to play in Theorem 3.5 and Theorem 3.7 the same role that were playing the sets \( R(u, v) \) in Theorem 3.1 and Theorem 3.2.

Now, for every \( x \in B^- B^+ \), let \( x = \mu(x) \cdot x^-+ x^0 \cdot x^+ \) be its Gauss decomposition, that is: \([x]_\pm \) belongs to the unipotent parts of the respective Borel subgroups and \([x]_0 \) to the Cartan part of \( G \). Let us denote \( \theta : G \rightarrow G \) the Cartan automorphism given by

\[
a^\theta = a^{-1}, \quad E^\theta = F^\theta, \quad F^\theta = E^\theta.
\]

**Definition 3.4.** Let \( w_1 \leq v, w_2 \in W \), \( i = i_1i_2 \) be a trivial \((w_1, w_2)_\circ\)-word and \( x \in X_i \), \( x_1 \in X_{i_1} \), \( x_2 \in X_{i_2} \) be such that \( x = \mu(x_1, x_2) \). We define the maps \( ev^\circ_{i_1} : X_{i_2} \rightarrow G \) and \( ev^\circ_{i_2} \), \( ev^\circ_{i_1} : X_{i_1} \rightarrow G \) by:

\[
ev^\circ_{i_1}(x) = ev_{i_1}(x) \prod_{j \in [1, \ell]} H^j(x^{-1}(x^{-1}))
\]

\[
ev^\circ_{i_2}(x) = ev_{i_2}(x_2)\mu_1(\mu_2(x_2))_{\leq 0} \quad \text{and} \quad ev^\circ_{i_1}(x) = ev_{i_1}(x_1)\mu_2(\mu_2(x_2))_{\leq 0} \quad \text{.}
\]

These maps are then extended to every \( i \in D(v) \) by setting

\[
ev^\circ_{i_1} = ev^\circ_{i_1} \circ \mu_1 \circ i_1i_2 \quad \text{and} \quad ev^\circ_{i_2} = ev^\circ_{i_2} \circ \mu_2 \circ i_1i_2 \quad \text{.}
\]

For every \( v \in W \) and \( i \in D(v) \), we define the twisted evaluation

\[
\hat{ev}_i : X_\mu_i \rightarrow (G, \pi_*), \quad x \mapsto ev^\circ_{i_1}(x)ev_{i_1}(x(\mathfrak{R}))\mu_1(\mu_2(x_2))_{\leq 0} \quad \text{.}
\]

where, for every double word \( i \), the seed \( X_\mu_i \) is the Poisson variety canonically associated to the seed \( [\mu_i]_\mathfrak{R} = (I(\mu_i), I_0(\mu_i), \pi(\mu_i)) \) defined by the values

\[
\eta(i)_{ij} = \begin{cases} 
\bar{\varepsilon}(i)_{ij} & \text{if } i, j \in I(\mu_i) \setminus I_0(\mu_i) \; ; \\
0 & \text{otherwise}.
\end{cases}
\]

Finally, for every \( t \in H \), let \( X_{i_1\mathfrak{R}}(t) \) be the subset of \( X_\mu_i \) obtained by fixing the cluster variables \( x(\mathfrak{R}) = \{(x_j) \mid j \in I_0(\mu_i) \} \) via the equality \( ev_{i_1}(x(\mathfrak{R})) = t \). Equation (3.4) implies that \( X_{i_1}(t) \) is a Poisson subvariety of \( X_{i_1\mathfrak{R}} \).

**Theorem 3.5.** [3] Theorem 7.9 For every \( v \in W \), \( t \in H \) and \( i \in D(v) \), the map \( \hat{ev}_i : X_\mu_i(t) \rightarrow (F_{\mu, v}, \pi_* \circ \mu) \) is a Poisson birational isomorphism onto a Zariski open set of \( F_{\mu, v} \).

**Remark 3.6.** The origin of the formula (3.3) comes from morphisms defined by Evens and Lu in [EL07], generalized in [3]. (The proof of Theorem 3.5 is based on [EL07] Corollary 5.11.)
Theorem 3.7. [B Theorem 8.12] Let \( v \in W \). For every \( i, j \in D(v) \), there exists a birational Poisson isomorphism \( \hat{\mu}_{i-j} : \mathcal{X}[i]_{|\mathcal{X}} \to \mathcal{X}[j]_{|\mathcal{X}} \) such that \( \hat{\epsilon}_{ij} = \hat{\epsilon}_{ij} \circ \hat{\mu}_{i-j} \).

Describing explicitly the birational Poisson isomorphism \( \hat{\mu}_{i-j} : \mathcal{X}[i]_{|\mathcal{X}} \to \mathcal{X}[j]_{|\mathcal{X}} \) asks us to extend the given combinatorics both on double words and on seed-\( \mathcal{X} \)-tori. Here is the construction.

3.2.1. Combinatorics on double words. We introduce new moves on double words in order to act transitively in \( D(v) \). From now on, let us regard the map \( i \mapsto i \) as an involution on \([1, \ell] \cup [1, \ell] \).

Definition 3.8. For every double word \( i = i_1 \ldots i_n \), let \( \mathcal{L}_{i_1}(i) \) (resp. \( \mathcal{R}_{i_n}(i) \)) be the double word obtained by changing the first letter (resp. last letter) of \( i \) such that:

\[
\mathcal{L}_{i_1}(i) = \overline{i_1}i_2 \ldots i_n \quad \text{and} \quad \mathcal{R}_{i_n}(i) = i_1 \ldots i_{n-1}\overline{i_n}.
\]

The involutive map \( i \mapsto \mathcal{L}_{i_1}(i) \) (resp. \( i \mapsto \mathcal{R}_{i_n}(i) \)) is called a left \( \tau \)-move (resp. right \( \tau \)-move).

Let us fix \( w_1 \leq v \in W \) and denote \( D_{w_1}(v) \) the set of \((w_1, w_2)\)-words for every \( w_2 \in W \). Therefore, the set \( D(v) \) is the union over all \( w_1 \leq v \in W \) of the sets \( D_{w_1}(v) \).

Lemma 3.9. [B] For every \( i, j \in D_{w_1}(v) \), there exist a sequence \( \varphi_{1-j} \) of generalized \( d \)-moves and right \( \tau \)-moves such that \( j = \varphi_{1-j}(i) \).

Definition 3.10. Let \( i = i_1 \ldots i_m \in R(1, w_0) \cup R(w_0, 1) \) be a positive or negative reduced word associated to \( w_0 \) and \( j = j_1 \ldots j_n \) be a double word. The following dual-move \( \Delta_j \) associated to the last letter of \( j \) transforms the product \( ji \) into the double word

\[
\Delta_j : ji \mapsto j_1 \ldots j_{n-1}\overline{j_n} \overline{i_m} \ldots \overline{i_1},
\]

where \( j = j_n \) if \( j_n \) positive and \( j = \overline{j_n} \) if \( j_n \) negative. It is easy to see that \( \Delta_j \circ \Delta_j \) is the identity map.

Definition 3.11. Let \( v \in W \) and \( i \in D(v) \) be a double word. A \( \tilde{d} \)-move on \( i \) is a generalized \( d \)-move; or a right \( \tau \)-move; or a dual-move \( \Delta_i \).

Here is an analog of the Tits theorem for the set \( D(v) \).

Lemma 3.12. [B] If \( i, j \in D(v) \) then there exists a sequence of \( \tilde{d} \)-moves relating \( i \) and \( j \).

3.2.2. Tropical mutations. The \( \tau \)-moves lead to a new type of mutations on seed \( \mathcal{X} \)-tori, called tropical mutations and defined in the following way. Let \( b_{ij} \) be the numerator of \( \varepsilon_{ij} \) for every \( i, j \in I \); so we have \( b_{ij} = \varepsilon_{ij} \) unless \( i, j \in I_0 \). Let us suppose that the denominator of \( \varepsilon_{ij} \) is the same for every \( i, j \in I_0 \). (Using the formulas (3.1) and (3.2), it is clear that it is the case for any seed \( I(\mathcal{I}) \) associated to a double word \( i \).)

Definition 3.13. Let \( I = (I_0, I, \varepsilon, d) \) be a seed such that \( I_0 \) is not empty. A cover \( \mathcal{C} \) on \( I \) is a family of sets \( I_1, \ldots, I_n \subset I_0 \) such that \( I_0 = \bigcup_{i=1}^n I_i \). (The union is not necessary disjoint.) For every \( k \in I_0 \), we denote \( I_0(k) \) the union

\[
I_0(k) := \bigcup_{\{i | k \in I_i \}} I_i.
\]
Definition 3.14. Let $\mathbf{I} = (I, I_0, \varepsilon, d)$ and $\mathbf{I}' = (I', I'_0, \varepsilon', d')$ be two seeds with covers, and $k \in I_0$. A tropical mutation in the direction $k$ is an involution $\mu_k : \mathbf{I} \rightarrow \mathbf{I}'$ satisfying the following conditions:

(i) $\mu_k(I_0(i)) = I'_0(i)$;
(ii) $d'_\mu_k(i) = d_i$;
(iii) $\varepsilon'_\mu_k(i)\mu_k(j) = \begin{cases} -\varepsilon_{ij} & \text{if } i = k \text{ or } j = k; \\ \varepsilon_{ij} & \text{if } i, j \in I_0(k) \setminus \{k\}; \\ \varepsilon_{ij} - \varepsilon_{ik}b_{kj} & \text{otherwise.} \end{cases}$

Tropical mutations induce involutive maps between the corresponding seed $\mathcal{X}$-tori, which are denoted by the same symbols $\mu_k$ and given by

$$x_{\mu_k(i)} = \begin{cases} x_k^{-1} & \text{if } i = k; \\ x_i x_k^{b_{ki}} & \text{if } i \in I_0(k) \setminus \{k\}; \\ x_i & \text{otherwise.} \end{cases}$$

(3.5)

For the remaining part of the paper, we suppose mutations and symmetries respect covers on seeds. A generalized cluster transformation linking two seeds (and two seed $\mathcal{X}$-tori) is then a composition of symmetries, mutations, and tropical mutations.

Let $\mathbf{i}$ be a double word and recall the subsets $I_0^S(\mathbf{i})$ and $I_0^R(\mathbf{i})$ of $I_0(\mathbf{i})$ defined in Subsection 3.1. From now on, the seed $\mathbf{I}(\mathbf{i}) = (I(\mathbf{i}), I_0(\mathbf{i}), \varepsilon(\mathbf{i}), d(\mathbf{i}))$ is given with the cover $I_0(\mathbf{i}) = I_0^S(\mathbf{i}) \cup I_0^R(\mathbf{i})$.

Lemma 3.15. [3] Proposition 5.11] The following tropical mutations are Poisson birational isomorphisms for every double word $\mathbf{i} = i_1 \ldots i_n$. They are called respectively left and right tropical mutations.

$$\mu_{(i_0)}^{(1)} : \mathcal{X}_1 \rightarrow \mathcal{X}^{i_1}(\mathbf{i}) \quad \text{and} \quad \mu_{(N^{i_n}(\mathbf{i}))}^{(n)} : \mathcal{X}_1 \rightarrow \mathcal{X}^{N^{i_n}(\mathbf{i})}(\mathbf{i}) .$$

Left and right tropical mutations are related to the geometry of the group $G$ in the following way.

Lemma 3.16. [3] Proposition 5.24] The following equalities are satisfied for every $u, v \in W$, every $(u, v)$-adapted double word $\mathbf{i} = i_1 \ldots i_n \in R(u, v)$, and every $\mathbf{x} \in \mathcal{X}_1$.

$$[ev_1(x)v^{-1}]_{\leq 0} = [ev_{N^{i_n}(\mathbf{i})}(x) \circ \mu_{(N^{i_n}(\mathbf{i}))}^{(i_n)}(x)v^{-1}]_{\leq 0}$$
$$[\bar{u}^{-1}ev_1(x)]_{\geq 0} = [\bar{s}_{i_n}\bar{u}^{-1}ev_{N^{i_n}(\mathbf{i})}(x) \circ \mu_{(i_n)}^{(1)}(x)]_{\geq 0} .$$

This result is the first step to get the combinatorics of the maps $b \mapsto \bar{b}v^{-1}]_{\leq 0}$ and $c \mapsto [\bar{u}^{-1}c]_{\geq 0}$, with $b \in G^{1, v}$ and $c \in G^{v, 1}$. To do that, we need more notations. For every positive reduced word $\mathbf{i} = i_1 \ldots i_n$, every negative reduced word $\mathbf{j} = j_1 \ldots j_n$, and every $k \in [1, n + 1]$, we then introduce the double words

$$i(k) = i(k)_-i(k)_+ \quad \text{where} \quad i(k)_+ = i_1 \ldots i_{k-1} \quad \text{and} \quad i(k)_- = \bar{i}_n \ldots \bar{i}_k,$$
$$j(k) = j(k)_-j(k)_+ \quad \text{where} \quad j(k)_+ = j_k \ldots j_1 \quad \text{and} \quad j(k)_- = \bar{j}_n \ldots \bar{j}_1 .$$

In particular, we will use the notations

$$\mathbf{i}^\Box := i(1) = \bar{i}_n \ldots \bar{i}_1 \quad \text{and} \quad \mathbf{j}^\Box := j(n + 1) = j_n \ldots j_1 .$$
For every positive reduced word $i = i_1 \ldots i_n$, every negative reduced word $j = j_1 \ldots j_n$, and every $k \in [1, n]$, we define the generalized cluster transformations $\zeta_{i(k)}: X_i \to X_i^{(k)}$, $\zeta_{j(k)}: X_j \to X_j^{(k)}$, $\zeta_i: X_i \to X_i^\varnothing$ by the following formulas

\[
\zeta_{i(k)} = \mu_{(N^k(i) \setminus \varnothing)}^{i_k} \circ \ldots \circ \mu_{N^k(i) \setminus \varnothing}^{i_k} \circ \mu_{N^k(i \setminus k)}^{i_k} \quad \text{and} \quad \zeta_i = \zeta_i^{(1)} \circ \ldots \circ \zeta_i^{(n)}.
\]

(3.7)

The following result is easily deduced from Theorem 3.2 and Lemma 3.16.

**Corollary 3.17.** Let $u, v \in W$, $i \in R(1, v)$ and $j \in R(u, 1)$. For every $x \in X_i, y \in X_j$ the following equalities are satisfied:

\[
[ev_i(x)v]^{-1}_0 = ev_{\varnothing} \circ \zeta_i(x) \quad \text{and} \quad [\hat{u}^{-1} ev_j(y)]_0 = ev_{\varnothing} \circ \zeta_j(y).
\]

(3.8)

3.2.3. The extended combinatorics on seed $X$-tori. Let us first remark that equation (3.4) implies that any cluster transformation $\mu_{i \to j}: X_i \to X_j$, for any $j$ linked to $i$ by composition of generalized $d$-moves, induces a cluster transformation $\mu_{[i]_\mathfrak{m} \to [j]_\mathfrak{m}}: X_{[i]_\mathfrak{m}} \to X_{[j]_\mathfrak{m}}$ given by

\[
x_{\mu_{[i]_\mathfrak{m} \to [j]_\mathfrak{m}}(j)} = \begin{cases} 
  x_{\mu_{i \to j}(j)} & \text{if } j \in I(i) \setminus J_0^\mathfrak{m}(i) \\
  x_j & \text{if } j \in J_0^\mathfrak{m}(i).
\end{cases}
\]

(3.9)

We have in particular the restriction on Poisson subvariety $\mu_{[i]_\mathfrak{m} \to [j]_\mathfrak{m}}: X_i(t) \to X_j(t)$ for any $t \in H$. It is easy to see that for every double word the identity map is a Poisson morphism from $X_{[i]_\mathfrak{m}}$ to $X_{[j]_\mathfrak{m}}$. We denote it $\mu_{[i]_\mathfrak{m} \to [j]_\mathfrak{m}}$ and extend the definition (3.9) by the formula

\[
\mu_{[i]_\mathfrak{m} \to [j]_\mathfrak{m}} := \mu_{[i]_\mathfrak{m} \to [j]_\mathfrak{m}} \circ \mu_{[j]_\mathfrak{m} \to [j]_\mathfrak{m}}.
\]

And because every elements $i, j \in D_{w_1}(v)$ can be obtained from each other by a sequence of generalized $d$-moves and right $\tau$-moves by Lemma 3.4, we deduce from Theorem 3.2 and the expression (3.3) of twisted evaluations that for every $i, j \in D_{w_1}(v)$, we have the equality

\[
\hat{e}v_i = \hat{e}v_j \circ \mu_{[i]_\mathfrak{m} \to [j]_\mathfrak{m}}.
\]

(3.10)

Therefore, the equation (3.10) allows us to associate, for every $w \leq v \in W$, a cluster $X$-torus $X_{w \leq v}$ to the set $D_v(v)$: it indeed contains every seed $X$-torus $X_{[i]_\mathfrak{m}}$ when $i \in D_w(v)$.

Next, we consider the combinatorics on seed $X$-tori related to dual moves to link all the cluster $X$-varieties $X_{w \leq v}$ for a fixed $v \in W$. To do that, we define for every double reduced word $j$, every positive word $i_+ \in R(1, w_0)$ and every $i \in [1, \ell]$, the map $\Xi_k : X_{[ji+, \overline{\mathfrak{k}}]_\mathfrak{m}} \to X_{[ji+, \mathfrak{k}]_\mathfrak{m}}$ given by

\[
x_{\Xi_k(j)} = \begin{cases} 
  x_{\xi_{i+}(j)}^{N^i(ji_+)} & \text{if } j < N^i(ji_+); \\
  x_{\xi_{i+}(j)}^{N^{-1}(ji_+)} & \text{if } j = N^i(ji_+) < N^i(ji+\overline{\mathfrak{k}}); \\
  x_{\xi_{i+}(j)} & \text{otherwise}.
\end{cases}
\]

(3.11)

Let $i \in [1, \ell]$ and $i$ be a double word such that we can apply the dual move $\Delta_i$ on it. Then the following product is a birational Poisson isomorphism.

\[
\Xi_i : X_{[i]_\mathfrak{m}} \to X_{[\Delta_i(i)]_\mathfrak{m}} \\
x \mapsto \mu_{[i]_\mathfrak{m} \to [\Delta_i(i)]_\mathfrak{m}} \circ \Xi_i \circ \mu_{[\mathfrak{i}]_\mathfrak{m} \to [\mathfrak{i}]_\mathfrak{m}}(x).
\]
Lemma 3.18. [3, Proposition 8.11] For every \( i \in \{1, \ell\} \) and every double reduced word \( \mathbf{i} \in R(s_i, w_0) \) starting with the letter \( i \), we have the equality \( \hat{\ev}_i = \hat{\ev}_{\Delta_i} \circ \Xi_{s_i} \).

This result is then immediately extended to any trivial \((w_1, 1)\)-words, for any \( w_1 \leq v \in W \) by using the amalgamation product on seed \( \mathcal{X} \)-tori.

Here is finally the construction of the birational Poisson isomorphism \( \hat{\mu}_{1-j} \) associated to any \( i, j \in D(v) \). We have seen that to any double words \( \mathbf{i}, \mathbf{i}' \in D(v) \), there exists a \( \delta \)-move \( \delta \) such that \( \delta : \mathbf{i} \to \mathbf{i}' \). From the previous results, it is natural to associate to the birational Poisson isomorphism \( \hat{\mu}_{1-j} \) given by

- the cluster transformation \( \mu_{[i]^{m_j}} \circ [j]^{m_j} \) if \( \delta \) is a generalized \( d \)-move;
- the identity map if \( \delta \) is a right \( \tau \)-move;
- the map \( \Xi_{s_i} \) if \( \delta \) is the dual-move \( \Delta_i \).

This definition is then extended to every \( i, j \in D(v) \) in the usual way: if \( \mathbf{i}, \mathbf{j} \) are double words linked by a sequence of \( \delta \)-moves and \( \mathbf{i} \to \mathbf{i}_1 \to \cdots \to \mathbf{i}_{n-1} \to \mathbf{j} \) is the associated chain of elements, the map \( \hat{\mu}_{1-j} \) will be the composition \( \hat{\mu}_{i_{n-1}-i_j} \circ \cdots \circ \hat{\mu}_{i_1-j} \), and Theorem 3.7 is derived from the equality (3.10) and Lemma 3.18.

We finish this subsection by considering the dual Poisson Lie-group \((BB_\ast, \pi_\ast)\) in \((G, \pi_\ast)\). For every double word \( \mathbf{i} \), let \( \mathcal{X}_1^{\ast} \subset \mathcal{X}_1^{\ast} \) be such that \( x_i \neq x_j \) for every \( i, j \in \mathcal{T}_0 \). It is a Poisson submanifold of \( \mathcal{X}_1^{\ast} \) because of (3.4). The decomposition (2.3) then leads to the following result.

Corollary 3.19. For every \( \mathbf{i} \in D(w_0) \), the map \( \hat{\ev}_1 : \mathcal{X}_1^{\ast} \to (BB_\ast, \pi_\ast) \) is a Poisson birational isomorphism on a Zariski open set of \( BB_\ast \) and the equality \( \hat{\ev}_1 = \hat{\ev}_1 \circ \hat{\mu}_{1-j} \) is satisfied for every \( j \in D(w_0) \).

4. The combinatorics of De-Concini-Kac-Procesi automorphisms

We add the left tropical mutations of Lemma 3.15 to the birational Poisson isomorphisms of the previous section in order to define an action of the Artin group associated to \( \mathfrak{g} \) on seed \( \mathcal{X} \)-tori. Applying twisted evaluations then leads to a generalization of the De-Concini-Kac-Procesi automorphisms associated to parabolic subgroups of \( W \).

4.1. Artin groups actions and Poisson automorphisms on seed \( \mathcal{X} \)-tori. Let us denote \( T_{ij}^{(m_i)} \) the product of \( m \) factors \( T_i T_j T_i \) and let \( m_{ij} := 2, 3, 4, 6 \) when \( a_{ij}, a_{ji}^\prime = 0, 1, 2, 3 \), respectively, where \( A = (a_{ij}) \) denotes the Cartan matrix associated to any semi-simple Lie algebra \( \mathfrak{g}_l \). We recall that the Artin group \( B_{g^l} \) associated to \( \mathfrak{g}_l \) is given by the presentation

\[ B_{g^l} = \langle T_1, \ldots, T_{\ell} \mid T_{ij}^{(m_{ij})} = T_{ji}^{(m_{ji})} \rangle. \]

We are going to show that tropical mutations induced by left \( \tau \)-moves, mutations and the previous saltations lead to Poisson automorphisms that define an action of the Artin group associated to \( \mathfrak{g} \) on cluster \( \mathcal{X} \)-varieties.

Let us remember the involution \( * : i \mapsto i^\ast \) on the indices of fundamental weights induced by the transformation \((-w_0)\) on \( \mathfrak{h}_\ast \) given in Subsection 3.2. Let \( W_I \) be the parabolic subgroup of \( W \) generated by a subset \( I \subset [1, \ell] \) stable under the involution \( * \), \( \mathfrak{g}_l^I \) be the associated semi-simple Lie algebra, and \( w_0(I) \) be the longest element of \( W_I \). In particular, for every \( j \in I \), there exist a reduced expression of \( w_0(I) \) starting with \( j \).

Let us then remark that, to every \( \mathbf{i} \in D(w_0(I)) \) and every letter \( j \in I \), we can associate via Theorem 3.7.a double reduced word \( \mathbf{i}_0 \in R(w_0(I), w_0) \) starting with \( j \) such that the
equality $\hat{c}v_i = \hat{c}v_{i0} \circ \hat{\mu}_{i \to i0}$ is satisfied. We use this double reduced word $i_0$ (although
the final result doesn’t depend of this particular choice) to define the Poisson birational
automorphism

$$T_j(i) : X_{[i]i_{10}} \to X_{[i]i_{10}} : x \mapsto \hat{\mu}_i(i_0) \circ \mu_{i \to i_0} \circ \mu_{i \to i_0}.$$  

(4.1)

Here is how we relate braids to cluster combinatorics.

**Theorem 4.1.** For every $I \subset [1, \ell]$ stable under the involution $\star$ and $i \in D(w_0(I))$, the
maps $T_j(i) : X_{[i]i_{10}} \to X_{[i]i_{10}}$, $j \in I$, define an action of $B_i$ on $X_{[i]i_{10}}$ given by

$$B_i X_{[i]i_{10}} \to X_{[i]i_{10}} : (T_j, x) \mapsto T_j(i)(x).$$

We need two lemmas to prove this theorem. To any set $I \subset [1, \ell]$ stable under the
involution $\star$, $w \leq w_0(I) \in W$, and any reduced decomposition $i_1 \ldots i_n$ of $w$, and any
double word $i \in D(w_0(I))$, we associate a Poisson automorphism $T_{w}(i)$ on $X_{[i]i_{10}}$ given by

$$T_{w}(i) = T_{i_0}(i) \circ \cdots \circ T_{i_1}(i).$$

For every $v \in W$, $k \in [1, \ell(v)]$ and $i_0 = i_1 i_2 \in D(v)$ such that $i_1 \in R(1,v)$ and $i_2 \in R(1,0)$, we recycle the notation of Subsection [3.2.2] by denoting $i_0(k) := i_1(k)i_2$. Let us
also introduce a variation of (3.7) by setting:

$\xi_{(i\to j)} := \mu_{(i\to j)} \circ \cdots \circ \mu_{i_1} \circ \mu_{i_0}$ and $\xi_{(i\to j)} := \xi_{i_0} \circ \cdots \circ \xi_{i_0}$.

**Lemma 4.2.** For every $I \subset [1, \ell]$ stable under the involution $\star$, every $w \leq w_0(I)$, every
$i \in D(w_0(I))$ and every $(\alpha, \beta)$-word $i_0 \in D(v)$ as above, we have the following equality

$$T_{w}(i) = \hat{\mu}_{i_0(\ell(w))} \circ \xi_{(i\to j)} \circ \mu_{i \to i_0}.$$ 

**Proof.** The proof of this lemma is done by induction on the length of $w \in W$, by using
the equality $\xi_{(i\to j)} = \xi_{i_0} \circ \xi_{(i\to j)}$. The first step of this induction, that is when $\ell(w) = 1$, comes from the definition (4.1) of the automorphism $T_j$.

**Lemma 4.3.** For every $v \in W$ and every reduced words $i, j \in R(1, v)$, or $i, j \in R(v, 1)$,
we have the equality $\mu_{\hat{\varphi}} \circ \varphi_{\hat{\varphi}} \circ \zeta_{i} = \zeta_{j} \circ \mu_{\hat{j}} \circ \zeta_{i}$. 

**Proof.** We suppose that $i, j \in R(1, v)$. Let us recall that the involution $\square$ maps double reduced words to double reduced words, and that the evaluation map $ev_j$ associated to any
double reduced word $j$ is birational because of Theorem 3.1. Therefore an equality $y = z$ between cluster variables on $X_{[i]}$ is satisfied if and only if the equality $ev_{[j]}(y) = ev_{[j]}(z)$ is satisfied on $G$. Now, it suffices to apply Theorem 3.2 and the second equation of (3.8) to obtain the following equality for every $x \in X_i$. The case $i, j \in R(v, 1)$ is proved in the same way.

$$ev_{[j]} \circ \varphi_{\hat{\varphi}} = [ev_{j} \circ \mu_{i \to j}(x) \circ \mu_{j \to i}]_{\leq 0} = ev_{[j]} \circ \varphi_{\hat{\varphi}} \circ \mu_{i \to j}(x) = ev_{[j]} \circ \mu_{j \to i} \circ \varphi_{\hat{\varphi}} \circ \mu_{i \to j}(x).$$

We can now prove Theorem 4.1.

**Proof.** Theorem 4.1 is clearly true if the set $I$ contains only one element, so let us take
$i, j \in I$ such that $i \neq j$. Let us recall that for every $I \subset [1, \ell]$ and every $w \in W_I$, we
have the relation $w \leq w_0(I)$. Therefore, there exist reduced expressions associated to the
element $w_0(I) \in W$ such that the $m_{i j}^{th}$ first letters are the strings $i(ij) := i j i \ldots$ and
Because of (3.9) and the equalities

\[
\hat{i} = \hat{i}.
\]

We introduce the notation \(\hat{i} \rightarrow j\) and \(\hat{j} \rightarrow i\). To prove the theorem, we have thus to prove the equality

\[
\tau_{w_{ij}}(i) = \tau_{w_{ji}}(i).
\]

To do it, we proceed in several steps, where each one proves the commutativity of a given diagram. First of all, applying Lemma 4.3 on the reduced words \(i(ij)\) and \(i(ji)\), we get the equality

\[
\mu_{i(ij)} \circ \tau_{i(ij)} = \tau_{i(ji) \circ \mu_{i(ji)}}.
\]

This relation is then extended to the double reduced words \(i_0\) and \(j_0\), because mutations and left tropical mutations commute with amalgamations done on the right. More precisely, let \(i_1\) and \(j_1\) be the double reduced words such that \(i_0 = i(ij)\) and \(j_0 = i(ji)\). We introduce the notation \(i_0^0 := (i(ij))\) and \(j_0^0 := (i(ji))\) and get

\[
\tau_{i_0}(\leq \ell(w_{ij})) = \mu_{i_0^0} \circ \tau_{j_0}(\leq \ell(w_{ji})) = \mu_{i_0} \tau_{j_0}.
\]

Because of (3.9) and the equalities \(\mu_{i_0} = \mu_{i_0^0}\) and \(\mu_{j_0} = \mu_{j_0^0}\), the previous equality implies the commutativity of the diagram

\[
\begin{array}{c}
\xymatrix{
\mathcal{X}_{[i_0]} \ar@/_/[r]_{\tau_{i_0}(\leq \ell(w_{ij}))} & \mathcal{X}_{[j_0]} \\
\mathcal{X}_{[j_0]} \ar@/^/[r]_{\tau_{j_0}(\leq \ell(w_{ji}))} & \mathcal{X}_{[i_0]} \\
\mathcal{X}_{[j_0]} \ar@/_/[u]_{\mu_{i_0^0}} & \mathcal{X}_{[i_0]} \ar@/^/[u]_{\mu_{j_0^0}} \\
\mathcal{X}_{[i_0]} \ar@/_/[u]_{\tau_{i_0}^0} & \mathcal{X}_{[j_0]} \ar@/^/[u]_{\tau_{j_0}^0}
}
\end{array}
\]

Next, let \(J\) be the maximal set, for the inclusion map, of pairwise disjoint elements of the set \(\{i, j, i^*, j^*\}\). (For example, we have \(J = \{i, j\}\) if the involution \(*\) is the identity map, \(J = \{i, j, j^*\}\) if \(i = i^* \neq j^* = j\), and so on). Lemma 4.2, applied on \(J\), then leads to the commutativity of the following diagrams.

\[
\begin{array}{c}
\xymatrix{
\mathcal{X}_{[i]} \ar@/_/[r]_{\tau_{w_{ij}}(i)} & \mathcal{X}_{[j]} \\
\mathcal{X}_{[j]} \ar@/^/[r]_{\tau_{w_{ji}}(i)} & \mathcal{X}_{[i]} \\
\mathcal{X}_{[i]} \ar@/_/[u]_{\mu_{i_0}} & \mathcal{X}_{[j]} \ar@/^/[u]_{\mu_{j_0}} \\
\mathcal{X}_{[j]} \ar@/_/[u]_{\mu_{i_0^0}} & \mathcal{X}_{[i]} \ar@/^/[u]_{\mu_{j_0^0}}
}
\end{array}
\]

Finally, for every \(v \in W\) and \(i, j, k \in D(v)\), the transitive equality \(\hat{\mu}_{i \rightarrow k} = \hat{\mu}_{j \rightarrow k} \circ \hat{\mu}_{i \rightarrow j}\) implies the commutativity of the following diagrams, where \(\text{Id}\) denotes the identity map on \(\mathcal{X}_{[i]}\).

\[
\begin{array}{c}
\xymatrix{
\mathcal{X}_{[i]} \ar@/_/[r]_{\mu_{i_0}} & \mathcal{X}_{[j]} \\
\mathcal{X}_{[j]} \ar@/^/[r]_{\mu_{j_0}} & \mathcal{X}_{[i]} \\
\mathcal{X}_{[i]} \ar@/_/[u]_{\mu_{i_0}} & \mathcal{X}_{[j]} \ar@/^/[u]_{\mu_{j_0}} \\
\mathcal{X}_{[j]} \ar@/_/[u]_{\mu_{i_0^0}} & \mathcal{X}_{[i]} \ar@/^/[u]_{\mu_{j_0^0}}
}
\end{array}
\]

We now incorporate the previous diagram in the synthesis diagram (4.5), via the following three steps procedure: 1) put the diagram (4.2) between the two diagrams constituting (4.3) and identify the arrows which have the same transformation as labeling; 2) put
the left diagram of (4.4) upside the new diagram thus obtained and identify the arrows which have the same transformation as labeling again; 3) put the right diagram of (4.4) downside the diagram, just as before and still identify the arrows which have the same transformation as labeling. The relation $T_{w_{ij}}(i) = T_{w_{ij}}(i)$ is then given by the boundary of the diagram (4.5).

\[ \xymatrix{ & \mathcal{X}_{[i|_R]} \ar[rr]^{\hat{\mu}_{i_{0} \rightarrow j_{0}}} \ar[dr]_{\hat{\mu}_{j_{0} \rightarrow i_{0}}} & & \mathcal{X}_{[i|_R]} \ar[dl]_{\hat{\mu}_{i_{0} \rightarrow j_{0}}} \ar[rr]^{\hat{\mu}_{j_{0} \rightarrow i_{0}}} & \mathcal{X}_{[i|_R]} \ar[dr]_{\hat{\mu}_{i_{0} \rightarrow j_{0}}} \ar[rr]^{\hat{\mu}_{j_{0} \rightarrow i_{0}}} & \mathcal{X}_{[i|_R]} \ar[dl]_{\hat{\mu}_{i_{0} \rightarrow j_{0}}} \ar[rr]^{\hat{\mu}_{j_{0} \rightarrow i_{0}}} & \mathcal{X}_{[i|_R]} \ar[dr]_{\hat{\mu}_{i_{0} \rightarrow j_{0}}} \ar[rr]^{\hat{\mu}_{j_{0} \rightarrow i_{0}}} & \mathcal{X}_{[i|_R]} \ar[dl]_{\hat{\mu}_{i_{0} \rightarrow j_{0}}} \ar[rr]^{\hat{\mu}_{j_{0} \rightarrow i_{0}}} & \mathcal{X}_{[i|_R]} \ar[dr]_{\hat{\mu}_{i_{0} \rightarrow j_{0}}} \ar[rr]^{\hat{\mu}_{j_{0} \rightarrow i_{0}}} & \mathcal{X}_{[i|_R]} \ar[dl]_{\hat{\mu}_{i_{0} \rightarrow j_{0}}} \ar[rr]^{\hat{\mu}_{j_{0} \rightarrow i_{0}}} & \mathcal{X}_{[i|_R]} \ar[dr]_{\hat{\mu}_{i_{0} \rightarrow j_{0}}} \ar[rr]^{\hat{\mu}_{j_{0} \rightarrow i_{0}}} & \mathcal{X}_{[i|_R]} \ar[dl]_{\hat{\mu}_{i_{0} \rightarrow j_{0}}} \ar[rr]^{\hat{\mu}_{j_{0} \rightarrow i_{0}}} & \mathcal{X}_{[i|_R]} } \]

\[ (4.5) \]

4.2. The De-Concini-Kac-Procesi automorphisms. Following [DCKP92] and [B02], we define for every $j \in [1, \ell]$ and $U_j = \exp(g_{a_j})$ the homomorphism $\xi_j : N_- \rightarrow U_j$ such that if $n_- \in N_-$ is factorized as a product of $u_\beta \in U_\beta$ for every negative root $\beta$ (each $\beta$ appearing only once), then $\xi_j(n_-) := u_{a_j}$. Let us recall that, as a set, the dual Poisson-Lie group $(G^*, \pi_G^*)$ is given by elements $(n_+ t, n_- t^{-1})$ such that $n_+ \in N_+$ and $t \in H$ and is isomorphic to the set of elements $n_+ t^2 n_-^{-1}$ in $G$, where we have still $n_+ \in N_+$ and $t \in H$. We denote $G^0$ this set; we therefore get a Poisson isomorphism $(G^*, \pi_G^*) \simeq (G^0, \pi_*)$. For every $j \in [1, \ell]$, we denote $b_{\beta_-} := \xi_j(n_-)^{-1}$ and recall the De-Concini-Kac-Procesi automorphism

\[ T_j : G^0 \rightarrow G^0 : n_+ t^2 n_-^{-1} \mapsto s_j \cdot b_{\beta_-} \cdot n_+ t^2 n_-^{-1} \cdot (s_j \cdot b_{\beta_-}^{-1})^{-1}. \]

The cluster combinatorics of the De Concini-Kac-Procesi Poisson automorphisms $T_j$ is established via the following results.

**Proposition 4.4.** Let $i$ be a double word, such that $i = i_1 i_2$ with $i_1, i_2 \in R(1, w_0)$, starting with the letter $j$, then $T_j = \hat{\nu} \circ \mu_{j_{0}} \circ \hat{\nu}_{i_{0}}^{-1}$. 

**Corollary 4.5.** We have the equality $T_j = \hat{\nu} \circ T_j(i) \circ \hat{\nu}_{i_{0}}^{-1}$ for every $i \in D(w_0)$.

Corollary 4.5 is deduced from Theorem 3.7, the formula (4.1) and Proposition 4.4. To prove Proposition 4.4, we need some preparation.

Let $u, v \in W$ and $i \in R(u, v)$. The double reduced word $i^* \in R(v^*, u^*)$ is obtained by transforming each letter $i$ of $[1, \ell] \cup [\ell, \ell]$, so if $i = i_1 \ldots i_n$ then $i^* = \overline{i_1} \ldots \overline{i_n}$. Starting with an elementary double word $i \in \{1, i, \overline{i}\}$, where $i \in [1, \ell]$, and then applying the properties of the amalgamated product, we easily prove the following lemma.
Lemma 4.6. Let \( u, v \in W \) and \( i \in R(u, v) \). For every cluster \( x \in \mathcal{X}_i \), let \( x^* \in \mathcal{X}_{i^*} \) be such that the equality \( \tilde{w}_0 \text{ev}_i(x) \tilde{w}_0^{-1} = \text{ev}_{i^*}(x^*) \) is satisfied. Then we have

\[
x^*_i(j) = \begin{cases} 
-x^{-1}_{j(i)} & \text{if } 0 = j \neq N^{i^*}(i) \text{ or } 0 \neq j = N^{i^*}(i); \\
x^{-1}_{j(i)} & \text{otherwise.}
\end{cases}
\]

A split of a seed \( \mathbf{I} \) is a pair of seeds \( (\mathbf{I}_1, \mathbf{I}_2) \) such that \( \mathbf{I} \) is their amalgamated product, that is \( \mathbf{I} = \mathbf{m}(\mathbf{I}_1, \mathbf{I}_2) \). An associated \( \mathcal{X} \)-split is a section of the amalgamation map \( \mathbf{m} : \mathcal{X}_{\mathbf{I}_1} \times \mathcal{X}_{\mathbf{I}_2} \to \mathcal{X}_{\mathbf{I}} \), i.e. a map \( \mathbf{s} : \mathcal{X}_{\mathbf{I}} \to \mathcal{X}_{\mathbf{I}_1} \times \mathcal{X}_{\mathbf{I}_2} \) such that the product \( \mathbf{m} \circ \mathbf{s} \) gives the identity map on \( \mathcal{X}_{\mathbf{I}} \). For every \( \mathcal{X} \)-split \( \mathbf{s} \) associated to the decomposition \( \mathbf{I} \to (\mathbf{I}_1, \mathbf{I}_2) \), we associate to any \( x \in \mathcal{X}_i \), some elements \( x_1 \in \mathcal{X}_{i_1} \) and \( x_2 \in \mathcal{X}_{i_2} \) given by \( \mathbf{s}(x) = (x_1, x_2) \).

Now, for every reduced word \( i = i_1 \ldots i_{\ell(w_0)} \in R(w_0, 1) \), and every \( k \in [1, \ell(w_0)] \), let us set \( w_{1:k} := s_{i_{k+1}} \ldots s_{i_{\ell(w_0)}} \). To every \( x \in \mathcal{X}_i \), we associate the following product of \( u_\beta \in U_\beta \) over negative roots, which is such that every negative root \( \beta \) appears exactly once.

\[
\tau_i(x) = \prod_{k=1}^{\ell(w_0)} w_{1:k}^{-\alpha_k}(x) (-x_{\zeta_{k-1}(u_k)}^{-1})^{w_{1:k}}, \quad \text{where } x_\tau(t) = \varphi_i \begin{pmatrix} 1 & 0 & 0 \\ t & 1 & 1 \end{pmatrix}.
\]

Here, the map \( \zeta_{ \leq k-1} : \mathcal{X}_i \to \mathcal{X}_{i(k-1)} \) is obtained by generalizing the formula (3.7):

\[
\zeta_{\leq k-1} = \zeta_{\delta(1)} \circ \cdots \circ \zeta_{\delta(k-1)}.
\]

In particular, the equality \( \zeta_{\leq \ell(w_0)} = \zeta_i \) is satisfied.

Lemma 4.7. [3 Lemma 9.10] Let \( i \) be a double word such that \( i = i_1 i_2 \) with \( i_1, i_2 \in R(1, w_0) \), and \( \mathbf{s} \) be a \( \mathcal{X} \)-split relative to the decomposition \( i \to (i_1, i_2) \). We have the equality \( [\text{ev}_i(x)]^{-1} = \tau_i(x^*_i) \).

Remark 4.8. The definition of \( \tau_i \) implies that the choice of the \( \mathcal{X} \)-split \( \mathbf{s} \) associated to the decomposition \( i \to (i_1, i_2) \) in the previous lemma doesn’t matter.

We can now prove Proposition 4.4.

**Proof.** Let \( \mathbf{s} \) be a \( \mathcal{X} \)-split relative to the decomposition \( i \to (i_1, i_2) \). Lemma 4.7 gives an expression of \( [\text{ev}_i(x)]^{-1} \) as a product of terms \( u_\beta \) where all the negative roots \( \beta \) appear exactly once. Now, let us remark that \( w_0 s_{i_1}^{-1}(\alpha_{i_1}) = \alpha_{i_1} \). Applying the definition of \( \xi_j \), Lemma 4.7 and the formula (4.7), we thus get

\[
b_j^i = \xi_j( [\text{ev}_i(x)]^{-1}) = x_j( -x_{s_j^i}^{-1} )^{-1} = x_j( x_j( -x_{s_j^i}^{-1} )^{-1} ) = x_j( -x_{s_j^i}^{-1} ) .
\]

Define \( j_1 \) and \( y \in \mathcal{X}_{j_1} \) such that \( j_1 = \mathcal{L}(i_1) \) and \( y = \mu_{(i_1)}(x_{(i_1)}) \). We have

\[
\hat{s}_j^{-1} \text{ev}_{j_1}(y) = x_j( -y_{(j_1)}^{-1} ) \text{ev}_{j_1} \circ \mu_{(j_1)}(y).
\]

Indeed, let us remember the map \( \varphi_j : \text{SL}(2, \mathbb{C}) \hookrightarrow G \) defined in Section 2. For any nonzero \( t \in \mathbb{C} \) and any \( i \in [1, \ell] \), let us denote

\[
x_{i}(t) = \varphi_{i} \begin{pmatrix} 1 & t & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad x_{\tau}(t) = \varphi_{i} \begin{pmatrix} 1 & 0 & 0 \\ t & 1 \end{pmatrix}.
\]

An elementary matrix calculus on \( \text{SL}(2, \mathbb{C}) \) leads to the following equality on \( G \), satisfied for every \( j \in [1, \ell] \).

\[
\hat{s}_j^{-1} x_{\tau}(t) = x_{\tau}(t^{-1}) \exp(\log(t)h_j) x_{\tau}(t^{-1}) .
\]
Using the definition of tropical mutation, and the fact that a left tropical mutation commutes with an amalgamation done on the right, we deduce the relation (4.8). Then, from (4.8), we get the following series of equalities
\[
\tilde{s}_j b_j^i \text{ev}_{H}(x_{(1)}) = \tilde{s}_j x_j^i (x_{(1)}) = \text{ev}_{H}(y) = \text{ev}_{H}(\mu_{\hat{\mu}}(x_{(1)})).
\]
This relation is then extended to the evaluation maps ev_{\hat{\mu}}, ev_{\hat{\mu}}^R and \tilde{e}_I by using (3.3) and the fact that a left tropical mutation commutes with an amalgamation done on the right. Thus, the definition (4.6) leads to the equality \( T_j \circ \tilde{e}_I = \tilde{e}_I \circ \mu_{\hat{\mu}} \).

5. The case \( G = \text{PGL}(2, \mathbb{C}) \).

To fix the ideas, we consider our construction in the case \( \text{PGL}(2, \mathbb{C}) \). Let us recall that the complex simple Lie group
\[
SL(2, \mathbb{C}) = \{ \left( \begin{array}{cc} t_{11} & t_{12} \\ t_{21} & t_{22} \end{array} \right) : t_{11}t_{22} - t_{12}t_{21} = 1, \ t_{ij} \in \mathbb{C} \}.
\]
has its Lie algebra \( \mathfrak{g} \) equal to the vector space \( \text{sl}(2, \mathbb{C}) \) of 2-squared complex matrices which have a zero trace. The Chevalley generators \( \{ e_1, f_1, h_1 \} \) and its related basis \( \{ e_1, f_1, h^1 \} \) are then given by the following matrices:
\[
e_1 = \left( \begin{array}{cc} 0 & 1 \\ 0 & 0 \end{array} \right), \ f_1 = \left( \begin{array}{cc} 0 & 0 \\ 1 & 0 \end{array} \right), \ h_1 = \left( \begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right), \ h^1 = \left( \begin{array}{cc} 1/2 & 0 \\ 0 & -1/2 \end{array} \right).
\]
Using the exponential map \( \exp : \mathfrak{g} \to G \), which, in this case, associates to a matrix \( M \in \mathfrak{g} \) the usual matrix \( \sum_{n=0}^{\infty} \frac{M^n}{n!} \in G \), we get the following generators of \( G \), the two last ones being associated to every non-zero complex number \( x \).
\[
E^1 = \left( \begin{array}{cc} 1 & 1 \\ 0 & 1 \end{array} \right), \ F^1 = \left( \begin{array}{cc} 1 & 0 \\ 1 & 1 \end{array} \right), \ H_1(x) = \left( \begin{array}{cc} x & 0 \\ 0 & x^{-1} \end{array} \right), \ H^1(x) = \left( \begin{array}{cc} x^{1/2} & 0 \\ 0 & x^{-1/2} \end{array} \right).
\]
Let us notice that \( H^1(x) \) is well-defined on \( \text{PGL}(2, \mathbb{C}) \), because of the identity
\[
H^1(x) = \left( \begin{array}{cc} x^{1/2} & 0 \\ 0 & x^{-1/2} \end{array} \right) \text{PGL}(2, \mathbb{C}) = \left( \begin{array}{cc} x & 0 \\ 0 & 1 \end{array} \right).
\]
Now, because there is only one simple root \( \alpha_1 \), the Weyl group \( W \) contains only two elements \( \{ 1, s_1 \} \) and the different double reduced words are the double words \( 1, 1, 1, 1, \ T_1, \ T_1, \), where \( 1 \) is the unity of the direct product \( W \times W \). Finally, the \( r \)-matrix \( r \in \mathfrak{g} \) associated to \( \text{sl}(2, \mathbb{C}) \) and its related elements \( r_{\pm} \in \mathfrak{g} \) are given by the following formulas.
\[
r = e_1 \wedge f_1, \ r_+ = \frac{1}{4} h_1 \otimes h_1 + e_1 \otimes f_1 \quad \text{and} \quad r_- = -\frac{1}{4} h_1 \otimes h_1 - f_1 \otimes e_1.
\]
For every \( i, j \in \{ 1, 2 \} \), let \( t_{ij} \) be the coordinate function on the matrices (5.1). Applying the formula (5.2) on the Semenov-Tian-Shansky Poisson bracket given by Proposition 2.1, it is easy to prove that in the matricial case, the Poisson bracket on \( (G, \pi_a) \) is given by the following equalities:
\[
\begin{align*}
\{ t_{11}, t_{12} \}_* &= t_{12}t_{22}, & \{ t_{11}, t_{21} \}_* &= -t_{21}t_{22}, \\
\{ t_{11}, t_{22} \}_* &= 0, & \{ t_{12}, t_{21} \}_* &= t_{11}t_{22} - t_{22}^2, \\
\{ t_{12}, t_{22} \}_* &= t_{12}t_{22}, & \{ t_{21}, t_{22} \}_* &= -t_{21}t_{22}.
\end{align*}
\]
Let us then consider the related evaluation maps. It is easy to check that the evaluation \( \hat{ev}_1 : \mathcal{X}_{[1]}_{\mathfrak{g}} \to (G, \pi_s) \) is Poisson. It is indeed given by the following expression:

\[
\hat{ev}_1(x_0, t) = \begin{pmatrix}
t^{1/2} + t^{-1/2} & -x_0t^{-1/2} \\
x_0^{-1/2}t^{1/2} & 0
\end{pmatrix}.
\]

The evaluations \( \hat{ev}_{\mathbb{T}} : \mathcal{X}_{\mathbb{T}}_{\mathfrak{g}} \to BB_- \) and \( \hat{ev}_{\mathbb{T}} : \mathcal{X}_{\mathbb{T}}_{\mathfrak{g}} \to BB_- \), parameterizing the subvariety \( BB_- \), are then obtained by the following formulas.

\[
\hat{ev}_{\mathbb{T}}(y_0, y_1, t) = \begin{pmatrix}
t^{-1/2}(1 + y_1) + t^{1/2} & -y_0y_1t^{-1/2} \\
y_0^{-1}(t^{1/2}(1 + y_1^{-1}) + t^{-1/2}(1 + y_1)) & -y_1t^{-1/2}
\end{pmatrix}.
\]

And it is straightforward to check that \( \hat{ev}_{\mathbb{T}} : \mathcal{X}_{\mathbb{T}}_{\mathfrak{g}} \to BB_- \) and \( \hat{ev}_{\mathbb{T}} : \mathcal{X}_{\mathbb{T}}_{\mathfrak{g}} \to BB_- \) are given by:

\[
\hat{ev}_{\mathbb{T}}(z_0, z_1, t) = \hat{ev}_{\mathbb{T}}(z_0, z_1, t) = \begin{pmatrix}
(1 + z_1^{-1})t^{1/2} + t^{-1/2} & -z_0((1 + z_1^{-1})t^{1/2} + (1 + z_1)t^{-1/2}) \\
z_0^{-1}z_1^{-1}t^{1/2} & -z_1^{-1}t^{-1/2}
\end{pmatrix}.
\]

It is easy to check that all these maps are Poisson when the matrices establishing the Poisson structure on the related seed \( \mathcal{X} \)-tori are given by

\[
\eta(11) = \eta(11) = \begin{pmatrix}
0 & -1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}, \quad \eta(11) = \eta(11) = \begin{pmatrix}
0 & 1 & 0 \\
-1 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}.
\]

We thus get two cluster \( \mathcal{X} \)-varieties for the variety \( BB_- \), denoted \( \mathcal{X}_e \) and \( \mathcal{X}_{\mathfrak{w}_0} \), and respectively associated to the cluster variables \((y_0, y_1, t)\) and \((z_0, z_1, t)\). They are linked in the following way: if the elements \( \hat{ev}_{\mathbb{T}}(y_0, y_1, t) \) and \( \hat{ev}_{11}(z_0, z_1, t) \) are equal, we quickly check with the expressions above that the map \( \phi : (y_0, y_1, t) \mapsto (z_0, z_1, t) \) is given by

\[
\begin{cases}
    z_0 = y_0(1 + y_1^{-1})^{-1}(1 + y_1^{-1}t)^{-1} \\
    z_1 = ty_1^{-1}
\end{cases}.
\]

In fact, we have the equality \( \phi = \Xi_{s_1} \), coming from the following formula for \( \Xi_{s_1} \).

\[
\Xi_{s_1}(y_0, y_1, t) = \mu_{\mathbb{T}_{\mathfrak{g}} \to [\mathbb{T}]_{\mathfrak{g}}} \circ \Xi_{1} \circ \mu_{\mathbb{T}_{\mathfrak{g}} \to [\mathbb{T}]_{\mathfrak{g}}}(y_0, y_1, t) = (y_0(1 + y_1^{-1})^{-1}(1 + y_1^{-1}t)^{-1}, y_1^{-1}t, t).
\]

Finally, we use tropical mutations to describe the De-Concini-Kac-Procesi Artin group action \((4.1)\) on \((\text{PGL}(2, \mathbb{C}), \pi_s)\). The cluster combinatorics is given by the following Poisson automorphism on the seed \( \mathcal{X} \)-torus \( \mathcal{X}_{[1]}_{\mathfrak{g}} \).

\[
\mathcal{T}_1(11) : \mathcal{X}_{[1]}_{\mathfrak{g}} \to \mathcal{X}_{[1]}_{\mathfrak{g}} : (z_0, z_1, t) \mapsto \Xi_{s_1} \circ \mu_{\mathbb{T}_{\mathfrak{g}}}^{-1}(z_0, z_1, t)
\]

satisfies \( \mathcal{T}_1(11)(z_0, z_1, t) = (z_0^{-1}(1 + z_1^{-1})^{-1}(1 + z_1^{-1}t)^{-1}, z_1^{-1}t, t) \).
Let us stress, however, that this birational Poisson isomorphism is not an involution; indeed, a straightforward computation gives the equality
\[ T^2(z_0, z_1, t) = (z_0z_1^{-2}t^2, z_1, t). \]
Therefore, the action of the center \( Z(B_9) \) of \( B_9 \) on \((BB_-, \pi_+)\) given by the De Concini-Kac-Procesi automorphism is not trivial.

References


[FG07a] Fock, V. V.; Goncharov, A. B. Cluster ensembles, quantization and the dilogarithm, [arXiv:math/0311245]


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