We define a Diophantine condition for interval exchange transformations (i.e.t.). When the number of intervals is two, that is for rotations on the circle, our condition coincides with the one in the classical Khinchin theorem, modulo the identification of a rotation with its rotation number. We prove that for i.e.t.s we have the same dichotomy of Khinchin theorem.
KHINCHIN THEOREM FOR INTERVAL EXCHANGE TRANSFORMATIONS.

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Abstract. We define a diophantine condition for interval exchange transformations (i.e.t.). When the number of intervals is two, that is for rotations on the circle, our condition coincides with the one in the classical Khinchin theorem, modulo the identification of a rotation with its rotation number. We prove that for i.e.t.s we have the same dichotomy of Khinchin theorem.

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1. Introduction.

A rotation on the circle is completely described by its rotation number and we can establish a very precise dictionary between the dynamical properties of the transformations in this class and the arithmetical properties of the rotation number. For example a rotation is periodic if and only if its rotation number is rational and conversely all irrational rotations are minimal. Besides this first coarse dichotomy
we can classify real numbers according to their diophantine properties, which in terms of rotations on the circle corresponds to give a measure of recurrence. In particular, for $\vartheta \in [0, 1)$ we can define the most general diophantine condition by

$$\{ n \vartheta \} < \varphi(n),$$

where $\{x\} := \min\{x - k; k \in \mathbb{Z}, k \leq x\}$, and where $\varphi : \mathbb{N} \to \mathbb{R}_+$ is a positive (vanishing) sequence. If $R_{\vartheta}$ is the rotation with rotation number $\vartheta$, then a solution $n \in \mathbb{N}$ of equation (1.1) corresponds to an iterate $R_{\vartheta}^n$ at (small) distance $\varphi(n)$ from the identity. Khinchin theorem is a classical result in arithmetic and establish a dichotomy between two opposite behaviors (see [K]).

**Theorem (Khinchin).** Let $\varphi : \mathbb{N} \to \mathbb{R}_+$ be a positive sequence such that $n \varphi(n)$ is monotone decreasing.

- If $\sum_{n=1}^{\infty} \varphi(n) < \infty$ then equation (1.1) has just finitely many solutions $n \in \mathbb{N}$ for almost any $\vartheta$.
- If $\sum_{n=1}^{\infty} \varphi(n) = \infty$ then for almost any $\vartheta$ equation (1.1) has infinitely many solutions $n \in \mathbb{N}$.

1.1. Interval exchange transformations. An alphabet is a finite set $A$ with $d \geq 2$ elements. An interval exchange transformation (also called i.e.t.) is a map $T$ from an interval $I$ to itself such that $T$ admits two finite partitions $P_t := \{I_\alpha^t\}_{\alpha \in A}$ and $P_b := \{I^b_\beta\}_{\beta \in A}$ into $d$ open intervals and for any $\alpha \in A$ the restriction of $T$ to the interval $I^t_\alpha$ is a translation with image the interval $I^b_\alpha$. The map $T : I \to I$ is therefore defined by rearranging (via translations) the intervals of the partition $P_t$ in a new order inside $I$ given by $P_b$ and is entirely defined by the following data:

1. The lengths of the intervals,
2. The order before and after rearranging.

The first are called **length data**, and are given by a vector $\lambda \in \mathbb{R}_+^A$, where $\lambda_\alpha$ denotes the length of $I^t_\alpha$ (which is equal to the length of $I^b_\alpha$) for any $\alpha \in A$. The second are called **combinatorial data** and are given by a pair of bijections $\pi = (\pi^t, \pi^b)$ from $A$ to $\{1, \ldots, d\}$. The meaning of $\pi$ is that for any $\alpha \in A$, if we count starting from the left, the interval $I^t_\alpha$ is in $\pi^t(\alpha)$-th position in $P_t$ and $I^b_\alpha$ is in $\pi^b(\alpha)$-th position in $P_b$. For any combinatorial datum $\pi$ let us call $\Delta_\pi := \{\pi\} \times \mathbb{R}_+^d$ the set of all i.e.t.s with combinatorial datum $\pi$. Let us consider any $T \in \Delta_\pi$ and let us write $T = (\pi, \lambda)$, where $\lambda$ is the corresponding length datum. For any $\alpha \in A$ with $\pi^t(\alpha) > 1$ we call $u^t_\alpha$ the left endpoint of $I^t_\alpha$. In general $T$ is not continuous at $u^t_\alpha$. Similarly for any $\beta \in A$ with $\pi^b(\beta) > 1$ we call $u^b_\beta$ the left endpoint of $I^b_\beta$. In general the inverse $T^{-1}$ of $T$ is not continuous at $u^t_\beta$. If we identify the interval $I$ with $(0, \sum_{\alpha \in A} \lambda_\alpha)$ then the position of the singularities $u^t_\alpha$ and $u^b_\beta$ is given by

$$u^t_\alpha := \sum_{\pi^t(\alpha') < \pi^t(\alpha)} \lambda_{\alpha'} \quad \text{and} \quad u^b_\beta := \sum_{\pi^b(\beta') < \pi^b(\beta)} \lambda_{\beta'}.$$

We say that the combinatorial datum $\pi$ is **admissible** if there is no proper subset $A' \subset A$ with $k < d$ elements such that $\pi^t(A') = \pi^b(A') = \{1, \ldots, k\}$. In the following always consider admissible combinatorial data. A **connection** for $T : I \to I$ is a triple $(\beta, \alpha, n)$ with $\pi^b(\beta) > 1$, $\pi^t(\alpha) > 1$ and $n \in \mathbb{N}$ such that $T^n u^b_\beta = u^t_\alpha$. In particular, if $T$ has no connections then $\pi$ is admissible.

Veech [Ve] and then Zorich [Z1] introduced a re-normalization map acting on i.e.t.s which generalizes the **Gauss map**. It happens that the **infinitely re-normalizable**
i.e.t.s are those without connections, exactly as irrational real numbers are the points where the Gauss map can be iterated infinitely many times. This reason lead us to think to i.e.t.s with connections as rational numbers. Another reason to trust this philosophy is Keane’s theorem (see [Ke]), which says that if $T$ has no connections, then it is minimal (we remark anyway that there is not an true dichotomy as for rotations, since there exist minimal i.e.t.s with connections).

**Khinchin type condition for i.e.t.s.** In order to define the diophantine condition that we study in this paper let us consider a positive sequence $\varphi : \mathbb{N} \to \mathbb{R}_+$ such that $n\varphi(n)$ is decreasing monotone. Let $T : I \to I$ be an i.e.t. without connections with combinatorial datum $\pi$. We consider triples $(\beta, \alpha, n)$ with $\pi^b(\beta) > 1$, $\pi^t(\alpha) > 1$ and $n \in \mathbb{N}$ such that

$$T^n(u^b_\beta) - u^t_\alpha < \varphi(n).$$

For any triple $(\beta, \alpha, n)$ as above let us call $I(\beta, \alpha, n)$ the open subinterval of $I$ whose endpoints are $T^n(u^b_\beta)$ and $u^t_\alpha$ (their reciprocal order does not matter).

**Definition 1.1.** Let $T$ be an i.e.t. with admissible combinatorial datum $\pi$ and $(\beta, \alpha, n)$ be a triple with $\pi^b(\beta) > 1$, $\pi^t(\alpha) > 1$ and $n \in \mathbb{N}$. We say that $(\beta, \alpha, n)$ is a reduced triple for $T$ if for any $k \in \{0, \ldots, n\}$ the pre-image $T^{-k}(I(\beta, \alpha, n))$ of $I(\beta, \alpha, n)$ does not contain in its interior any singularity $u^t_{\alpha'}$ for $T$ or any singularity $u^b_{\beta'}$, for $T^{-1}$ (where $\alpha', \beta' \in A$ and $\pi^b(\beta') > 1, \pi^t(\alpha') > 1$).

**Definition 1.2.** Given a function $\varphi : \mathbb{N} \to \mathbb{R}_+$ as before and an admissible $\pi$, an interval exchange transformation $T \in \Delta_\pi$ is said

- mod-$\varphi$-Diophantine if equation (1.2) has just finitely many solutions.
- mod-$\varphi$-Liouville if for any pair of letters $\beta, \alpha$ with $\pi^b(\beta) > 1$ and $\pi^t(\alpha) > 1$ there exists infinitely many triples $(\beta, \alpha, n)$ reduced for $T$ which are solution of equation (1.2).

We proved the following generalization of Khinchin theorem.

**Theorem 1.3.** Let us consider a positive sequence $\varphi : \mathbb{N} \to \mathbb{R}_+$ such that $n\varphi(n) : \mathbb{N} \to \mathbb{R}_+$ is decreasing monotone. For any admissible combinatorial datum $\pi_0$ we have the following dichotomy:

a: If $\sum_{n=1}^{\infty} \varphi(n) < +\infty$ then almost any i.e.t. $T \in \Delta_{\pi_0}$ is mod-$\varphi$-Diophantine.

b: If $\sum_{n=1}^{\infty} \varphi(n) = +\infty$ then almost any i.e.t. $T \in \Delta_{\pi_0}$ is mod-$\varphi$-Liouville.

I.e.t.s are related to translation surfaces and to the Teichmüller flow on their moduli space (see [Ve], [Ma1] and [Z2]). In particular in [Mar2] we prove a version of theorem 1.3 for translation surfaces which implies a refinement of Masur’s logarithmic law for the Teichmüller flow on strata of the moduli space of translation surfaces (see [Ma2] for Masur’s original result).

**Linear involutions** are a natural generalization of i.e.t.s introduced in [DaNo] by Danthony and Nogueira. In [BoLa] Boissy and Lanneau related linear involutions to half-translation surfaces, which are the general contest where Masur’s logarithmic law holds. We believe that the techniques introduced in this paper can be extended to linear involutions and we ask if a generalization of theorem 1.3 can be proved for them (more precisely for the subclass of linear involutions which are relevant for half-translation surfaces, as it is explained in [BoLa]). Our guess is also motivated by the paper of Avila and Resende ([A,R]), where the authors generalize some results of [A,G,Y] which play an important role in the proof of theorem 1.3.
Boshernitzan and Chaika studied shrinking target properties for i.e.t.s related to our diophantine condition in definition 1.2 (see [B,Ch] and [Ch]). Their results are obtained independently from this article and use different techniques. In particular in [Ch] it is proven the following result.

**Theorem (Chaika).** Let \( \varphi : \mathbb{N} \to \mathbb{R}_+ \) be a positive sequence such that \( n \varphi(n) \) is decreasing monotone and \( \sum_{n=1}^{\infty} \varphi(n) = \infty \). Then for almost any i.e.t. \( T : I \to I \) with admissible combinatorial datum, for any \( x \in I \) and for almost any \( y \in I \), there are infinitely many \( n \in \mathbb{N} \) such that

\[
|T^n(x) - y| < \varphi(n).
\]

### 1.2. Decomposition of the problem.

Theorem 1.3 is an example of application to a dynamical problem of the so called Borel-Cantelli lemma (see [Bi]).

**Theorem.** Let \((X, \mathcal{F})\) be a probability space and let \((X_n)_{n \in \mathbb{N}}\) be a countable family of events in \( X \).

- If \( \sum_{n=1}^{\infty} \mathbb{P}(X_n) < +\infty \) then almost any \( x \in X \) belongs to finitely many events \( X_n \).
- On the other hand, if \( \sum_{n=1}^{\infty} \mathbb{P}(X_n) = +\infty \) and the events \( X_n \) are each other independent, then almost any \( x \in X \) belongs to infinitely many \( X_n \).

In applications to dynamics the events \( X_n \) are usually related to the iterates of some map on the space \( X \). In this case they are almost never independent in the probabilistic sense and the main difficulty is to develop some weak form of independence for them in order to make the Borel-Cantelli argument work.

In our case, for any triple \((\beta, \alpha, n)\) as in theorem 1.3 the set of those \( T \) such that \((\beta, \alpha, n)\) is reduced for \( T \) and satisfies equation (1.2) defines an event in \( \Delta_{\pi_0} \).

Roughly speaking our strategy is to prove that the probability of such event is of the same order of \( \varphi(n) \) and that, when the triple \((\beta, \alpha, n)\) varies, we have some weak form of independence for the family of the associated events. Even if this program is perfectly meaningful without any notion of dynamics, in order to pursue it we need to define a dynamical system on the space of all i.e.t.s with the same number \( d \) of intervals. The dynamic we refer to is the so-called Rauzy-Veech map, which has been introduced in [Ve] and then modified in [Z1] in order to have better ergodic properties. Unfortunately, for a generic \( T \), we are not able to translate the properties of being mod\( \varphi \)-Diophantine or mod\( \varphi \)-Liouville into equivalent ones for its orbit. We can just establish a necessary condition for the orbits of a mod\( \varphi \)-Diophantine \( T \) which holds for the convergent case (i.e. when \( \sum_{n=1}^{\infty} \varphi(n) < +\infty \)) and conversely, in the divergent case (i.e. when \( \sum_{n=1}^{\infty} \varphi(n) = +\infty \)), we can establish a sufficient condition for the orbits of \( T \) in order to have that that \( T \) is mod\( \varphi \)-Liouville. The techniques in the two cases are different, therefore the proof of part a) and of part b) of theorem 1.3 have to be treated separately. We also have to introduce a normalization on length data of i.e.t.s. A first reason is that the Borel-Cantelli lemma requires a probabilistic setting, whereas for any combinatorial datum \( \pi \) the cone \( \Delta_\pi \) has infinite lebesgue measure. On the other hand the Rauzy-Veech map has interesting recurrence properties just at projective level, that is on the space of rays in \( \Delta_\pi \). For the reasons above we will consider i.e.t.s \( T \) acting on an interval \( I \) with length one. On this codimension-one subspace of \( \Delta_{\pi_0} \) we will state separately two criterions. The first is proposition 1.4 which is sufficient to get part a) of theorem 1.3, the second is proposition 1.5 which is sufficient to get part b) of theorem 1.3.
1.2.1. Normalization of lengths. For any vector \( \lambda \in \mathbb{R}_+^d \) we introduce the notation
\[
\| \lambda \| := \sum_{\alpha \in \mathcal{A}} \lambda_\alpha \quad \text{and} \quad \hat{\lambda} := \frac{\lambda}{\| \lambda \|}.
\]
If the i.e.t. \( T \) is defined by the data \((\pi, \lambda)\), we call \( \hat{T} \) the i.e.t. whose combinatorial and length data are \((\pi, \hat{\lambda})\). For any admissible combinatorial datum \( \pi \) we introduce the \((d-1)-\)simplex \( \Delta_\pi^{(1)} := \{ T = (\pi, \lambda) \in \Delta_\pi; \| \lambda \| = 1 \} \). The map \( T \mapsto (\hat{T}, \| \lambda \|) \) establishes an homeomorphism between \( \Delta_\pi \) and \( \Delta_\pi^{(1)} \times \mathbb{R}_+ \) and by Fubini’s theorem the \( d \)-dimensional lebesgue measure \( \text{Leb}_d \) on \( \Delta_\pi \) is equivalent to the product of the \((d-1)-\)dimensional lebesgue measure \( \text{Leb}_{d-1} \) on \( \Delta_\pi^{(1)} \) with \( \text{Leb}_1 \) on \( \mathbb{R}_+ \).

We fix an admissible combinatorial datum \( \pi_0 \) as in theorem 1.3 we consider an i.e.t. \( T = (\pi_0, \lambda) \) in \( \Delta_{\pi_0} \) and the corresponding normalized i.e.t. \( \hat{T} = (\pi_0, \hat{\lambda}) \) in \( \Delta_\pi^{(1)} \). If \( u_\beta^b \) and \( u_\alpha^t \) are the singularities for \( T \) we denote \( \hat{u}_\beta^b \) and \( \hat{u}_\alpha^t \) the corresponding singularities for \( \hat{T} \). We have \( |\hat{T}^n \hat{u}_\beta^b - \hat{u}_\alpha^t| = \| \hat{\lambda} \|^{-1} |T^n u_\beta^b - u_\alpha^t| \), therefore
\[
|T^n u_\beta^b - u_\alpha^t| < \varphi(n) \Leftrightarrow |\hat{T}^n \hat{u}_\beta^b - \hat{u}_\alpha^t| < \| \hat{\lambda} \|^{-1} \varphi(n).
\]

1.2.2. Convergent case. For any positive integer \( M \) we consider the set \( \Delta_{\pi_0}(> 1/M) \) of those \( T \in \Delta_{\pi_0} \) whose length datum \( \lambda \) satisfies \( \| \lambda \| > 1/M \). If a triple \((\beta, \alpha, n)\) is solution of equation (1.2) for \( T \in \Delta_{\pi_0}(> 1/M) \), then for the normalized \( \hat{T} \) we necessarily have \( |\hat{T}^n \hat{u}_\beta^b - \hat{u}_\alpha^t| < M \varphi(n) \). We observe that if \( \sum_{n=1}^{\infty} \varphi(n) < +\infty \) then the same holds for the re-scaled sequence \( M \varphi(n) \). If we assume that part a) of theorem 1.3 holds on \( \Delta_\pi^{(1)} \) with respect to the measure \( \text{Leb}_{d-1} \), then the argument above, together with Fubini’s theorem, implies that it holds on \( \Delta_{\pi_0}(> 1/M) \) with respect to the measure \( \text{Leb}_d \). Since \( M \) is arbitrary we have that the following proposition is a sufficient condition to get part a) of theorem 1.3.

**Proposition 1.4.** Let \( \pi_0, (\beta, \alpha) \) and \( \varphi \) be respectively a combinatorial datum, a pair of letters and a positive sequence as in theorem 1.3. If \( \sum_{n=1}^{\infty} \varphi(n) < +\infty \), then for almost any \( T \in \Delta_{\pi_0}(> 1/M) \) there exists just finitely many triples \((\beta, \alpha, n)\) which satisfy equation (1.2).

1.2.3. Divergent case. For any positive integer \( M \) we consider the set \( \Delta_{\pi_0}(< M) \) of those \( T \in \Delta_{\pi_0} \) whose length datum \( \lambda \) satisfies \( \| \lambda \| < M \). We first observe that a triple \((\beta, \alpha, n)\) is reduced for \( T \) if and only if it is reduced for \( \hat{T} \). Moreover if a reduced \((\beta, \alpha, n)\) satisfies \( |\hat{T}^n \hat{u}_\beta^b - \hat{u}_\alpha^t| < (1/M) \varphi(n) \), then it is a reduced solution of equation (1.2) for \( T \in \Delta_{\pi_0}(> 1/M) \). As in the convergent case, the re-scaled sequence \((1/M) \varphi\) has divergent series if and only if \( \varphi \) has divergent series. Arguing as before, if we assume that part b) of theorem 1.3 holds on \( \Delta_\pi^{(1)} \) with respect to the measure \( \text{Leb}_{d-1} \), then it holds on \( \Delta_{\pi_0}(< M) \) with respect to the measure \( \text{Leb}_d \). Since \( M \) is arbitrary we have that the following proposition is a sufficient condition to get part b) of theorem 1.3.

**Proposition 1.5.** Let \( \pi_0, (\beta, \alpha) \) and \( \varphi \) be respectively a combinatorial datum, a pair of letters and a positive sequence as in theorem 1.3. If \( \sum_{n=1}^{\infty} \varphi(n) = \infty \), then for almost any \( T \in \Delta_{\pi_0}^{(1)} \) there exists just finitely many triples \((\beta, \alpha, n)\) which satisfy equation (1.2).
1.3. Contents of this article. This article is devoted to the proof of theorem 1.3. As we explained, the convergent case is treated independently from the divergent one and we are reduced to prove two independent statements: proposition 1.4 for the former and proposition 1.5 for the latter.

In Section 2, we recall the basic theory of i.e.t.s. In paragraph 2.1 we describe the Rauzy-Veech map and the Zorich’s acceleration, in particular we introduce Rauzy classes. In paragraph 2.2 we describe a combinatorial operation on Rauzy classes introduced in [A,G,Y] and called reduction, the formalism introduced in this paragraph is used in paragraphs 3.3, 4.1, 5.1 and 5.2. The normalized Rauzy-Veech map is a piecewise linear-projective map, in paragraph 2.3 we describe the connected components of its domains and we give a formula for their volume. The volume has unbounded distortion under iteration of the map and in paragraph 2.3.1 we state a result proved in [A,G,Y] on the control of the distortion. We also explain how the distortion can be interpreted in terms of conditional probability.

Section 3. treats the convergent case, that is the proof of proposition 1.4. We first give a short argument in order to reduce ourselves to the case of reduced triples, then for any $(\beta, \alpha, n)$ as in theorem 1.3 we introduce the set $I(\pi_0, \beta, \alpha, n)$ of those $T$ in $\Delta^{(1)}_{\pi_0}$ such that the triple $(\beta, \alpha, n)$ is reduced for $T$ and equation (1.2) is satisfied. The main task is to prove that $\text{Leb}_{d-1}(I(\pi_0, \beta, \alpha, n))$ is of the same order of $\varphi(n)$. Lemma 3.2 in paragraph 3.1 is one of the main steps of the proof: it says that for a triple $(\beta, \alpha, n)$ reduced for $T$ the quantity $|T^n u^b_\beta - u^t_\alpha|$ equals to $|u^{(r).b}_\beta - u^{(r).t}_\alpha|$, where $u^{(r).b}_\beta$ and $u^{(r).t}_\alpha$ are the singularities of some $T^{(r)}$ in the orbit of $T$ under the Rauzy-Veech map. In paragraph 3.2 we show that the locus of those $T$ such that $u^{(r).b}_\beta = u^{(r).t}_\alpha$ is a codimension-one subset of $\Delta^{(1)}_\pi$ with a complicated shape and we reduce the estimation of $\text{Leb}_{d-1}(I(\pi_0, \beta, \alpha, n))$ to an estimation of the $(d-2)$-volume of this codimension one subset. The estimate is proved in lemma 3.6 in paragraph 3.3.

In Section 4, we treat the divergent case, that is we prove proposition 1.5. The proof is obtained by a sequence of tree sufficient criterions, each one implying the precedent. The main idea in paragraph 4.1 is to consider the orbit $(T^{(r)})_{r \in N}$ of $T$ under the Rauzy-Veech map and to find good instants $r$ such that $|T^n u^b_\beta - u^t_\alpha|$ equals the length of some interval of $T^{(r)}$. The idea is developed in lemmas 4.3 and 4.4. They work in parallel, since they hold under some combinatorial conditions on the pair $(\beta, \alpha)$ (explained in definition 4.1 and one applies when the other fails and vice-versa. The final result is proposition 4.5 which is the first sufficient condition. In paragraph 4.2 we convert proposition 4.5 into a shrinking target criterion: to any pair $(\beta, \alpha)$ as in theorem 1.3 we associate a map $F_\eta$, which essentially coincides with the first return of the Rauzy-Veech map to a proper section $\Delta^{(1)}_\pi$ (see definition 4.10 and equation 4.15). The property that we require is that almost any orbit under $F_\eta$ enters infinitely many times into a sequence of sets, or targets, whose $(d-1)$-volume shrinks to zero. In paragraph 4.3 we state and prove our final sufficient criterion, that is proposition 4.24. It is formulated as the divergent part of the Borel-Cantelli lemma, where the events $X_k$ have the form $F^{-k}_\eta (E_k)$ and the sets $E_k$ are subsets of the shrinking targets in proposition 4.24. It is important that any $E_k$ is measurable with respect to the sigma-algebra generated by the connected components of the domain of the map $F_\eta$. This is quite a delicate point and it is treated in paragraphs 4.3 and 4.4.1.
In Section 5, we state and prove two general results for i.e.t.s that we had to develop to prove Proposition 1.5. The first is Theorem 5.1 which affirms that for any pair of letters \((\beta, \alpha)\) as in Theorem 1.3 any Rauzy class contains an element \(\pi\) where the two letters are in some required reciprocal position. Theorem 5.1 provides the combinatorial property that we need to apply Proposition 4.5. The second general result is Theorem 5.2: we cut any \(\Delta\) along an hyper-plane parallel to any \((d-2)\)-hyper-face, at any height, then consider the band in \(\Delta^{(1)}\) delimited by the hyper-plane and the hyper-face. The theorem says that the total volume of those connected components of the domain of \(F_\eta\) that are included in the band is proportional to the volume of the band. Theorem 5.2 plays a crucial role passing from Proposition 4.12 to Proposition 4.21.

Acknowledgements. The results in this paper were obtained in my Ph-D thesis. I would like to thank Jean-Christophe Yoccoz for many discussions and for his help in revising this work. I am also grateful to Stefano Marmi for many discussions and to Giovanni Forni and Pascal Hubert for many precious remarks.

2. Background theory.

2.1. The algorithm of Rauzy and Veech. In this paragraph we give a brief survey of the basic properties of the algorithm of Rauzy and Veech and we develop the notation that we use in the following. We follow [M.M.Y] and [A.G.Y].

Let \(\pi = (\pi^t, \pi^b)\) and \(\lambda\) define an interval exchange transformation \(T : I \rightarrow I\). Let \(\epsilon \in \{t, b\}\), where the letter \(t\) stands for top and the letter \(b\) for bottom. If \(\epsilon = t\) we put \(1 - \epsilon := b\) and if \(\epsilon = b\) we put \(1 - \epsilon := t\). Let us call \(\alpha_t\) and \(\alpha_b\) the two letters in \(\mathcal{A}\) such that respectively \(\pi^t(\alpha_t) = d\) and \(\pi^b(\alpha_b) = d\). The rightmost singularity of \(T\) is therefore \(u_{\alpha_t}^t\) and the rightmost singularity of \(T^{-1}\) is \(u_{\alpha_b}^b\). We suppose that

\[
(2.1) \quad u_{\alpha_t}^t \neq u_{\alpha_b}^b
\]

and we consider the value of \(\epsilon \in \{t, b\}\) such that

\[
(2.2) \quad u_{\alpha_t}^\epsilon < u_{\alpha_{1-\epsilon}}^\epsilon.
\]

With this definition of \(\epsilon\) we say that \(T\) is of type \(\epsilon\). We also say that the letter \(\alpha_\epsilon\) is the winner of \(T\) and \(\alpha_{1-\epsilon}\) is the loser. We consider the subinterval of \(I\)

\[
\tilde{I} := I \cap (0, u_{\alpha_{1-\epsilon}}^t)
\]

and we define \(\tilde{T} : \tilde{I} \rightarrow \tilde{I}\) as the first return map of \(T\) to \(\tilde{I}\). It is easy to check that \(\tilde{T}\) is an interval exchange transformation. The combinatorial datum \(\tilde{\pi} = (\tilde{\pi}^t, \tilde{\pi}^b)\) of \(\tilde{T}\) is given by:

\[
\tilde{\pi}^\epsilon(\alpha) = \pi^\epsilon(\alpha) \forall \alpha \in \mathcal{A}
\]

\[
\tilde{\pi}^{1-\epsilon}(\alpha) = \pi^{1-\epsilon}(\alpha) \quad \text{if} \quad \pi^{1-\epsilon}(\alpha) \leq \pi^{1-\epsilon}(\alpha_\epsilon)
\]

\[
\tilde{\pi}^{1-\epsilon}(\alpha_{1-\epsilon}) = \pi^{1-\epsilon}(\alpha_\epsilon) + 1
\]

\[
\tilde{\pi}^{1-\epsilon}(\alpha) = \pi^{1-\epsilon}(\alpha) + 1 \quad \text{if} \quad \pi^{1-\epsilon}(\alpha_\epsilon) < \pi^{1-\epsilon}(\alpha) < d.
\]

The length datum \(\tilde{\lambda}\) of \(\tilde{T}\) is given by:

\[
(2.3) \quad \tilde{\lambda}_\alpha = \lambda_\alpha \quad \text{if} \quad \alpha \neq \alpha_\epsilon
\]

\[
\tilde{\lambda}_{\alpha_\epsilon} = \lambda_{\alpha_\epsilon} - \lambda_{\alpha_{1-\epsilon}}.
\]
When $T = (\pi, \lambda)$ satisfies condition (2.1), equations (2.2) and (2.3) define a map $T \mapsto Q(T) := \tilde{T}$ that we call the Rauzy-Veech induction map. We introduce two operations $R^\ell$ and $R^b$ on the set of admissible combinatorial data as follows: if $\epsilon$ is the type of $T$ and $\pi$ is its combinatorial datum, then we set $R^\ell(\pi) := \tilde{\pi}$, where $\tilde{\pi}$ is the combinatorial datum of $\tilde{T}$. It is easy to check that if $\pi$ is an admissible combinatorial datum then both $R^\ell(\pi)$ and $R^b(\pi)$ are admissible.

**Definition 2.1.** Let us call $\mathcal{S}$ the set of all the admissible combinatorial data $\pi$ over some alphabet $A$. The maps $R^\ell$ and $R^b$ from $\mathcal{S}$ to itself are called the Rauzy elementary operations.

- A Rauzy class is a minimal non-empty subset $\mathcal{R}$ of $\mathcal{S}$ which is invariant under $R^\ell$ and $R^b$.
- A Rauzy diagram is a connected oriented graph $\mathcal{D}$ whose vertexes are the elements of $\mathcal{R}$ and whose oriented arcs, or arrows, correspond to Rauzy elementary operations $\pi \mapsto R^\ell(\pi)$ between elements of $\mathcal{R}$.
- An arrow corresponding to $R^\ell$ is called a top arrows and we say that $\alpha_t$ is its winner and $\alpha_0$ is its loser. Conversely an arrow corresponding to $R^b$ is called a bottom arrow and we say that $\alpha_t$ is its loser and $\alpha_0$ is its winner.
- A concatenation of compatible arrows in a Rauzy diagram is called a Rauzy path. The set of all Rauzy paths connecting elements of $\mathcal{R}$ is denoted $\Pi(\mathcal{R})$. If a path $\gamma$ is concatenation of $r$ simple arrows, we say that $\gamma$ has length $r$. Length one paths are arrows, we also identify the elements of $\mathcal{R}$ with trivial paths, that is length zero paths.
- A partial ordering $\prec$ is defined on $\Pi(\mathcal{R})$ saying that $\nu \prec \gamma$ iff $\gamma$ begins with $\nu$. A subfamily $\Gamma$ of $\Pi(\mathcal{R})$ is called disjoint iff for any two elements $\eta$ and $\nu$ of $\Gamma$ we have neither $\eta \prec \gamma$ nor $\gamma \prec \eta$.

With the notation above, recalling that for any combinatorial datum $\pi$ we defined $\Delta_{\pi} = \{\pi\} \times \mathbb{R}^A_+$, we denote $\Delta(\mathcal{R}) := \bigsqcup_{\pi \in \mathcal{R}} \Delta_{\pi}$ the set of all the intervals exchange transformations with combinatorial datum in the Rauzy class $\mathcal{R}$.

2.1.1. **Linear action.** For any Rauzy class $\mathcal{R}$ and any path $\gamma \in \Pi(\mathcal{R})$ we define a linear map $B_{\gamma} \in \text{SL}(d, \mathbb{Z})$ as follows. If $\gamma$ is trivial then $B_{\gamma} = \text{id}$. If $\gamma$ is an arrow with winner $\alpha$ and loser $\beta$ then we set $B_{\gamma} e_\alpha = e_\gamma e_\alpha + e_\beta$ and $B_{\gamma} e_\xi = e_\xi$ for all $\xi \in A \setminus \{\alpha\}$, where $\{e_\xi\}_{\xi \in A}$ is the canonical basis of $\mathbb{R}^A$. We extend the definition to paths so that $B_{\gamma_1 \gamma_2} = B_{\gamma_2} B_{\gamma_1}$.

Let us fix some element $\pi$ in the Rauzy class $\mathcal{R}$. For any $\gamma \in \Pi(\mathcal{R})$ starting at $\pi$ we define the simplicial sub-cone $\Delta_{\gamma} \subset \Delta_{\pi}$ by

$$\Delta_{\gamma} = \{\pi\} \times^t B_{\gamma}(\mathbb{R}^A_+),$$

where $^t B_{\gamma}$ denotes the trasposed of the matrix $B_{\gamma}$ defined above. For the same $\gamma$ we also define the vector $q^\gamma \in \mathbb{N}^A$ by

$$q^\gamma := B_{\gamma} \bar{1},$$

where $\bar{1}$ denotes the vector of $\mathbb{N}^A$ that has all entries equal to 1.

2.1.2. **Iteration of the algorithm.** When $T \in \Delta(\mathcal{R})$ is such that the $r$-th iterated of $Q$ is defined, we have an explicit formula for $T^{(r)} := Q^r(T)$.

**Lemma 2.2.** Let $\gamma \in \Pi(\mathcal{R})$ be a path in the Rauzy diagram with length $r$ and let $B_{\gamma}$ and $\Delta_{\gamma}$ be respectively the matrix and the simplicial cone defined in paragraph
Then for any $T \in \Delta$, the \( r \)-th iterated of $Q$ is defined and the length datum $\lambda^{(r)}$ of $Q^{r}(T)$ is given by the formula

$$
\lambda^{(r)} = t \cdot B_{\gamma}^{-1} \lambda.
$$

**Proof:** Let us first consider an arrow $\gamma$. We call $\pi$ its starting point, $\alpha$ its winner and $\beta$ its loser. We suppose that $\gamma$ is of type top, the other case being identic. Let us consider $T = (\pi, \lambda)$ in $\Delta_{\gamma}$. By the definition of the matrix $B_{\gamma}$ we have

$$
B_{\gamma} = \text{id} + E_{\beta, \alpha}
$$

where $E_{\beta, \alpha}$ is the matrix whose entry in the column $\alpha$ and row $\beta$ is 1 and all the others are 0. It follows that $t \cdot B_{\alpha} \mathbb{R}_{+}$ is the open half cone of $\mathbb{R}_{+}$ of those $\lambda$ such that $\lambda_{\alpha} > \lambda_{\beta}$. Since $\pi^{t}(\alpha) = \pi^{h}(\beta) = d$ this last condition is equivalent to the condition (2.1) and therefore $Q$ is defined on $T$. Moreover equation (2.3) says that for the length datum $\lambda^{(1)}$ of $Q(T)$ we have $\lambda^{(1)} = t \cdot B_{\gamma}^{-1} \lambda$.

The proof continues by induction on $r$. Let us suppose that the lemma is proved for any concatenation $\gamma_{1}...\gamma_{r-1}$ of $r - 1$ arrows. We consider a path $\gamma = \gamma_{1}...\gamma_{r-1}\gamma_{r}$ starting at $\pi$, where $\gamma_{r}$ is an arrow which can be concatenated to $\gamma_{r-1}$. Let us consider any $T = (\pi, \lambda) \in \Delta_{\gamma}$ such that $T \in \Delta_{\gamma}$. By (2.1) and therefore $Q$ is defined on $T$. Moreover equation (2.3) says that for the length datum $\lambda^{(1)}$ of $Q(T)$ we have $\lambda^{(1)} = t \cdot B_{\gamma}^{-1} \lambda$.

The proof continues by induction on $r$. Let us suppose that the lemma is proved for any concatenation $\gamma_{1}...\gamma_{r-1}$ of $r - 1$ arrows. We consider a path $\gamma = \gamma_{1}...\gamma_{r-1}\gamma_{r}$ starting at $\pi$, where $\gamma_{r}$ is an arrow which can be concatenated to $\gamma_{r-1}$. Let us consider any $T = (\pi, \lambda) \in \Delta_{\gamma}$ such that $T \in \Delta_{\gamma}$. By (2.1) and therefore $Q$ is defined on $T$. Moreover equation (2.3) says that for the length datum $\lambda^{(1)}$ of $Q(T)$ we have $\lambda^{(1)} = t \cdot B_{\gamma}^{-1} \lambda$.

The proof continues by induction on $r$. Let us suppose that the lemma is proved for any concatenation $\gamma_{1}...\gamma_{r-1}$ of $r - 1$ arrows. We consider a path $\gamma = \gamma_{1}...\gamma_{r-1}\gamma_{r}$ starting at $\pi$, where $\gamma_{r}$ is an arrow which can be concatenated to $\gamma_{r-1}$. Let us consider any $T = (\pi, \lambda) \in \Delta_{\gamma}$ such that $T \in \Delta_{\gamma}$. By (2.1) and therefore $Q$ is defined on $T$. Moreover equation (2.3) says that for the length datum $\lambda^{(1)}$ of $Q(T)$ we have $\lambda^{(1)} = t \cdot B_{\gamma}^{-1} \lambda$.

We denote $\Delta_{1}(R)$ the domain of $Q$, whose connected components are the simplicial cones $\Delta_{\gamma}$ associated to arrows $\gamma \in \Pi(R)$. One easily checks that if $\gamma$ is an arrow starting at $\pi$ and ending at $\pi'$ then the matrix $t \cdot B_{\gamma}^{-1}$ establishes an homeomorphism between $\Delta_{\gamma}$ and $\Delta_{\gamma'}$. Let $\Delta_{1}(R)$ be the domain of $Q$. Lemma (2.2) says that the connected components of $\Delta_{1}(R)$ are naturally labeled by paths $\gamma \in \Pi(R)$ of length $r$. If $\gamma$ is such a path ending at $\pi'$ then $Q_{r} : \Delta_{\gamma} \to \Delta_{\gamma'}$ is a homeomorphism. The set $\Delta_{\infty}(R) := \bigcap_{k \in \mathbb{N}} \Delta_{k}(R)$ is the set of those i.e.t. $T$ such that the map $Q$ can be applied infinitely many times. Being the intersection of countably many sets of full lebesgue measure, $\Delta_{\infty}(R)$ has full lebesgue measure.

The complement of $\Delta_{\infty}(R)$ is the set of those i.e.t. $T$ such that for some $r \in \mathbb{N}$ the $r$-th element $T^{(r)}$ of the $Q$-orbit violates the condition in equation (2.1), that is the algorithm stops. For the complement the following characterization holds.

**Lemma.** When applied to an i.e.t. $T$ the Rauzy algorithm $Q$ eventually stops if and only if $T$ has a connection.
For any $\gamma \in \overline{B_\gamma} \cap \mathbb{R}$, it follows that for any $\beta \in A$ there are exactly $[B_\gamma]_{\beta, \alpha}$ integers $i \in \{0, \ldots, R\}$ such that $T^i(I_{\alpha}^{(r), t}) \subset I_\beta^t$, where $[B_\gamma]_{\beta, \alpha}$ is the entry of $B_\gamma$ in row $\alpha$ and column $\beta$. It follows that the return time $R$ of $I_{\alpha}^{(r), t}$ to $I^{(r)}$ under iteration of $T$ is given by the entry $q^r_{\alpha}$ of the vector $q^r = B_\gamma \mathbf{1}$.

Let $u_{\alpha}^{(r), t}$ and $u_{\beta}^{(r), b}$ be singularities for $T^{(r)}$ and $(T^{(r)})^{-1}$. Since $T^{(r)}$ is the first return map of $T$ to $I^{(r)}$, then for $n \in \mathbb{N}$ we have $T^nu_{\beta}^{b} \subset I^{(r)}$ if and only if there exists some $l$ such that $T^nu_{\beta}^{b} = (T^{(r)})^lu_{\alpha}^{(r), b}$. In particular, denoting $l = l(r, \alpha)$ the first time such that $T^lu_{\beta}^{b} \in I^{(r)}$, we have

$$u_{\beta}^{(r), b} = T^lu_{\alpha}^{(r), b}.$$  

For $u_{\alpha}^{(r), t}$ the discussion is the same. If $h = h(r, \alpha)$ is the smallest integer such that $T^{-h}u_{\beta}^{t} \in I^{(r)}$ then

$$u_{\alpha}^{(r), t} = T^{-h}u_{\alpha}^{(r), t}.$$  

For any $\alpha \in A$ the definition of the integer $h(r, \alpha)$ implies $T^{h(r, \alpha)}(I_{\alpha}^{(r), t}) \subset I^{(r)}$. Then applying $T$ we have $T^{h(r, \alpha)+1}(I_{\alpha}^{(r), t}) \subset I^{(r)}$. Finally iterating $l(r, \alpha)$ times we get $T^{h(r, \alpha)+1+l(r, \alpha)}(I_{\alpha}^{(r), t}) = I_{\alpha}^{(r), b}$. For any $\alpha \in A$ the positive integers $l(r, \alpha)$ and $h(r, \alpha)$ satisfy

$$h(r, \alpha) + 1 + l(r, \alpha) = q^r_{\alpha}.$$  

2.1.4. Normalized Rauzy Veech algorithm and Zorich’s acceleration. The Rauzy-Veech algorithm has interesting recurrence properties just at projective level. We introduce a normalization on the sum of the lengths of the intervals. For $\lambda \in \mathbb{R}_{++}^A$ we recall the notation $||\lambda|| := \sum_{\alpha \in A} \lambda_{\alpha}$ and $\bar{\lambda} := ||\lambda||^{-1}\lambda$. For any combinatorial datum $\pi$ in some $\mathcal{R}$ we write

$$\Delta^{(1)}_{\pi} := \{(\pi, \lambda) \in \Delta_{\pi}; ||\lambda|| = 1\}.$$  

The normalized length datum of an i.e.t. $T \in \Delta^{(1)}_{\pi}$ will be often denoted $\tilde{\lambda}$. For any Rauzy class $\mathcal{R}$ the set of all normalized i.e.t.s with combinatorial datum in $\mathcal{R}$ is denoted $\Delta^{(1)}(\mathcal{R}) := \bigsqcup_{\pi \in \mathcal{R}} \Delta^{(1)}_{\pi}$. From now on, when applying the map $Q$, we will not worry about its domain and we will say that it is defined on $\Delta^{(1)}(\mathcal{R})$ modulo a set of measure zero.

**Definition 2.3.** Let $\mathcal{R}$ be a Rauzy class over an alphabet $A$. The normalized Rauzy-Veech algorithm is the map $Q : \Delta^{(1)}(\mathcal{R}) \to \Delta^{(1)}(\mathcal{R})$ defined by

$$\tilde{Q}(\pi, \lambda) := (\bar{\pi}, \frac{\bar{\lambda}}{||\lambda||}),$$  

where $(\bar{\pi}, \bar{\lambda}) = Q(\pi, \lambda)$ is the Rauzy-Veech algorithm introduced in paragraph 2.1.7.

If $T \in \Delta^{(1)}(\mathcal{R})$ is an i.e.t. without connections, for any $r \in \mathbb{N}$ we denote $\hat{T}^{(r)} := \tilde{Q}^r(T)$. For any $r$ let $\gamma_r$ be the simple arrow associated to the step $\hat{T}^{(r)} = \tilde{Q}(\hat{T}^{(r-1)})$ of the algorithm. We obtain a sequence $\gamma_1, \gamma_2, \ldots, \gamma_r$ of simple arrows. We denote $\gamma(T, r)$ the concatenation $\gamma_1...\gamma_r$ of the first $r$ arrows in the sequence. We have $\gamma(T, r) \prec \gamma(T, r + 1)$ with respect to the ordering $\prec$ in definition 2.1. Then we define $\gamma(T, \infty)$ as the half infinite path in $\Pi(\mathcal{R})$ such that $\gamma(T, r) \prec \gamma(T, \infty)$ for all $r > 0$. 

Veech proved that $\hat{Q}$ has an unique invariant measure which is absolutely continuous with respect to the lebesgue measure, nevertheless this measure is not finite (see [Ve]). Zorich introduced an acceleration of $\hat{Q}$ with a finite invariant measure (see [Z1]). For an i.e.t. $T$ without connections we define the integer $N(T)$ as the minimum of those $r \in \mathbb{N}$ such that the type of $T$ is different from the type of $\hat{Q}^r(T)$.

**Definition 2.4.** The Zorich’s acceleration is the map $Z : \Delta^{(1)}(\mathcal{R}) \to \Delta^{(1)}(\mathcal{R})$ defined by

$$Z(T) := \hat{Q}^{N(T)}(T),$$

where $Q^N$ is the $N$-th iterated of the Rauzy-Veech algorithm introduced in paragraph 2.1 and $\hat{Q}^N$ is its normalized version.

The following is one of the main results in the ergodic theory of i.e.t.s ([Z1]).

**Theorem (Zorich).** For any Rauzy class $\mathcal{R}$ the Zorich map $Z$ in definition 2.4 has an unique invariant measure $\mu$ which is absolutely continuous with respect to the lebesgue measure on $\Delta^{(1)}(\mathcal{R})$. Moreover $\mu$ is finite and ergodic.

### 2.2. Reduction of Rauzy classes.

In this paragraph we describe a combinatorial operation on Rauzy classes called reduction, which has been introduced in [A,G,Y] generalizing a previous simpler version appearing in [A,V]. We closely follow §5 of [A,G,Y].

#### 2.2.1. Decorated Rauzy classes.

Let $\mathcal{R}$ be a Rauzy class with alphabet $\mathcal{A}$ and $\mathcal{A}' \subset \mathcal{A}$ be a proper subset. An arrow is called $\mathcal{A}'$-colored if its winner belongs to $\mathcal{A}'$. A path $\gamma \in \Pi(\mathcal{R})$ is $\mathcal{A}'$-colored if it is a concatenation of $\mathcal{A}'$-colored arrows.

For an element $\pi \in \mathcal{R}$ we say that $\pi$ is $\mathcal{A}'$-trivial if the last letters on both the top and the bottom rows of $\pi$ do not belong to $\mathcal{A}'$, $\pi$ is $\mathcal{A}'$-intermediate if exactly one of those letters belongs to $\mathcal{A}'$ and finally $\pi$ is $\mathcal{A}'$-essential if both letters belong to $\mathcal{A}'$. An $\mathcal{A}'$-decorated Rauzy class $\mathcal{R}_* \subset \mathcal{R}$ is a maximal subset whose elements can be joined by an $\mathcal{A}'$-colored path. We let $\Pi_*(\mathcal{R}_*)$ be the set of $\mathcal{A}'$-colored paths starting (and ending) at permutations in $\mathcal{R}_*$.

A decorated Rauzy class is called trivial if it contains a trivial element, in this case $\mathcal{R}_* = \{\pi\}$ and $\Pi_*(\mathcal{R}_*) = \{\pi\}$, recalling that vertices are identified with zero-length paths. A decorated Rauzy class is called essential if it contains an essential element. Any essential decorated Rauzy class contains intermediate elements.

Let $\mathcal{R}_*$ be an essential decorated Rauzy class and let $\mathcal{R}^{\text{ess}}_* \subset \mathcal{R}_*$ be the subset of essential elements. Let $\Pi^{\text{ess}}_*(\mathcal{R}_*)$ be the set of paths in $\Pi_*(\mathcal{R}_*)$ starting and ending at elements of $\mathcal{R}^{\text{ess}}_*$. An arc is a minimal non-trivial path in $\mathcal{R}_*^{\text{ess}}$, all arrows in the same arc are of the same type and have the same winner, so winner and type of an arc are well defined. The losers in an arc are all distinct, moreover the first loser is in $\mathcal{A}'$ and the others are not. Any element in $\mathcal{R}_*^{\text{ess}}$ is the starting point of a top and of a bottom arc and also the ending point of a top and a bottom arc.

If $\gamma \in \Pi_*(\mathcal{R}_*)$ is an arrow then there exist unique paths $\gamma_s$ and $\gamma_e$ in $\Pi_*(\mathcal{R}_*)$ such that $\gamma_s \gamma_e \gamma$ is an arc, called the completion of $\gamma$. If $\pi$ is intermediate the completion of the $\mathcal{A}'$-colored arrow starting at $\pi$ is the only arc passing through $\pi$.

If $\pi \in \mathcal{R}_*$ we define $\pi^{\text{ess}}$ as follows. If $\pi$ is essential then $\pi^{\text{ess}} = \pi$, if $\pi$ is intermediate let $\pi^{\text{ess}}$ be the end of the arc passing through $\pi$.

To $\gamma \in \Pi_*(\mathcal{R}_*)$ we associate an element $\gamma^{\text{ess}} \in \Pi^{\text{ess}}_*(\mathcal{R}_*)$ as follows. For a trivial path $\pi \in \mathcal{R}_*$ we use the previous definition of $\pi^{\text{ess}}$. Assuming that $\gamma$ is an arrow we distinguish two cases:
(1) If $\gamma$ starts at an essential element, we let $\gamma^{\text{ess}}$ be the completion of $\gamma$.
(2) Otherwise, we let $\gamma^{\text{ess}}$ be the endpoint of the completion of $\gamma$.

We extend the definition to paths $\gamma \in \Pi_\ast(R_\ast)$ by concatenation. Notice that if $\gamma \in \Pi^{\text{ess}}(R_\ast)$ then $\gamma^{\text{ess}} = \gamma$.

2.2.2. Reduction of Rauzy classes. Given a permutation $\pi$ on the alphabet $A$, even not admissible, whose top and bottom rows end with different letters, we obtain the admissible end of $\pi$ by deleting as many letters from the top and bottom rows of $\pi$ as necessary to obtain an admissible permutation. The resulting permutation belongs to some Rauzy class $R''$ on some alphabet $A'' \subset A$.

Let $R_\ast$ be an essential decorated Rauzy class, and let $\pi \in R_\ast^{\text{ess}}$. Delete all the letters not belonging to $A'$ from the top and bottom rows of $\pi$. The resulting permutation $\pi'$ is not necessary admissible, but since $\pi$ is essential the letters in the end of the top and bottom rows of $\pi'$ are distinct. Let $\pi^{\text{red}}$ be the admissible end of $\pi'$. We call $\pi^{\text{red}}$ the reduction of $\pi$. We extend the operation of reduction from $R_\ast^{\text{ess}}$ to $R_\ast$, defining the reduction of $\pi \in R_\ast$ as the reduction of $\pi^{\text{ess}}$.

If $\gamma \in \Pi^{\text{ess}}(R_\ast)$ is an arc starting at $\pi_s$ and ending in $\pi_e$, then the reductions of $\pi_s$ and $\pi_e$ belong to the same Rauzy class and we define $\gamma^{\text{red}}$ as the arrow which joins $\pi_s^{\text{red}}$ with $\pi_e^{\text{red}}$. $\gamma^{\text{red}}$ has the same type and the same winner of the arc $\gamma$ and its loser is the first loser of $\gamma$. It follows that the set of reductions of all $\pi \in R_\ast$ is a Rauzy class $R^{\text{red}}$ on some alphabet $A'' \subset A' \subset A$. We define the reduction of a path $\gamma \in \Pi_\ast(R_\ast)$ as follows. If $\gamma$ is a trivial (zero-length) path or an arc, it is defined as above. We extend the definition to the case $\gamma \in \Pi^{\text{ess}}(R_\ast)$ by concatenation. In general we let the reduction of $\gamma$ to be equal to the reduction of $\gamma^{\text{ess}}$. Restricted to essential elements the operation of reduction give a bijection $\text{red}: R_\ast^{\text{ess}} \rightarrow R^{\text{red}}$. If we think to elements $\pi \in R$ as trivial paths we can extend the previous operation to a bijection $\text{red}: \Pi^{\text{ess}}(R_\ast) \rightarrow \Pi(R^{\text{red}})$ compatible with concatenation on the set of arcs.

2.2.3. Drift in essential decorated Rauzy classes. Let $R_\ast \subset R$ be an essential $A'$-decorated Rauzy class. For $\pi \in R_\ast$, let $\alpha_l(\pi)$ (respectively $\alpha_0(\pi)$) be the rightmost letter in the top (respectively in the bottom) row of $\pi$ that belongs to $A \setminus A'$. Let $d_t(\pi)$ (respectively $d_b(\pi)$) be the position of $\alpha_l(\pi)$ (respectively of $\alpha_0(\pi)$). Let $d(\pi) := d_t(\pi) + d_b(\pi)$. An essential element of $R_\ast$ is thus some $\pi$ such that $d_t(\pi) < d_b(\pi) < d$. If $\pi_s$ is an essential element of $R_\ast$ and $\gamma$ is an arrow starting at $\pi_s$ and ending at $\pi_e$ then

(1) $d_t(\pi_s) = d_t(\pi_e)$ or $d_b(\pi_e) = d_b(\pi_s) + 1$, the second possibility happening if and only if $\gamma$ is a bottom whose winner precedes $\alpha_l(\pi_s)$ in the top of $\pi_s$.
(2) $d_t(\pi_e) = d_t(\pi_s)$ or $d_b(\pi_s) = d_b(\pi_e) + 1$, the second possibility happening if and only if $\gamma$ is a top whose winner precedes $\alpha_0(\pi_s)$ in the bottom of $\pi_s$.

In particular $d(\pi_e) = d(\pi_s)$ or $d(\pi_e) = d(\pi_s) + 1$. In the second case we say that $\gamma$ is drifting. Let $R^{\text{red}}(\pi)$ be the reduction of $\pi$ and let $A'' \subset A' \subset A$ be the alphabet of $R^{\text{red}}$. If $\pi \in R_\ast$ is essential, then there exists some $\alpha \in A''$ which either precedes $\alpha_l(\pi)$ in the top row of $\pi$ or precedes $\alpha_0(\pi)$ in the bottom row of $\pi$, we call such an $\alpha$ good. Indeed, if $\gamma \in \Pi_\ast(R_\ast)$ is a path starting at $\pi$, ending with a drifting arrow and minimal with this property, then the winner of the last arrow of $\gamma$ belongs to $A''$ and either precedes $\alpha_l(\pi)$ in the top of $\pi$ (if the drifting arrow is a bottom) or precedes $\alpha_0(\pi)$ in the bottom of $\pi$ (if the drifting arrow is a top).
Note that if $\gamma \in \Pi^\text{ess}(R_\ast)$ is an arrow starting and ending at essential elements $\pi_s, \pi_e$ then a good letter for $\pi_s$ is also a good letter for $\pi_e$. Moreover, if $\gamma$ is not drifting then the winner of $\gamma$ is not a good letter for $\pi_s$.

2.2.4. Standard decomposition of separated paths. An arrow is called $(A \setminus A')$-separated if both its winner and its loser belong to $A'$. A path $\gamma \in \Pi_\ast(R_\ast)$ is $(A \setminus A')$-separated if it is a concatenation of $(A \setminus A')$-separated arrows. We also say that a Rauzy path $\gamma$ is complete (or $A$-complete) if for any letter $\alpha \in A$ there exists an arrow composing $\gamma$ having $\alpha$ as winner.

If $\gamma \in \Pi(R)$ is a non-trivial maximal $(A \setminus A')$-separated path then there exists an essential $A'$-decorated Rauzy class $R_\ast \subset R$ such that $\gamma \in \Pi_\ast(R_\ast)$. Moreover, if $\gamma = \gamma_1...\gamma_n$ then any arrow $\gamma_i$ starts at an essential element $\pi_i \in R_\ast^\text{ess}$ (and $\gamma_n$ ends at an intermediate element of $R_\ast$ by maximality).

**Remark 2.5.** Let $r := d(\pi_n) - d(\pi_1)$. Let $\gamma = \gamma^{(1)}...\gamma^{(r)}\gamma^r$, where the $\gamma^i$ are drifting arrows and $\gamma^{(i)}$ are (possibly trivial) concatenation of non drifting arrows. If $\alpha$ is a good letter for $\pi_1$, then it follows that $\alpha$ is not the winner of any arrow in any $\gamma^{(i)}$. The reduction of any $\gamma^{(i)}$ are therefore non-complete paths in $\Pi(R^{\text{red}})$.

2.3. Lebesgue measure and distortion under $\hat{Q}$. Let $\pi$ be an admissible combinatorial datum in some Rauzy class $R$ over the alphabet $A$ and consider the associated simplex $\Delta_\pi^{(1)}$ of those $T \in \Delta_\pi$ with $||\lambda|| = 1$. We call $\text{Leb}_{d-1}$ the lebesgue measure on $\Delta_\pi^{(1)}$ normalized in order to give measure one to it. For any finite path $\gamma \in \Pi(R)$ starting at $\pi$ we define a sub-simplex of $\Delta_\pi^{(1)}$ by

$$\Delta_\gamma^{(1)} := \Delta_\gamma \cap \Delta_\pi^{(1)}.$$  

We can identify $\Delta_\gamma^{(1)}$ with the standard simplex $\Delta^{(1)} := \{\lambda \in \mathbb{R}_+^d; ||\lambda|| = 1\}$, modulo this identification the vertices of $\Delta^{(1)}$ are the vectors $\{(1/q_0^\gamma)^t B_\gamma e_\alpha\}_{\alpha \in A}$. Recalling that $B_\gamma \in \text{SL}(d, \mathbb{Z})$ for any $\gamma \in \Pi(R)$, we have the nice formula

$$(2.4) \quad \text{Leb}_{d-1}(\Delta_\gamma^{(1)}) = \prod_{\alpha \in A} (q_0^\gamma)^{-1}.$$  

Let $\Gamma$ be a family of disjoint Rauzy paths starting at $\pi$. Disjointness of $\Gamma$ is equivalent to say that the simplices $\Delta_\gamma^{(1)}$ for $\gamma \in \Gamma$ are each other disjoint. In this case we have

$$\text{Leb}_{d-1}(\bigcup_{\gamma \in \Gamma} \Delta_\gamma^{(1)}) = \sum_{\gamma \in \Gamma} \text{Leb}_{d-1}(\Delta_\gamma^{(1)}).$$

Given a proper sub-alphabet $A'$ of $A$ with $d'$ letters (thus $2 \leq d' \leq d-1$) we call $\Delta_{\pi, A'}^{(1)}$ the hyper-face of $\Delta_\pi^{(1)}$ whose extremal points are the vectors $e_\alpha$ with $\alpha \in A'$. It is a $(d' - 1)$-simplex. We call $\text{Leb}_{d'-1}$ the lebesgue measure on it normalized in order to have $\text{Leb}_{d'-1}(\Delta_{\pi, A'}^{(1)}) = 1$. If $\gamma$ is a Rauzy path starting at $\pi$ we denote $\Delta_{\gamma, A'}^{(1)}$ the hyper-face of $\Delta_\gamma^{(1)}$ spanned by the vectors $\{(1/q_0^\gamma)^t B_\gamma e_\alpha\}_{\alpha \in A'}$. With the normalization introduced above we have

$$\text{Leb}_{d'-1}(\Delta_{\gamma}^{(1)}) = \prod_{\alpha \in A'} (q_0^\gamma)^{-1}.$$
2.3.1. Probabilistic interpretation and distortion estimate. Let us consider \( \pi \) in some Rauzy class \( \mathcal{R} \) and a path \( \gamma \in \Pi(\mathcal{R}) \) starting at \( \pi \). In view of lemma 2.2 we can interpret \( \text{Leb}_{d-1}(\Delta^{(1)}_\pi) \) as the probability of the event \( \{ T \in \Delta^{(1)}_\pi; \gamma \prec \gamma(T, \infty) \} \).

We may look at orbits of \( \hat{Q} \) not only from the beginning, but also from their middle. This is to say, for some fixed integer \( i_0 > 0 \), we split any orbit into two parts, that is \( \{ \hat{T}^{(i)} \}_{1 < i_0} \) and \( \{ \hat{T}^{(i)} \}_{i_0} \). Then we fix \( \pi \in \mathcal{R} \) and we consider those orbits such that \( \hat{T}^{(i_0)} \) belongs to \( \Delta^{(1)}_\pi \). Restarting the algorithm \( \hat{Q} \) at \( \hat{T}^{(i_0)} \) we consider \( \{ \hat{T}^{(i)} \}_{1 < i_0} \) as its past and \( \{ \hat{T}^{(i)} \}_{i_0} \) as its future. If \( \nu \in \Pi(\mathcal{R}) \) is a finite Rauzy path ending in \( \pi \), it is natural to consider the probability, for \( \hat{T}^{(i_0)} \in \Delta^{(1)}_\pi \), that \( \gamma(\hat{T}^{(i_0)}, \infty) \) begins with \( \gamma \) once we know that \( \gamma(T, i_0) \) ends with \( \nu \). We denote \( P_\nu(\Delta^{(1)}_\gamma) \) such probability. According to lemma 2.2 all the orbits such that \( \gamma(T, i_0) \) ends with \( \nu \) are characterized by the condition \( \hat{T}^{(i-1)} \in \Delta^{(1)}_\nu \) for some common instant \( i-1 < i_0 \). Then they all follow the steps of the algorithm prescribed by the path \( \nu \) till they arrive in \( \Delta^{(1)}_\pi \) at the instant \( i_0 \) (in particular \( i_0 - i-1 \) coincides with the length of \( \nu \)). The condition on the future \( \gamma \prec \gamma(\hat{T}^{(i_0)}, \infty) \) is equivalent to \( \hat{T}^{(i_0)} \in \Delta^{(1)}_\gamma \), that is to say \( \hat{T}^{(i-1)} \in \Delta^{(1)}_\nu \), thus for \( P_\nu(\Delta^{(1)}_\gamma) \) we have the formula

\[
P_\nu(\Delta^{(1)}_\gamma) = \frac{\text{Leb}_{d-1}(\Delta^{(1)}_\gamma)}{\text{Leb}_{d-1}(\Delta^{(1)}_\nu)} = \frac{\prod_{\alpha \in A} q^\nu_{\alpha}}{\prod_{\alpha \in A} q^\nu_{\alpha}}.
\]

Observing that \( B_\gamma = B_\nu \), and therefore \( q^\nu = q^\nu \), we are led to consider the following generalization. For any vector \( q \) in \( \mathbb{R}^A \) we write \( N(q) := \prod_{\alpha \in A} q_{\alpha} \), then we define

\[
P_q(\Delta^{(1)}_\gamma) := \frac{N(q)}{N(B_\gamma q)}.
\]

For \( A' \subset A \) and \( q \in \mathbb{R}^A \) we put \( M_{A'}(q) := \max_{\alpha \in A'} q_{\alpha} \). In the trivial case \( A' = A \) we simply denote \( M(q) := M_A(q) \). We also call \( \Pi_\pi(\mathcal{R}) \) the set of those \( \gamma \in \Pi(\mathcal{R}) \) which start at \( \pi \). In \( [A,G,Y] \) it is proved the following theorem.

**Theorem** (Avila-Gouezel-Yoccoz). There exist a pair of constant \( C > 0 \) and \( \theta > 1 \), depending only on the number of intervals \( d \), with the following property. Let \( A' \subset A \) be a non-empty proper subset, \( m \) and \( M \) be integers with \( 0 \leq m \leq M \) and \( q \) be any vector in \( \mathbb{R}^A \). Then we have the estimates

\[
P_q(\gamma \in \Pi_\pi(\mathcal{R}); M(B_\gamma q) > 2^M M(q)) \leq C(m + 1)^\theta 2^{-m},
\]

(2.6)

\[
P_q(\gamma \in \Pi_\pi(\mathcal{R}); M(B_\gamma q) < 2^m M(q)) \leq C(m + 1)^\theta 2^{-m}
\]

(2.7)

\[
P_q(\gamma \in \Pi_\pi(\mathcal{R}); \gamma \text{ is not complete } ; M(A'(B_\gamma q)) > 2^M M(q)) \leq C(M + 1)^\theta 2^{-M}.
\]

**Note.** If fact the complete result in \( [A,G,Y] \) contains a complete distortion estimates similar to these ones, but we don’t need them in our work.

3. The convergent case.

This section is devoted to the proof of proposition 1.4. Let \( \pi_0 \) and \( (\beta, \alpha) \) be respectively a combinatorial datum and a pair as in theorem 1.3. For any \( n \in \mathbb{N} \) we call \( I(\pi_0, \beta, \alpha, n) \) the set of those \( T \in \Delta^{(1)}_{\pi_0} \) such that the triple \( (\beta, \alpha, n) \) is reduced for \( T \) and equation (1.2) is satisfied, that is such that \( |T^nu^b - u^a| < \varphi(n) \).

Consider \( T \in \Delta^{(1)}_{\pi_0} \) such that the triple \( (\beta, \alpha, n) \) satisfies equation (1.2). If \( (\beta, \alpha, n) \) is not reduced for \( T \) then there exists an integer \( k \in \{0, \ldots, n\} \) and a letter
$Z \in \mathcal{A}$ such that $T^{-k}I(\beta, \alpha, n)$ contains in its interior either $u_{Z}^{t}$ or $u_{Z}^{b}$, moreover we can suppose that $k$ is minimal with one of the two properties. The first case implies

$|T^{n-k}u_{Z}^{b} - u_{Z}^{t}| < |T^{n}u_{Z}^{b} - u_{Z}^{t}|$, thus $|T^{n-k}u_{Z}^{b} - u_{Z}^{t}| < \varphi(n-k)$, since $\varphi$ is monotone. Similarly the second case implies $|u_{Z}^{b} - T^{-k}u_{Z}^{t}| < |T^{n}u_{Z}^{b} - u_{Z}^{t}|$, so $|T^{k}u_{Z}^{b} - u_{Z}^{t}| < \varphi(k)$ by minimality of $k$. In both the two cases we pass from $(\beta, \alpha, n)$ to an other triple $(\beta', \alpha', n')$ satisfying equation (1.2) with $n' < n$. Applying iteratively the argument we get a triple reduced for $T$ which still satisfies equation (1.2).

Now let us suppose that for the i.e.t. $T$ there exist infinitely many triples which are solutions of equation (1.2), but which are not necessarily reduced. With the argument above we get a sequence of reduced solutions $\{(\beta_{k}, \alpha_{k}, n_{k})\}_{k \in \mathbb{N}}$ for $T$. Finally there exist at least one pair $(\beta', \alpha')$ appearing infinitely many times in the sequence.

According to the first part of the Borel-Cantelli lemma, if for any pair $(\beta, \alpha)$ as in theorem 1.3 we show that $\sum_{n=1}^{\infty} I(\pi_{0}, \beta, \alpha, n) < +\infty$, then for almost any $T$ in $\Delta_{w}^{(1)}$ there exist just finitely many triples $(\beta, \alpha, n)$ which are reduced for $T$ and satisfy (1.2). Then it follows form the discussion above that the following proposition holds.

**Proposition 3.1.** In order to prove proposition 1.4 it is enough to prove that there exists a positive constant $C = C(d)$ depending only on the number of letters of $\mathcal{A}$ such that for any $n \in \mathbb{N}$ we have

$$\text{Leb}_{d-1}(I(\pi_{0}, \beta, \alpha, n)) < C\varphi(n).$$

3.0.2. Notation. Let us consider an admissible combinatorial datum $\pi$ in the same Rauzy class $\mathcal{R}$ of $\pi_{0}$. Letting $L$ vary in $\mathcal{A}$ we denote $e_{L}$ the correspondent vectors of the standard basis of $\mathbb{R}^{d}$. For a pair of letters $\beta$ and $\alpha$ such that $\pi(\beta) > 1$ and $\pi(\alpha) > 1$ we define the vectors with integer coordinates

$$w_{\beta}^{b}(\pi) := \sum_{\pi(\beta') < \pi(\beta)} e_{\beta'} \text{ and } w_{\alpha}^{t}(\pi) := \sum_{\pi(\alpha') < \pi(\alpha)} e_{\alpha'},$$

then we set $w_{\beta, \alpha}(\pi) := w_{\beta}^{b}(\pi) - w_{\alpha}^{t}(\pi)$. The property of these vectors is that the singularities for the i.e.t. $T = (\pi, \lambda)$ are given by $u_{\beta}^{b} = \langle w_{\beta}^{b}(\pi), \lambda \rangle$ and $u_{\alpha}^{t} = \langle w_{\alpha}^{t}(\pi), \lambda \rangle$. Therefore

$$u_{\beta}^{b} - u_{\alpha}^{t} = \langle w_{\beta, \alpha}(\pi), \lambda \rangle$$

and in particular we have that the set of those $T \in \Delta_{w}^{(1)}$ such that $u_{\beta}^{b} = u_{\alpha}^{t}$ coincides with $\Delta_{w}^{(1)} \cap (w_{\beta, \alpha}(\pi))_{1}^{-}$, where $(w_{\beta, \alpha}(\pi))_{1}^{-}$ denotes the hyperplane normal to $w_{\beta, \alpha}(\pi)$. We observe that for $L \in \mathcal{A}$ we have $\langle w_{\beta, \alpha}(\pi), e_{L} \rangle = 0$ or $\pm 1$ and all the tree values are attained. Therefore, if for some letter $\beta'$ we have $\langle w_{\beta, \alpha}(\pi), e_{\beta'} \rangle \neq 0$ then there exists some other letter $\alpha'$ such that $\langle w_{\beta, \alpha}(\pi), e_{\beta'} + e_{\alpha'} \rangle = 0$. It follows that $\Delta_{w}^{(1)} \cap (w_{\beta, \alpha}(\pi))_{1}^{-}$ is a convex sub-set of $\Delta_{w}^{(1)}$ with dimension $d - 2$.

Finally we introduce the antisymmetric matrix $\Omega_{\pi} \in \text{hom}(\mathbb{Z}^{d}, \mathbb{Z}^{d})$ defined by

$$(\Omega_{\pi})_{\beta, \alpha} = -1 \text{ if } \pi(\beta) < \pi(\alpha), \pi(\beta) > \pi(\alpha) \text{ and let us define the vector } \delta = \delta(\pi, \lambda) := \Omega_{\pi} \lambda. \text{ Then } T \text{ acts by } T|_{\Omega_{\pi}}(x) = x + \delta_{L}.$$
We can interpret the vector $\delta$ as a piecewise constant function $\delta: I \to \mathbb{R}$ defined by
$$\delta(x) = \delta_L \text{ if } x \in I_L^r.$$ Similarly we can also define a function $e: I \to \mathbb{N}^d$ by $e(x) = e_L$ if $x \in I_L^r$. The two functions are related by $\delta(x) = -\langle \Omega_\pi e(x), \lambda \rangle$.

3.1. **A necessary condition for the orbits of $Q$.** Let us fix a combinatorial datum $\pi_0$, a pair of letters $(\beta, \alpha)$ and a positive integer $n$ as in proposition 2.1. In this paragraph we describe a necessary condition which has to be satisfied elements $T$ in $I(\pi_0, \beta, \alpha, n)$. We consider the non-normalized version $Q$ of the Rauzy-Veech algorithm and we write $T^{(r)} = Q'(T)$.

**Lemma 3.2.** If the triple $(\beta, \alpha, n)$ is reduced for $T$ then there exists a finite Rauzy path $\gamma' = \gamma'(\beta, \alpha, n, T)$ starting at $\pi_0$ with length $r$ such that $T \in \Delta^{(1)}_\pi$ and for the singularities $u^{(r),t}_\alpha$ and $u^{(r),b}_\beta$ of $T^{(r)}$ we have
$$|T^n u^{b}_\beta - u^{t}_\alpha| = |u^{(r),t}_\alpha - u^{(r),b}_\beta|.$$ **Proof:** We consider $l \in \{0, ..., n\}$ such that $T^{-l} u^{b}_\beta = \min\{T^{-i} u^{b}_\beta; i \in \{0, ..., n\}\}$. Since the triple $(\beta, \alpha, n)$ is reduced for $T$, then for the same $l$ we have $T^{-l} u^{t}_\alpha = \min\{T^{-i} u^{t}_\alpha; i \in \{0, ..., n\}\}$ and
$$|T^n u^{b}_\beta - u^{t}_\alpha| = |T^{-l} u^{b}_\beta - T^{-l} u^{t}_\alpha|.$$ We consider the orbit $T^{(i)}$ of $T$ under $Q$ and for any $i$ we look at the interval $I^{(i)}$ where the i.e.t. $T^{(i)}$ acts. We consider the integer $r$ defined by
$$r := \max\{i \geq 1; T^{-i} u^{b}_\beta \in I^{(i)} \text{ and } T^{-i} u^{t}_\alpha \in I^{(i)}\}.$$ The discussion in paragraph 2.1.3 in the background implies that for this $r$ we have $T^{-l} u^{b}_\beta = u^{(r),b}_\beta$ and $T^{-l} u^{t}_\alpha = u^{(r),t}_\alpha$. The lemma is proved. \hfill $\square$

We call $\Gamma' = \Gamma' (\pi_0, \beta, \alpha, n)$ the set of all paths $\gamma'(\beta, \alpha, n, T)$ starting at $\pi_0$ and given by the lemma, where $T$ varies among all the elements of $\Delta^{(1)}_\pi$, for which $(\beta, \alpha, n)$ is a reduced triple. The set $\Gamma'$ is not disjoint, since there are different i.e.t.s $T$ and $T'$ in $\Delta^{(1)}_\pi$ such that $\gamma'(\beta, \alpha, n, T) < \gamma'(\beta, \alpha, n, T')$ strictly.

**Definition 3.3.** We call $\Gamma = \Gamma(\pi_0, \beta, \alpha, n)$ the subset of paths $\gamma \in \Gamma'$ which are minimal with respect to the ordering $\prec$. For any $\pi$ in the same Rauzy class of $\pi_0$ we denote $\Gamma_\pi$ the set of paths $\gamma \in \Gamma$ which end in $\pi$.

By minimality the set $\Gamma$ is automatically disjoint. If $T \in \Delta^{(1)}_\pi$ has $(\beta, \alpha, n)$ as a reduced triple, we call $\gamma = \gamma(\beta, \alpha, n, T)$ the path in $\Gamma$ such that $T \in \Delta^{(1)}_\pi$. Let us fix any $\pi$ in the same Rauzy class of $\pi_0$ and any $\gamma \in \Gamma_\pi$. Let $r$ be the length of $\gamma$ and $T = \langle \pi_0, \lambda \rangle$ be any element in $\Delta^{(1)}_\pi$. For the singularities $u^{(r),t}_\alpha$ and $u^{(r),b}_\beta$ of $T^{(r)}$ we have $|T^n u^{b}_\beta - u^{t}_\alpha| = |u^{(r),t}_\alpha - u^{(r),b}_\beta|$, moreover by minimality this is no more true for any $T^{(i)}$ with $i < r$. On the other hand, for any $T \in \Delta_\gamma$, the length datum of $T^{(r)}$ is $\lambda^{(r)} = t B^{-1}_\gamma \lambda$, therefore we can write (even if $(\beta, \alpha, n)$ is not reduced for $T$)
$$|u^{(r),t}_\alpha - u^{(r),b}_\beta| = |\langle \lambda^{(r)}, w_{\beta, \alpha}(\pi) \rangle| = |\langle \lambda, B^{-1}_\gamma w_{\beta, \alpha}(\pi) \rangle|.$$ By the discussion above, for the same $\pi$ and $\gamma$ in $\Gamma_\pi$, we have

$$\mathcal{I}(\pi_0, \beta, \alpha, n) \cap \Delta^{(1)}_\gamma \subset \{ \lambda \in \Delta^{(1)}_\gamma; |\langle \lambda, B^{-1}_\gamma w_{\beta, \alpha}(\pi) \rangle| < \phi(n) \}.$$
3.2. A global estimate.

Lemma 3.4. There exists a constant $c_1 = c_1(d)$ depending only on the numbers $d$ of letters of $A$ such that for any $\pi$ in the same Rauzy class of $\pi_0$ and any $\gamma$ in $\Gamma_{\pi}$ we have

$$\|B_{\gamma}^{-1}w_{\beta,\alpha}(\pi)\|_2 \geq c_1 n,$$

where $\| \cdot \|_2$ denotes the euclidean norm.

Proof: As we said above, for any $T = (\pi_0, \lambda)$ in $\Delta_{\pi_0}^{(1)}$ we can write $|T^n u^b_{\beta} - u^e_{\alpha}| = |(B_{\gamma}^{-1}w_{\beta,\alpha}(\pi), \lambda)|$ (even if $(\beta, \alpha, n)$ is not reduced for $T$). On the other hand we can also write

$$T^n u^b_{\beta} = u^b_{\beta} + \delta(u^b_{\beta}) + \ldots + \delta(T^{n-1}u^b_{\beta}) = \langle w^b_{\beta}(\pi_0), \lambda \rangle + \langle \Omega_{\pi_0} e(u^b_{\beta}), \lambda \rangle + \ldots + \langle \Omega_{\pi_0} e(T^{n-1}u^b_{\beta}), \lambda \rangle = \langle w^b_{\beta}(\pi_0) + \Omega_{\pi_0} S_n e(u^b_{\beta}), \lambda \rangle,$$

where $S_n e$ denotes the Birkhoff sum of the function $e : I \to \mathbb{N}^d$ introduced in paragraph 3.0.2. If we define the vector $v = v(\beta, n, \lambda) := S_n e(u^b_{\beta})$ we can write

$$\langle w_{\beta,\alpha}(\pi_0) + \Omega_{\pi_0} v, \lambda \rangle = \langle B_{\gamma}^{-1}w_{\beta,\alpha}(\pi), \lambda \rangle.$$

Since for fixed values of $\beta$ and $n$ the function $\lambda \mapsto v(\beta, n, \lambda)$ is locally constant, letting $\lambda$ vary in a small open set of $\Delta_{\pi}$ where the value $v$ of $S_n e(u^b_{\beta})$ is constant, we get

$$w_{\beta,\alpha}(\pi_0) + \Omega_{\pi_0} v = B_{\gamma}^{-1}w_{\beta,\alpha}(\pi).$$

Now we observe that for any $\lambda \in \Delta_{\pi_0}^{(1)}$ there exists a letter $L$ such that for the vector $\delta = \Omega_{\pi_0} \lambda$ we have $|\delta_L| \geq 1/d$. Since the normalized vector $n^{-1}v$ belongs to $\Delta_{\pi_0}^{(1)} \cap \mathbb{Q}^d$ we can apply the argument to it, therefore the exists some $L \in A$ such that $|\langle \Omega_{\pi_0} e, e_L \rangle| > n/d$. We get the lemma with $c_1 = 1/(d + \epsilon)$ for some small $\epsilon$. □

Let us consider any Rauzy path $\gamma$ starting at $\pi_0$ and ending at $\pi$. For any letter $L$ of $A$ we have

$$(B_{\gamma}^{-1}w_{\beta,\alpha}(\pi), t B_{\gamma} e_L) = \langle w_{\beta,\alpha}(\pi), e_L \rangle.$$  \hspace{1cm} (3.3)

The value of the scalar product above is always 1, 0 or $-1$ and all the values are attained. We consider $L$ such that the scalar product is 0 or $-1$. Since the vector $t B_{\gamma} e_L$ is in the positive cone $\mathbb{R}_+^d$ we get that $B_{\gamma}^{-1}w_{\beta,\alpha}(\pi)$ is outside of it. This implies that intersection $(B_{\gamma}^{-1}w_{\beta,\alpha}(\pi))^\perp \cap \Delta_{\pi_0}^{(1)}$ is always a convex subset of $\Delta_{\pi_0}^{(1)}$ with dimension $d - 2$. Moreover there exists a positive constant $c_2 = c_2(d)$ depending only on $d$ such that

$$|\cos \angle (\tilde{\Gamma}, B_{\gamma}^{-1}w_{\beta,\alpha}(\pi))| \geq c_2.$$

It follows that the projection from the ray $< B_{\gamma}^{-1}w_{\beta,\alpha}(\pi) >$ to $(\tilde{\Gamma})^\perp$ has norm always less than $1/c_2$ and therefore that for any $T = (\pi_0, \lambda)$ in $\Delta_{\pi_0}^{(1)}$ we have

$$\text{dist}(\lambda, (B_{\gamma}^{-1}w_{\beta,\alpha}(\pi))^\perp \cap \Delta_{\pi_0}^{(1)}) \leq \frac{|\langle \lambda, B_{\gamma}^{-1}w_{\beta,\alpha}(\pi) \rangle|}{c_2 \|B_{\gamma}^{-1}w_{\beta,\alpha}(\pi)\|_2}.$$  \hspace{1cm} (3.4)

Furthermore, if $\pi$ is in the same Rauzy class of $\pi_0$ and $\gamma$ belongs to $\Gamma_{\pi}$, lemma 3.4 implies that for $T = (\pi_0, \lambda)$ in $\Delta_{\pi_0}^{(1)}$ we have

$$\text{dist}(\lambda, (B_{\gamma}^{-1}w_{\beta,\alpha}(\pi))^\perp \cap \Delta_{\pi_0}^{(1)}) \leq \frac{|\langle \lambda, B_{\gamma}^{-1}w_{\beta,\alpha}(\pi) \rangle|}{c_3 n},$$  \hspace{1cm} (3.5)
where the constant is given by $c_3 := c_2 c_1$ and it depends only from the number $d$ of letters of $A$. As explained in paragraph 3.0.2, the intersection $(w_{\beta,\alpha}(\pi))^\perp \cap \Delta_\gamma^{(1)}$ is a convex subset of $\Delta_\gamma^{(1)}$ with dimension $d - 2$, thus equation (3.3) implies that $\Delta_\gamma^{(1)} \cap (B_\gamma^{-1} w_{\beta,\alpha}(\pi))^\perp$ is a non-empty convex subset of $\Delta_\gamma^{(1)}$ with dimension $d - 2$.

**Lemma 3.5.** There exists a constant $C_1 = C_1(d)$ such that

$$\text{Leb}_{d-1}(I(\pi_0, \beta, \alpha, n)) < \frac{C_1 \varphi(n)}{n} \sum_{\pi \in \mathbb{R}} \sum_{\gamma \in \Gamma_\pi} \text{Leb}_{d-2}(\Delta_\gamma^{(1)} \cap (B_\gamma^{-1} w_{\beta,\alpha}(\pi))^\perp).$$

**Proof:** Recalling equations (3.2) and (3.4) we get

$$I(\pi_0, \beta, \alpha, n) = \bigcup_{\pi \in \mathbb{R}} \bigcup_{\gamma \in \Gamma_\pi} I(\pi_0, \beta, \alpha, n) \cap \Delta_\gamma^{(1)} \subset$$

$$\bigcup_{\pi \in \mathbb{R}} \bigcup_{\gamma \in \Gamma_\pi} \{ \lambda \in \Delta_\gamma^{(1)} ; |(\lambda, B_\gamma^{-1} w_{\beta,\alpha}(\pi))| < \varphi(n) \} \subset$$

$$\bigcup_{\pi \in \mathbb{R}} \bigcup_{\gamma \in \Gamma_\pi} \{ \lambda \in \Delta_\gamma^{(1)} ; \text{dist}(\lambda, (B_\gamma^{-1} w_{\beta,\alpha}(\pi))^\perp \cap \Delta^{(1)}_{\pi_0}) < \frac{\varphi(n)}{c_3 n} \}$$

For any $\pi$ and any $\gamma$ in $\Gamma_\pi$ the set $\Delta_\gamma^{(1)} \cap (B_\gamma^{-1} w_{\beta,\alpha}(\pi))^\perp$ is a convex subset of $(B_\gamma^{-1} w_{\beta,\alpha}(\pi))^\perp \cap \Delta^{(1)}_{\pi_0})$. Thus we have

$$\text{Leb}_{d-1}(\lambda \in \Delta_\gamma^{(1)} ; \text{dist}(\lambda, (B_\gamma^{-1} w_{\beta,\alpha}(\pi))^\perp \cap \Delta^{(1)}_{\pi_0}) < \frac{\varphi(n)}{c_3 n} \leq$$

$$\frac{2 \varphi(n)}{c_3 n} \text{Leb}_{d-2}(\Delta_\gamma^{(1)} \cap (B_\gamma^{-1} w_{\beta,\alpha}(\pi))^\perp)$$

and the lemma follows putting $C_1 := 2/c_3$. \qed

### 3.3. A local uniform estimate

This paragraph is devoted to the proof of the following lemma:

**Lemma 3.6.** Let us consider a fixed triple $(\beta, \alpha, n)$ and a fixed element $\pi_0$. There exists a positive constant $C_2 = C_2(d)$ depending only from $d$ such that for any $\pi$ in the same Rauzy class of $\pi_0$ and for any $\gamma$ in $\Gamma_\pi$ we have

$$\text{Leb}_{d-2}(\Delta_\gamma^{(1)} \cap (B_\gamma^{-1} w_{\beta,\alpha}(\pi))^\perp) \leq C_2 n \text{Leb}_{d-1}(\Delta_\gamma^{(1)})$$

Combining lemma 3.6 with the result in lemma 3.5 and recalling that $\Gamma$ is a disjoint family we get that equation (3.1) holds, where the constant $C$ is given by $C = C_1 C_2$. Then proposition 1.4 holds.

Now we pass to the proof of lemma 3.6. Our strategy is the following: for any $\pi$ and any $\gamma$ in $\Gamma_\pi$ we decompose the convex set $\Delta_\gamma^{(1)} \cap (B_\gamma^{-1} w_{\beta,\alpha}(\pi))^\perp$ into maximal $(d - 2)$—simplices, that is simplices having their vertexes in the set of extremal points of $\Delta_\gamma^{(1)} \cap (B_\gamma^{-1} w_{\beta,\alpha}(\pi))^\perp$. Equation (3.3) implies that such decomposition is combinatorially the same for all $\gamma \in \Gamma_\pi$ and coincides with the one for $\Delta_\gamma^{(1)} \cap (w_{\beta,\alpha}(\pi))^\perp$. Once $\Delta_\gamma^{(1)} \cap (B_\gamma^{-1} w_{\beta,\alpha}(\pi))^\perp$ is decomposed into maximal simplices, we give an uniform estimate for any simplex in the decomposition. The lemma will follow since the number of simplices is uniformly bounded for all $\gamma$ and depends only from $d$. 

3.3.1. Decomposition in $(d-2)$-simplices. We fix any $\pi$ in the same Rauzy class of $\pi_0$ and any pair of letters $\beta$ and $\alpha$ such that $\pi^i(\beta) > 1$ and $\pi^j(\alpha) > 1$. We want to describe the set $\text{Ext}(\beta, \alpha, \pi)$ of the extremal points of $\Delta_\pi^{(1)} \cap (w_{\beta, \alpha}(\pi))^\perp$.

Let $S = S(\pi, \beta, \alpha) \subset A$ be the set of letters $L$ of $A$ which satisfy the condition
\begin{equation}
\pi^i(L) < \pi^j(\alpha) \quad \text{or} \quad \pi^i(L) < \pi^k(\beta).
\end{equation}

It is easy to check that a letter $L$ belongs to $S$ if and only if $\langle e_L, w_{\beta, \alpha}(\pi) \rangle = 0$, therefore the set $\{e_L\}_{L \in S}$ is exactly the set of the elements in $\text{Ext}(\beta, \alpha, \pi)$ which are also extremal points of $\Delta_\pi^{(1)}$ (the notation $S$ stands for *singletons*).

The other extremal points of $\Delta_\pi^{(1)} \cap (w_{\beta, \alpha}(\pi))^\perp$ have the form $2^{-1}(e_{\beta'} + e_{\alpha'})$, where $\beta'$ and $\alpha'$ are letters in $A \setminus S$ such that $\beta' \neq \alpha'$ and $\langle e_{\alpha'} + e_{\beta'}, w_{\beta, \alpha}(\pi) \rangle = 0$. It is easy to check that these conditions for the pair $(\beta', \alpha')$ are equivalent to
\begin{equation}
\begin{aligned}
\pi^i(\alpha') &< \pi^j(\alpha) & &\text{and} & &\pi^k(\alpha') &> \pi^l(\beta)
\end{aligned}
\end{equation}

We call $P = P(\pi, \beta, \alpha)$ the set of pairs of letters $(\beta', \alpha')$ satisfying equation (3.6) (the notation $P$ stands for *pairs*). We also denote $A(P)$ the set of those letters which appear in the pairs of $P$ and $p$ the cardinality of $A(P)$. It follows from the discussion in paragraph 3.3.2 that we have $A(P) = A \setminus S$. Note that the set $S$ may be empty, but $P$ always contains at least one element. With the notation above we can write
\[\text{Ext}(\beta, \alpha, \pi) = \{e_L\}_{L \in S} \cup \{\frac{e_{\beta'} + e_{\alpha'}}{2}\}_{(\beta', \alpha') \in P}.\]

Now we describe the maximal simplices, that is the $(d-2)$-simplices contained in $\Delta_\pi^{(1)} \cap (w_{\beta, \alpha}(\pi))^\perp$ which have their vertexes in $\text{Ext}(\beta, \alpha, \pi)$. Any such simplex, let us call it $\Delta$, is determined by the set of its vertexes. We observe that $e_L$ is an extremal point of $\Delta$ for any $L \in S$. Indeed if this is not true, since $L \notin A(P)$, the simplex $\Delta$ is contained in the hyper-face $\Delta_\pi^{(1)} \setminus \{e_L\}$ spanned by the vertexes $\{e_X\}_{X \neq L}$. This is absurd because $\Delta_\pi^{(1)} \setminus \{e_L\} \cap (w_{\beta, \alpha}(\pi))^\perp$ has dimension $d-3$ and it cannot contain any $(d-2)$-simplex. It follows that to give a maximal simplex $\Delta$ the only freedom is in the choice of a subset $P'$ of $P$ with $p-1$ elements such that the corresponding vectors $(e_{\beta'} + e_{\alpha'})$ are affine independent, that is a subset $P'$ which satisfies
\begin{equation}
\exists(P') = p-1 \quad \text{and} \quad \dim \text{Span}\{(e_{\beta'} + e_{\alpha'})\}_{(\beta', \alpha') \in P'} = p-2.
\end{equation}

Given a subset $P'$ of $P$ which satisfies equation (3.7) we denote $\Delta(P')$ the maximal $(d-2)$-simplex of $\Delta_\pi^{(1)} \cap (w_{\beta, \alpha}(\pi))^\perp$ determined by $P'$. We denote $A(P')$ the set of those letters which appear in the pairs of $P'$ and we note that equation (3.7) implies that $A(P') = A(P)$. Indeed, if there is a letter $L \in A(P)$ which does not appear in any pair of $P'$, then with the same argument as above we prove that $\Delta(P')$ is contained in $\Delta_\pi^{(1)} \setminus \{e_L\} \cap (w_{\beta, \alpha}(\pi))^\perp$, which is absurd since such set has dimension $d-3$.

Now we fix any path $\gamma \in \Gamma_\pi$ and denote $\text{Ext}(\beta, \alpha, \pi, \gamma)$ the set of extremal points of $\Delta_\pi^{(1)} \cap (B^{-1}_{\gamma} w_{\beta, \alpha}(\pi))^\perp$. Equation (3.3) implies
\[\text{Ext}(\beta, \alpha, \pi, \gamma) = \left(\frac{t B_{\gamma} e_L}{q_L}\right)_{L \in S} \cup \left(\frac{t B_{\gamma} (e_{\beta'} + e_{\alpha'})}{q_{\beta'} + q_{\alpha'}}\right)_{(\beta', \alpha') \in P'}.\]
As above, for any subset $P'$ of $P$ which satisfy equation (3.7), we call $\Delta(P', \gamma)$ the $(d-2)$-simplex whose vertexes are the points

$$\left(\frac{tB_\gamma e_L}{q_L^\gamma}\right)_{L \in S} \cup \left(\frac{tB_\gamma (e_\beta + e_\alpha')}{q_{\beta'} + q_{\alpha'}'}\right)_{(\beta', \alpha') \in P'}.$$

All the maximal $(d-2)$-simplices in $\Delta^{(1)}_\gamma \cap (B^{-1}_\gamma w_{\beta', \alpha}(\pi))^\perp$ are obtained in this way. Finally the volume of a maximal simplex in the decomposition is given by the formula

$$\text{Leb}_{d-2}(\Delta(P', \gamma)) = \prod_{L \in S} (q_L^\gamma)^{-1} \prod_{(\beta', \alpha') \in P'} (q_{\beta'}^\gamma + q_{\alpha'}^\gamma)^{-1}.$$  

### 3.3.2. Estimation for any $(d-2)$-simplex

Before entering into the details we recall that paths in $\Gamma$ are minimal by definition. This means that if $\gamma$ is a path in $\Gamma$ with length $r$ and $T = (\pi_0, \lambda)$ is in $\Delta^{(1)}_\gamma$, then for $\lambda^{(r)} = tB^{-1}_\gamma \lambda$ we have $|T^n u_\beta^{(r)} - u_\alpha^{(r)}| = |u_\beta^{(r)} - u_\alpha^{(r)}|$ and this condition does not hold for any $T^{(1)}$ with $i < r$. It follows that for any $\gamma \in \Gamma$ the last loser is either $\beta$ (in a top arrow) or $\alpha$ (in a bottom arrow). The discussion is the same for the two cases, therefore from now on we suppose without losing in generality that $\gamma$ ends with a top arrow with loser $\beta$. If we call $W$ the letter which wins against $\beta$ in this last arrow, then the ending point $\pi$ of $\gamma$ has the form

$$\pi = \left( \ldots \alpha \ldots W \beta \ldots \right).$$

We observe that $W \in A \setminus S$, that is it does not satisfy equation (3.5), therefore any subset $P'$ of $P$ satisfying equation (3.7) contains a pair where the letter $W$ appears. Moreover, since $\beta$ loses against $W$ in the last arrow of $\gamma$, we have $u_\beta^{(r)} = T^{(r-1)}(u_\beta^{(r-1)}, b)$, where the singularity $u_\beta^{(r-1), b}$ belongs to the interval $I_W^{(r-1), t}$ in the domain of $T^{(r-1)}$. Since we have $u_\beta^{(r), b} = T^{n-l}u_\beta^{(r)}$ for some $l \in \{0, \ldots, n\}$, then the vector $q^{(r)} = q^\gamma$ of the return times to $I^{(r)}$ satisfies

$$q_W^{(r)} = q_W^{(r-1)} \leq n.$$

Now let us consider any $\gamma \in \Gamma_x$, any subset $P'$ of $P$ satisfying equation (3.7) and the associated maximal $(d-2)$-simplex $\Delta(P', \gamma)$. Recalling that $\text{Leb}_{d-1}(\Delta^{(1)}_\gamma) = (d!)^{-1}(\prod_{L \in A} q_L^\gamma)^{-1}$ and using equation (3.5) for the volume of $\Delta(P', \gamma)$, we get that the condition

$$\text{Leb}_{d-2}(\Delta(P', \gamma)) \leq d n \text{Leb}_{d-1}(\Delta^{(1)}_\gamma)$$

is equivalent to

$$n \prod_{(\beta', \alpha') \in P'} (q_{\beta'}^\gamma + q_{\alpha'}^\gamma) \geq \prod_{L \in A} q_L^\gamma.$$

The argument to prove equation (3.10) is purely combinatorial and to explain it we have to introduce a combinatorial notion. Let us consider a subset $P'$ of $P$ satisfying equation (3.7). We say that a subset $P_0$ of $P'$ is isolated if $A(P_0) \cap A(P' \setminus P_0) = \emptyset$, where $A(P_0)$ and $A(P' \setminus P_0)$ denote respectively the set of the letters appearing in the pairs of $P_0$ and the set of letters appearing in the pairs of $P' \setminus P_0$. We denote $p_0$ the number of elements of $A(P_0)$ and $r_0$ the number of elements of $P_0$. We have $2 \leq p_0 \leq p - 2$ and $1 \leq r_0 < p_0$. The key point is that
if \( P' \) satisfies equation (3.7) then it does not contain isolated subsets. In order to prove this let us consider any subset \( P_0 \) of \( P' \) and let us suppose that \( P_0 \) is isolated. We call \( \Delta(P' \setminus P_0) \) the hyper-face of \( \Delta(P') \) whose vertices are the points \( e_{\beta'} + e_{\alpha'} \) for \((\beta', \alpha') \in P' \setminus P_0 \). We have \( \dim(\Delta(P' \setminus P_0)) = p - r_0 - 2 \). We call \( \tilde{\Delta}_{\mathcal{A}(P' \setminus P_0)}^{(1)} \) the hyper-face of \( \Delta_{\mathcal{A}(P' \setminus P_0)}^{(1)} \) whose vertices are the points \( e_L \) for \( L \in \mathcal{A}(P' \setminus P_0) \). We have \( \dim(\tilde{\Delta}_{\mathcal{A}(P' \setminus P_0)}^{(1)}) = p - p_0 - 1 \). Then we observe that for any \( L \in \mathcal{A}(P' \setminus P_0) \) we have \( \langle w_{\beta,\alpha}(\pi), e_L \rangle \neq 0 \), therefore \( \dim(\langle w_{\beta,\alpha}(\pi) \rangle_{\mathcal{A}(P' \setminus P_0)}^{(1)}) = p - p_0 - 2 \). Finally we simply observe that assuming that \( P_0 \) is isolated implies that the sub-simplex \( \Delta(P' \setminus P_0) \) is contained in \( \langle w_{\beta,\alpha}(\pi) \rangle_{\mathcal{A}(P' \setminus P_0)}^{(1)} \) and this is absurd since \( p - p_0 - 2 < p - r_0 - 2 \).

Now we prove equation (3.10). Since \( W \in \mathcal{A}(P') \) then there exists a letter \( L_1 \neq W \) such that \((W, L_1) \in P' \) and for this pair we obviously have

\[
q_{W}^\gamma + q_{L_1}^\gamma > q_{L_2}^\gamma.
\]

We set \( P_1 := \{(L_1, W)\} \) and \( \mathcal{A}(P_1) = \{W, L_1\} \). Since \( P_1 \) is not isolated there exists a letter \( L_2 \not\in \mathcal{A}(P_1) \) such that either \((L_2, L_1) \in P' \) or \((L_2, W) \in P' \) (we do not care about the order in which letters appears in pairs of \( P' \); this is not a problem since condition (3.9) implies that there is just one possible order). As before we obviously have

\[
\text{either } q_{L_2}^\gamma + q_{L_1}^\gamma > q_{L_2}^\gamma \text{ or } q_{L_2}^\gamma + q_{L_1}^\gamma > q_{L_2}^\gamma.
\]

According to the two cases we set either \( P_2 := P_1 \cup \{(W, L_2)\} \) or \( P_2 := P_1 \cup \{(L_1, L_2)\} \) and we argue again that \( P_2 \) is not isolated. Going on iteratively we get a strictly increasing sequence of sets \( P_1 \subset P_2 \subset \ldots \subset P_{p-1} = P' \) and a sequence of letters \( L_0 = W, L_1, \ldots, L_{p-1} \) without repetitions such that for all \( i = 1, \ldots, p - 1 \) the only pair in \( P_i \setminus P_{i-1} \) contains the letter \( L_i \). It follows that

\[
\prod_{\beta, \alpha \in \mathcal{P}} (q_{\beta'}^\gamma + q_{\alpha'}^\gamma) > \frac{1}{q_{W}^\gamma}
\]

and recalling that \( q_{W}^\gamma \leq n \) we get equation (3.10). Lemma 3.6 is proved.

4. The divergent case.

This section is entirely devoted to the proof of proposition 1.5. Let \( \varphi : \mathbb{N} \rightarrow \mathbb{R}_+ \) be a positive sequence such that \( n \varphi(n) \) is monotone decreasing and \( \sum_{n=1}^{\infty} \varphi(n) = \infty \) and let \((\beta, \alpha)\) be any pair of letters with \( \pi_0^\beta(\beta) > 1 \) and \( \pi_0^\alpha(\alpha) > 1 \). We want to prove that for almost any \( T \in \Delta_{\mathcal{A}(P_0)}^{(1)} \) there exist infinitely many triples \((\beta, \alpha, n)\) reduced for \( T \) which are solution of equation (1.2), that is such that \( |T^n(u_0^\beta) - u_0^\alpha| < \varphi(n) \).

4.1. Sufficient condition for the orbits of \( Q \).

4.1.1. Properties \( A \) and \( B \). We introduce two combinatorial properties of pairs \((\beta, \alpha)\) with \( \pi_0^\beta(\beta) > 1 \) and \( \pi_0^\alpha(\alpha) > 1 \). They are called property \( A \) and property \( B \) and depend only from the Rauzy class \( \mathcal{R} \) of \( \pi_0 \).

Definition 4.1. Let \((\beta, \alpha)\) be an ordered pair of letters with \( \pi_0^\beta(\beta) > 1 \), \( \pi_0^\alpha(\alpha) > 1 \).
• We say that \((\beta, \alpha)\) has property A if there exists some \(\pi = \pi(\beta, \alpha) \in \mathcal{R}\) such that
\[
\pi^t(\alpha) = \pi^b(\beta) = d
\]
that is we have
\[
\pi = \left( \ldots \alpha \; \ldots \beta \right).
\]

• We say that \((\beta, \alpha)\) has property B if there exists some \(\pi = \pi(\beta, \alpha) \in \mathcal{R}\) and some letter \(V \in \mathcal{A}\) such that
\[
\{ \xi \in \mathcal{A} \mid \pi^t(\xi) < \pi^t(\alpha) \} \cup \{ V \} = \{ \xi \in \mathcal{A} \mid \pi^b(\xi) < \pi^b(\beta) \}
\]
\[
\pi^t(V) = \pi^b(\alpha) = d.
\]

When \((\beta, \alpha)\) has property B and \(\pi\) is the combinatorial datum given above, we call \(L\) the letter such that \(\pi^l(L) = \pi^b(\beta) - 1\) and \(\pi^t(L) < \pi^t(\alpha)\) (observe that in this case that the admissibility of \(\pi\) implies \(L \neq V\)). We have
\[
\pi = \left( \ldots \; L \; \ldots \alpha \; \ldots \; V \; \ldots \; L \; \beta \; \ldots \alpha \right).
\]

Remark 4.2. In paragraph 5.1 we state and prove a combinatorial property of Rauzy classes, namely theorem 5.3, which implies that any pair \((\beta, \alpha)\) has above has either property A or property B (or both) in definition 4.1.

4.1.2. Steps of the algorithm which give reduced triples. We write \(\gamma(r)\) to denote a Rauzy path of length \(r\). In this paragraph we consider the non-normalized version of the Rauzy-Veech algorithm \(Q : \Delta(\mathcal{R}) \to \Delta(\mathcal{R})\). For any \(T = (\pi_0, \lambda) \in \Delta_{\pi_0}\) without connections we call \((\pi(r), \lambda(r))\) the pair of combinatorial and length data of \(T(r) = Q^r(T)\).

Lemma 4.3. Let \((\beta, \alpha)\) be a pair with property A in definition 4.1 and let \(\pi = \pi(\beta, \alpha)\) be an element in \(\mathcal{R}\) such that \(\pi^b(\beta) = \pi^t(\alpha) = d\). Then there exists a finite path \(\eta\) in \(\Pi(\mathcal{R})\) having \(\pi\) in third to last position and such that the following holds. For any path \(\gamma(r)\) ending with \(\eta\) we have an integer \(n = n(\gamma(r))\) with \(n \leq \|q^r(\gamma)\|\) such that for any \(T \in \Delta_{\gamma(r)}\) the triple \((\beta, \alpha, n)\) is reduced for \(T\) and
\[
\lambda^{(r)} = |T^n u^b_\beta - u^l_\alpha|.
\]

Proof: Let \(\gamma^r_\alpha\) be the top arrow with winner \(\alpha\) connecting \(\pi\) to \(R^t(\pi)\). Then let \(\gamma^r_W\) be the bottom arrow starting from \(R^t(\pi)\) with loser \(\alpha\), where \(W\) is the letter which wins against \(\alpha\) in this second arrow. Let us consider the concatenation \(\gamma^r_\alpha \gamma^r_W\) of these two arrows and let \(\eta\) be any path ending with \(\gamma^r_\alpha \gamma^r_W\) (therefore \(\pi\) is in third to last position in \(\eta\)).

Let \(\gamma(r)\) be a path with length \(r\) as in the hypothesis and take \(T \in \Delta_{\gamma(r)}\). Let us consider \(T^{(r-2)} = Q^{r-2}(T)\) and the two singularities \(u^{(r-2), b}_\beta\) and \(u^{(r-2), l}_\alpha\). According to paragraph 2.1.3 in the background we have two positive integers \(l(r-2, \beta)\) and \(h(r-2, \alpha)\) depending from \(\gamma(r-2)\) such that \(u^{(r-2), b}_\beta = T^{(r-2, \beta)} u^{(r-2), l}_\alpha\) and \(u^{(r-2), l}_\alpha = T^{-h(r-2, \alpha)} u^{(r-2), b}_\beta\). We set
\[
n := l(r - 2, \beta) + h(r - 2, \alpha).
\]

The discussion in paragraph 2.1.3 also implies \(n < \|q^{(r-2)}\|\), so \(n < \|q^{(r)}\|\). Since \(\gamma(r)\) ends with \(\eta\) then \(T^{(r-2)} \in \Delta_{\gamma^r_\alpha \gamma^r_W}\) in particular \(T^{(r-2)} \in \Delta_{\gamma^r_\alpha}\), thus lemma 2.2.
in the background implies that for $T^{(r-1)} = Q(T^{(r-2)})$ we have
\[ \lambda_\alpha^{(r-1)} = |u_\beta^{(r-2),b} - u_\alpha^{(r-2),t}|. \]
Furthermore $T^{(r-1)} \in \Delta_{\eta_W}$, thus the winner of $T^{(r-1)}$ is $W$ and in particular we have $\lambda_\alpha^{(r-1)} = \lambda_\alpha^{(r)}$. Summing up, we proved that
\[ \lambda_\alpha^{(r)} = |T^{l(r-2,\beta)}u_\beta^b - T^{-h(r-2,\alpha)}u_\alpha^t|. \]

Let us consider the sub-interval $J^{(r-2)}(\beta, \alpha)$ of $I^{(r-2)}$ whose endpoints are $u_\beta^{(r-2),b}$ and $u_\alpha^{(r-2),t}$. In order to complete the proof of the lemma it is enough to prove that for $-l(r-2,\beta) \leq i \leq h(r-2,\alpha)$ the iterates $T^i(J^{(r-2)}(\beta, \alpha))$ do not contain neither any singularity $u_\alpha^t$ of $T$ nor any singularity $u_\beta^b$ of $T^{-1}$.

Since $q_\alpha^{(r-2)}$ is the return time of $I_\alpha^{(r-2),t}$ to $I^{(r-2)}$ under iteration of $T$, then the iterates $T^i(I_\alpha^{(r-2),t})$ do not contain any singularity of $T$ or of $T^{-1}$ for $0 \leq i \leq q_\alpha^{(r-2)}$ (see paragraph 2.1.3). In our case the condition $T^{(r-2)} \in \Delta_{\eta_W}$ implies $J^{(r-2)}(\beta, \alpha) \subset I_\alpha^{(r-2),t}$ and recalling that $0 \leq h(r-2,\alpha) < q_\alpha^{(r-2)}$ we get the required property of $T^i(J^{(r-2)}(\beta, \alpha))$ for $0 \leq i \leq h(r-2,\alpha)$.

Similarly $q_W^{(r-1)}$ is the return time of $I_W^{(r-1),b}$ to $I^{(r-1)}$ under iteration of $T^{-1}$, so the iterates $T^{-i}(I_W^{(r-1),b})$ do not contain any singularity of $T$ or of $T^{-1}$ for $0 \leq i \leq q_W^{(r-1)}$. To finish we just observe that in our case the condition $T^{(r-1)} \in \Delta_{\eta_W}$ implies $J^{(r-2)}(\beta, \alpha) \subset I_W^{(r-1),b}$. Moreover we have $l(r-2,\beta) < q_W^{(r-1)}$, because otherwise we could extend continuously $T^{-q_W^{(r-1)}}$ to $u_\beta^{(r-2)}$ and get $T^i u_\beta^{(r-2)} \in I^{(r-2)}$ with $l = l(r-2,\beta) - q_W^{(r-1)}$, which is impossible. It follows that the required property of $T^i(J^{(r-2)}(\beta, \alpha))$ holds also for $-l(r-2,\beta) \leq i \leq 0$. The lemma is proved.

Lemma 4.4. Let $(\beta, \alpha)$ be a pair with property $B$ and consider an element $\pi = \pi(\beta, \alpha)$ of $R$ as in definition 4.1. Let $V$ and $L$ be the associated letters. Then there exists a finite path $\eta$ in the Rauzy diagram of $R$ having $\pi$ in second to last position and ending with a bottom arrow with winner $\alpha$. Moreover for any Rauzy path $\gamma(r)$ ending with $\eta$ we have an integer $n = n(\gamma(r))$ with $n \leq \|q^{(r)}\|$ such that for $T \in \Delta_{\gamma(r)}$ we have
\begin{equation}
\lambda_\pi^{(r)} = |T^n u_\beta^b - u_\alpha^t|.
\end{equation}
Furthermore if $\lambda_\pi^{(r)} < \lambda_L^{(r)}$, then the triple $(\beta, \alpha, n)$ is reduced for $T$.

Proof: We consider the bottom arrow $\gamma_\alpha^b$ with winner $\alpha$ starting at $\pi$ and we fix any finite path $\eta$ having $\gamma_\alpha^b$ as last arrow (thus $\pi$ is in second to last position in $\eta$).

Let $\gamma(r)$ be a path with length $r$ as in the hypothesis and take any $T \in \Delta_{\gamma(r)}$. Then consider the two singularities $u_\beta^{(r-1),b}$ and $u_\alpha^{(r-1),t}$ associated to $T^{(r-1)}$. Since $\pi$ is second to last in $\eta$ and $\gamma(r)$ ends with $\eta$, then $T^{(r-1)} \in \Delta_\pi$ and the combinatorics of $\pi$ implies
\[ \lambda_\pi^{(r-1)} = |u_\beta^{(r-1),b} - u_\alpha^{(r-1),t}| = |T^{l(r-1,\beta)}u_\beta^b - T^{-l(r-1,\alpha)}u_\alpha^t|. \]
where as in lemma 4.3 we denote \( l(r - 1, \beta) \) and \( h(r - 1, \alpha) \) the positive integers defined in paragraph 2.1.3. We set
\[
n := l(r - 1, \beta) + h(r - 1, \alpha)
\]
and according to paragraph 2.1.3 we have \( n < \|q(r-1)\| < \|q(r)\| \). Since \( T(r-1) \in \Delta_{\beta} \), then \( \alpha \) is the winner of \( T(r-1) \) and we have \( \lambda_V^{(r)} = \lambda_V^{(r-1)} \). Summing up we get
\[
\lambda_V^{(r)} = |T^{l(r-1,\beta)}u_\beta - T^{-l(r-1,\alpha)}u_\alpha|.
\]
As in lemma 4.3 we consider the interval \( J^{(r-1)}(\beta, \alpha) \) whose endpoints are \( u^{(r-1),b}_\alpha \) and \( u^{(r-1),i}_\alpha \). To prove the lemma it is enough to prove that the iterates \( T^i(J^{(r-1)}(\beta, \alpha)) \) do not meet any singularity of \( T \) or any singularity of \( T^{-1} \) for \( -l(r-1, \beta) \leq i \leq h(r-1, \alpha) \). The condition \( T^{(r-1)} \in \Delta_{\beta} \) implies \( \lambda_V^{(r-1)} > \lambda_{V^{(r-1)}}^{(r)} \) and it follows from the combinatorics of \( \pi \) that \( u^{(r-1),i}_\alpha < u^{(r-1),b}_\alpha < u^{(r-1),i}_\alpha + \lambda^{(r-1)} \). Therefore we have \( J^{(r-1)}(\beta, \alpha) \subset I^{(r-1),i}_\alpha \) and arguing as in lemma 4.3 we get that the condition on \( T^i(J^{(r-1)}(\beta, \alpha)) \) holds for any \( 0 \leq i \leq h(r-1, \alpha) \). On the other hand we assume as hypothesis that \( \lambda_V^{(r)} < \lambda_V^{(r-1)} \), which is equivalent to \( \lambda_V^{(r-1)} < \lambda_L^{(r-1)} \). As above the last condition implies \( J^{(r-1)}(\beta, \alpha) \subset I^{(r-1),b}_\beta \). Arguing again as in lemma 4.3 we get that the condition on \( T^i(J^{(r-1)}(\beta, \alpha)) \) holds for any \( -l(r-1, \beta) \leq i \leq 0 \). The lemma is proved.

4.1.3. The sufficient condition. We recall that for \( T \in \Delta_{\alpha}^{(1)} \) without connections we denote \( \gamma(T, \infty) \) the half infinite Rauzy path in the Rauzy diagram generated by \( T \) and \( \gamma(T, r) \) the concatenation of the first \( r \) arrows of \( \gamma(T, \infty) \).

**Proposition 4.5.** Let \( (\beta, \alpha) \) be a pair with property A and \( \eta \) be a path given by lemma 4.3. For any \( T \in \Delta_{\alpha}^{(1)} \) without connections, in order to have infinitely many triples \( (\beta, \alpha, n) \) reduced for \( T \) and solutions of equation (1.2), it is enough to have infinitely many instants \( r \in \mathbb{N} \) such that \( \gamma(T, r) \) ends with \( \eta \) and
\[
\lambda_V^{(r)} \leq \frac{1}{\|\lambda^{(r)}\|} \varphi(\|q^{(r)}(T, r)\|).
\]

Let \( (\beta, \alpha) \) be a pair with property B, let \( V \) and \( L \) be the associated letters and let \( \eta \) be a path given by lemma 4.4. For any \( T \in \Delta_{\alpha}^{(1)} \) without connections, in order to have infinitely many triples \( (\beta, \alpha, n) \) reduced for \( T \) and solutions of equation (1.2), it is enough to have infinitely many instants \( r \in \mathbb{N} \) such that \( \gamma(T, r) \) ends with \( \eta \) and
\[
\lambda_V^{(r)} \leq \min\{\lambda^{(r)}_L, \frac{1}{\|\lambda^{(r)}\|} \varphi(\|q^{(r)}(T, r)\|)\}
\]

Note: Theorem 5.1 implies that any pair \( (\beta, \alpha) \) with \( \pi_0^{(\alpha)}(\alpha) > 1 \) and \( \pi_0^{(\beta)}(\beta) > 1 \) has either property A or property B (or both), therefore, according to the two cases, we can associate to it a path \( \eta \) respectively as in lemma 4.3 or as in lemma 4.4. It follows that proposition 4.5 always applies.

**Proof:** Let \( \eta \) be a path given by lemma 4.3 or by lemma 4.4 and let \( \gamma \) be any path in \( \Pi(R) \) ending with \( \eta \). In both cases the triple \( (\beta, \alpha, n) \) associated to \( \gamma \) satisfies \( n < \|q^\gamma\| \). Moreover, since by assumption \( n\varphi(n) \) is decreasing monotone, then \( \varphi(n) \) is and therefore \( \varphi(\|q^\gamma\|) < \varphi(n) \). The proposition is proved.
4.2. Sufficient shrinking target criterion.

4.2.1. First return to a neat path. Let $T \in \Delta^{(1)}_{\pi_0}$ be an i.e.t. without connections. Let $\eta$ be any finite path of length $l$, that is $\eta$ is the concatenation $\eta_1..\eta_l$ of $l$ elementary arrows. For $r \in \mathbb{N}$ the path $\gamma(T, r)$ ends with $\eta$ if and only if $\tilde{T}^{(r-l)} \in \Delta^{(1)}_{\eta}$, according to lemma 2.2 in the background. Therefore, motivated by proposition 4.5 we look for instants $r$ such that the iterates $\tilde{T}^{(r)} = \tilde{Q}^r(T)$ belong to $\Delta^{(1)}_{\eta}$. Since the map $\tilde{Q}$ is recurrent, for a generic $T$ this happens infinitely many times.

**Definition 4.6.** We say that a finite Rauzy path $\eta$ is neat if any time that we can write $\eta = \eta_1 \eta_2 = \eta_3 \eta_4$ either $\eta = \eta_1$ or $\eta_1$ is trivial.

**Lemma 4.7.** Let us consider any finite Rauzy path $\eta$ and the associated simplicial cone $\Delta_{\eta}$. Let $l = l(\eta)$ be the number of elementary arrows which compose $\eta$. Then $\eta$ is neat if and only if for any $T \in \Delta_{\eta}$ we have $T^{(i)} \notin \Delta_{\eta}$ for all $i \in \{1, \ldots, l - 1\}$.

**Proof:** Let us first suppose that $\eta$ is not neat, which means that there exist three non trivial paths $\eta_1, \eta_2, \eta_3$ such that $\eta = \eta_1 \eta_2 = \eta_2 \eta_3$. We consider the sub-cone $\Delta_{\eta_3}$ of $\Delta_{\eta}$ and any $T \in \Delta_{\eta_3}$. Let $i$ be the length of $\eta_1$. Since $\eta_1$ is not trivial then $1 \leq i \leq l$ and we have $T^{(i)} \in \Delta_{\eta_3} = \Delta_{\eta}$.

In order to see the other implication we recall that for the first $L$ steps of the algorithm $\tilde{Q}$ applied to $T$ are given by the $l$ arrows $\eta_1, \ldots, \eta_l$ composing $\eta$. Let us suppose that for some $1 \leq i \leq L - 1$ we have $T^{(i)} \in \Delta_{\eta}$. This means that $\eta$ begins with $\eta_{i+1} \ldots \eta_l$, on the other hand $\eta_{i+1} \ldots \eta_l$ is also the ending part of $\eta$, so $\eta$ is not neat. The lemma is proved.

Let $\eta : \pi_0 \rightarrow \pi_1$ be a neat path starting at $\pi_0$ and ending in $\pi_1$. We consider the sub-simplex $\Delta^{(1)}_{\eta}$ of $\Delta^{(1)}_{\pi_0}$ and we define the first entering map of $\tilde{Q}$ in $\Delta^{(1)}_{\eta}$, that is the map $R_{\eta} : \Delta^{(1)}(\mathcal{R}) \rightarrow \Delta^{(1)}_{\eta}$ defined by $R_{\eta}(T) := \tilde{Q}^{E(T)}(T)$, where

$$E(T) := \min\{k \in \mathbb{N}^*: \tilde{Q}^k(T) \in \Delta^{(1)}_{\eta}\}.$$ 

By recurrence of $\tilde{Q}$ the map $R_{\eta}$ is defined almost everywhere on $\Delta^{(1)}(\mathcal{R})$. We denote $R_{\eta, \pi_1} := R_{\eta}|_{\Delta^{(1)}_{\pi_1}}$ the restriction of $R_{\eta}$ to the simplex $\Delta^{(1)}_{\pi_1}$. Then we consider the homeomorphism $\tilde{Q}_{\eta} : \Delta^{(1)}_{\eta} \rightarrow \Delta^{(1)}_{\pi_1}$ defined by

$$\tilde{Q}_{\eta}(\pi_0, \lambda) = (\pi_1, \frac{t B^{-1}_{\eta} \lambda}{\|t B^{-1}_{\eta} \lambda\|}).$$

$\tilde{Q}_{\eta}$ can be concatenated with $R_{\eta, \pi_1}$. We define the map $\mathcal{F}_{\eta} : \Delta^{(1)}_{\pi_1} \rightarrow \Delta^{(1)}_{\pi_1}$ by

$$\mathcal{F}_{\eta}(T) := \tilde{Q}_{\eta} \circ R_{\eta, \pi_1}(T).$$

Let $\Gamma^\nu$ be the set of Rauzy paths $\gamma$ starting and ending in $\pi_1$ which contain $\eta$ and are minimal with this property with respect to the ordering $\prec$. In other words the elements of $\Gamma^\nu$ are the paths $\gamma$ which admit a decomposition

(4.6) $\gamma = \nu \eta$,

with $\nu \in \Pi(\mathcal{R})$ and have the property that for no proper sub-path $\gamma'$ with $\gamma' \prec \gamma$ the decomposition in equation (4.6) is possible. The simplices $\Delta^{(1)}_{\gamma}$ with $\gamma \in \Gamma^\nu$ are
exactly the connected components of the domain of the map $F_\eta$. On each connected component $F_\eta$ acts as a projective linear map, that is, if we write $T = (\pi_1, \lambda)$, then

$$T \in \Delta^{(1)}_\gamma \Leftrightarrow F_\eta(T) = (\pi_1, \frac{tB^{-1}_\gamma \lambda}{\|tB^{-1}_\gamma \lambda\|}).$$

For $k \in \mathbb{N}$ let us introduce the set $\Gamma^{(k),\eta}$ of those finite paths $\gamma_k$ starting and ending in $\pi_1$ which contain exactly $k$ distinct copies of the path $\eta$ and are minimal with respect to the ordering $\prec$ (we observe that by minimality all these paths end with $\eta$). The connected components of the $k$-th iterated $F^{(k)}_\eta$ of $F_\eta$ are exactly the simplices $\Delta^{(1)}_{\gamma_k}$ with $\gamma_k \in \Gamma^{(k),\eta}$. For any $\gamma_k$ in $\Gamma^{(k),\eta}$ and for any $T$ in $\Delta^{(1)}_{\gamma_k}$ we have $F^k_\eta(T) = (\pi_1, \frac{tB\lambda}{\|tB\lambda\|})$. Since $F_\eta$ is defined almost everywhere, for all $k \in \mathbb{N}$ we have

$$\Delta^{(1)}_{\gamma_k} = \bigcup_{\gamma_k \in \Gamma^{(k),\eta}} \Delta^{(1)}_{\gamma_k} \mod 0.$$

### 4.2.2. Uniform control of the speed of shrinking.

Let us come back to $\pi_0$ appearing in proposition 1.5. For a generic (i.e. recurrent) $T$ in $\Delta^{(1)}_{\pi_0}$ let us consider the sequence $(r_k)_{k\in\mathbb{N}}$ of the instants $r_k(T)$ such that the initial segment $\gamma(T, r_k)$ of $\gamma(T, \infty)$ ends with $\eta$. In particular $\tilde{T}^{(r_k)}$ belongs to $\Delta^{(1)}_{\pi_1}$, therefore let us write $\tilde{T}^{(r_k)} = (\pi_1, \lambda^{(r_k)})$. Both equations (4.3) and (4.4) in proposition 4.5 require to compare the length on an interval of $\tilde{T}^{(r_k)}$ with the quantity

$$\|\lambda^{(r_k)}\|^{-1} \varphi(\|q^{\gamma_k(\pi, \lambda)}\|),$$

where $\lambda^{(r_k)}$ denotes the length datum of the non-normalized i.e.t. $T^{(r_k)} = Q^{r_k}(T)$. What makes things complicated is that the expression above does not depend just from $k$, but also from $T$ itself. In this section we apply well known results on the ergodic theory of the Rauzy-Veech and Zorich algorithms to get a lower bound for it. We denote $O(T)$ the orbit $\{\tilde{T}^{(i)}\}_{i\in\mathbb{N}}$ of $T$ under $Q$.

**Lemma 4.8.** There exists a constant $\theta > 1$ (depending on $\eta$) such that for almost any $T \in \Delta^{(1)}_{\pi_0}$ there exists an integer $N$ such that for any $k \geq N$ we have

$$\frac{1}{\|\lambda^{(r_k)}\|} \leq \theta^k \text{ and } \|q^{\gamma(T, r_k)}\| \leq \theta^k.$$

**Proof:** For any finite path $\gamma$ in the Rauzy diagram and any $T \in \Delta^{(1)}_\gamma$ without connections let us denote with $\tilde{\gamma}(T)$ the shortest segment of the Zorich’s path generated by $T$ which begins with $\gamma$. For a generic $T$ and any $k \in \mathbb{N}$, if $r_k(T)$ are the instants defined above, we have

$$\|q^{\gamma(T, r_k)}\| \leq \|q^{\tilde{\gamma}(T, r_k)}\| \text{ and } \|\lambda^{(r_k)}\| \geq \|tB^{-1}_{\tilde{\gamma}(T, r_k)} \lambda\|.$$

For any finite path $\gamma$ in the Rauzy diagram let us call $\zeta(\gamma)$ its Zorich’s time, that is $\zeta(\gamma)$ is the number of Zorich’s elementary steps composing $\gamma$. Denoting $\nu > 1$ the maximal Lyapunov exponent of the Zorich cocycle, for almost any $T \in \Delta^{(1)}_{\pi_0}$ and any $k$ big enough we have

$$\|q^{\tilde{\gamma}(T, r_k)}\| \leq \nu^{\zeta(\tilde{\gamma}(T, r_k))} \text{ and } \frac{1}{\|tB^{-1}_{\tilde{\gamma}(T, r_k)} \lambda\|} \leq \nu^{\zeta(\tilde{\gamma}(T, r_k))}.$$
Now let us consider the Zorich map \( Z \) (see definition \[2.4\]), the positive integer function \( Z : \Delta^{(1)}(\mathbb{R}) \to \mathbb{N}^* \) defined by \( Z(T) := \min\{i \in \mathbb{N}^* : \zeta_i(T) \in \Delta^{(1)}_{\eta} \} \) and the first entering map \( \zeta_{\eta} : \Delta^{(1)}(\mathbb{R}) \to \Delta^{(1)}_{\eta} \) of the Zorich map \( Z \) into the simplex \( \Delta^{(1)}_{\eta} \), that is \( Z_{\eta}(T) := Z(\zeta_{\eta}(T)). \) Finally we consider the \( \text{Birkhoff sum} \) \( S_k Z : \Delta^{(1)}(\mathbb{R}) \to \mathbb{N}^* \) of the function \( Z \) with respect to the map \( Z_{\eta} \).

Consider the maps \( R_{\eta} : \Delta^{(1)}(\mathbb{R}) \to \Delta^{(1)}_{\eta} \) and \( Q_{\eta} : \Delta^{(1)}_{\eta} \to \Delta^{(1)}_{\eta} \) introduced in paragraph \[4.2.1\]. The instants \( r_k(T) \) defined above are such that \( \hat{T}^{(r_k)} = \hat{Q}_{\eta} \circ R^{k}_{\eta}(T). \) By definition of first entering map, the finite segment of the orbit \( O(T) \) between \( T \) and \( R^{k}_{\eta}(T) \) intersects \( \Delta^{(1)}_{\eta} \) exactly \( k \) times. Then the segment between \( R^{k}_{\eta}(T) \) and \( Q_{\eta} \circ R^{k}_{\eta}(T) \) never intersects \( \Delta^{(1)}_{\eta} \) since \( \eta \) is neat and lemma \[4.7\] applies. Finally we argue that \( Z \) is an acceleration of \( Q \), therefore for any \( k \in \mathbb{N} \) we have \( \zeta(\hat{\gamma}(T, r_k)) \leq S_k Z(T). \)

Since the Zorich’s map \( Z \) is ergodic (Zorich’s theorem in \[Z1\]), there exists a positive constant \( C \) such that \( \lim_{k \to \infty} k^{-1} S_k Z(T) = C \) for almost any \( T \in \Delta^{(1)}_{\eta} \) (the constant \( C \) being equal to \( 1/\mu(\Delta^{(1)}_{\eta}) \), where \( \mu \) is the invariant measure for the Zorich’s map). Therefore for almost any \( T \in \Delta^{(1)}_{\eta} \) there exists an integer \( N \) such that for any \( k \geq N \) we have \( S_k Z(T) \leq Ck \) (in fact this is true modulo a small change of \( C \), that is taking \( C + \epsilon \) instead of \( C \) for any choice of \( \epsilon > 0 \)). We deduce that for any \( k \in \mathbb{N} \) sufficiently big we have

\[
(4.9) \quad \zeta(\hat{\gamma}_k(\pi, \lambda)) \leq Ck.
\]

Now we put together the results of equations \[4.7\], \[4.8\] and \[4.9\] and defining \( \theta := \nu C \) we get that for almost any \( T \in \Delta^{(1)}_{\eta} \), for any \( k \in \mathbb{N} \) sufficiently big we have

\[
\|q^{\gamma(T, r_k)}\| \leq \theta^k \quad \text{and} \quad \frac{1}{\|q^{\gamma(T, r_k)}\|} \leq \theta^k.
\]

The proposition is proved. \( \square \)

It is easy to check that the sequence \( \varphi \) can be interpolated by a positive function on \( \mathbb{R}_+ \), that we still denote \( \varphi \), such that \( t \varphi(t) \) is monotone decreasing for \( t \in \mathbb{R}_+ \) (for example the interpolation \( \lfloor t \varphi(t) / t \rfloor \) works, where \( \lfloor t \rfloor \) denotes the integer part of \( t \)). If \( \theta > 1 \) is the constant appearing in lemma \[4.8\] we get a monotone decreasing sequence setting

\[
(4.10) \quad \psi_k := \theta^k \varphi(\theta^k).
\]

**Lemma 4.9.** Let \( \varphi \) be a sequence such that \( n \varphi(n) \) is decreasing monotone. If \( \sum_{n=1}^{\infty} \varphi(n) = +\infty \), then the sequence \( \psi_k \) defined by \[4.10\] has divergent series.

For a generic \( T \in \Delta^{(1)}_{\eta} \) let \( r_k \) and \( \gamma(T, r_k) \) be the instants and the path defined above. Recalling that \( n \varphi(n) \) is decreasing monotone, lemma \[4.8\] implies that for any \( k \) high enough we have

\[
(4.11) \quad \psi_k \leq \frac{1}{\|q^{\gamma(T, r_k)}\|} \varphi(\|q^{\gamma(T, r_k)}\|).
\]

**4.2.3. Sufficient shrinking target property.** In this paragraph we state a sufficient criterion to prove proposition \[4.3\]

**Definition 4.10.** Let \( (\beta, \alpha) \) be a pair with \( \pi^1_0(\alpha) > 1 \) and \( \pi^1_0(\beta) > 1 \). A \textit{pre-reference path} for \( (\beta, \alpha) \) is a neat path \( \eta : \pi_0 \to \pi_1 \) starting at \( \pi_0 \) and ending in \( \pi_1 \).
which is chosen accordingly to lemma 4.3 if \((\beta, \alpha)\) has property A, or accordingly to lemma 4.4 if \((\beta, \alpha)\) has property B.

Remark 4.11. Lemmas 4.3 and 4.4 just specify the ending part (the last arrow or the last two) of the path \(\eta\) that they provide, whereas they leave complete freedom in the choice of its beginning. This make possible to arrange an appropriate \(\eta\) which moreover starts at \(\pi_0\) and is neat. Later on (in definition 4.22) we will introduce more properties that \(\eta\) has to satisfy, so we will take advantage of the great freedom we leave in the definition of pre-reference path.

Let \(\pi_0\) and \((\beta, \alpha)\) be respectively a combinatorial datum and a pair as in proposition 4.1. Let \(\eta: \pi_0 \rightarrow \pi_1\) be a pre-reference path for \((\beta, \alpha)\) as in definition 4.10 and let \(F_\eta: \Delta_{\pi_1}^{(1)} \rightarrow \Delta_{\pi_1}^{(1)}\) be the map defined in equation 4.5.

**Proposition 4.12.** If the pair \((\beta, \alpha)\) has property A and \(\eta\) is given by lemma 4.3 then in order to get proposition 4.10 it is sufficient that for almost any \(T \in \Delta_{\pi_1}^{(1)}\) there exist infinitely many \(k \in \mathbb{N}\) such that

\[
F_\eta^k(T) \in \{ (\pi_1, \lambda) \in \Delta_{\pi_1}^{(1)}; \lambda_\alpha < \psi_k \}.
\]

If the pair \((\beta, \alpha)\) has property B, if \(V\) and \(L\) are the associated letters, and if \(\eta\) is given by lemma 4.3 then in order to get proposition 4.13 it is sufficient that for almost any \(T \in \Delta_{\pi_1}^{(1)}\) there are infinitely many \(k \in \mathbb{N}\) such that

\[
F_\eta^k(T) \in \{ (\pi_1, \lambda) \in \Delta_{\pi_1}^{(1)}; \lambda_\alpha < \min \{ \lambda_L, \psi_k \} \}.
\]

**Proof:** We recall that our aim is to get, for almost any \(T \in \Delta_{\pi_0}^{(1)}\), infinitely many triples \((\beta, \alpha, n)\) reduced for \(T\) and solutions of \(|T^n u_\beta^k - u_\alpha^k| < \varphi(n)\). For a recurrent \(T\) in \(\Delta_{\pi_0}^{(1)}\), that is for a generic element, the orbit \(\mathcal{O}(T)\) intersects \(\Delta_{\eta}^{(1)}\) infinitely many times. Let us consider such a \(T\). As above we denote \(\langle r_k \rangle_{k \in \mathbb{N}}\) the sequence of instants \(r_k(T)\) such that \(\gamma(T, r_k)\) ends with \(\eta\). The paths \(\gamma(T, r_k)\) are all distinct and totally ordered by \(\prec\) and we consider the instant \(r_1\) corresponding to the minimal one. If \(l\) denotes the number of elementary arrows composing \(\eta\), by definition we can decompose \(\gamma(T, r_1)\) as

\[
\gamma(T, r_1) = \gamma(T, r_1 - l)\eta,
\]

where the decomposition above is no more possible for any \(\gamma(T, i)\) with \(i < r_1\). In particular \(\hat{T}(r_1 - l) \in \Delta_{\eta}^{(1)}\) and \(\hat{T}(r_1) \in \Delta_{\pi_1}^{(1)}\).

Let us consider the map \(F_\eta\) and let us recall the definition of the set \(\Gamma^{(k-1), \eta}\) of those finite paths \(\gamma_{k-1}\) starting and ending at \(\pi_1\) such that \(\Delta_{\gamma_{k-1}}^{(1)}\) is a connected component of the domain of \(F_\eta^{k-1}\) (see paragraph 4.23). If we denote \(\gamma_{k-1}(\hat{T}(r_1))\) the path of \(\Gamma^{(k-1), \eta}\) such that \(\hat{T}(r_1)\) belongs to \(\Delta_{\gamma_{k-1}}^{(1)}(\hat{T}(r_1))\), we can write

\[
\gamma(T, r_k) = \gamma(T, r_1)\gamma_{k-1}(\hat{T}(r_1)),
\]

that is, for any instant \(r_k\) defined as above we have \(\hat{T}(r_k) = F_\eta^{k-1}(\hat{T}(r_1))\). Let us write it as \(\hat{T}(r_k) = (\pi_1, \hat{\lambda}(r_k))\).

Let us suppose that \((\beta, \alpha)\) has property A and \(\eta\) is given by lemma 4.3 Proposition 4.3 implies that it is sufficient to have infinitely many instants \(r_k\) such that

\[
\frac{1}{\lambda(k)} \varphi(\|q \gamma(T, k)\|),
\]
then combining it with the estimate in lemma 4.8 the first part of the propositions follows. Let us suppose that \((\beta, \alpha)\) has property B and \(\eta\) is given by lemma 4.4. In this case proposition 4.5 implies that it is sufficient to have infinitely many instants \(r_k\) such that
\[
\hat{\lambda}_V^{(r_k)} < \min\{\hat{\lambda}_V^{(r_k)}, \frac{1}{\|\lambda_\eta^{(k)}\|}\}
\]
and again the estimate in lemma 4.8 implies that the second part of the proposition follows. The proposition is proved. \(\Box\)

4.3. Refined shrinking target. In order to translate the shrinking target property in proposition 4.12 into the setting of the Borel-Cantelli lemma we need that the family of the shrinking targets is a subset of the sigma-algebra generated by the connected components of the domain of the map \(F_\eta\). We parameterize the family of targets with a positive (small) real number \(\epsilon\). If the pair \((\beta, \alpha)\) has property A the targets are the sets of the form \(\{T \in \Delta^{(1)}_{\pi_1}; \lambda_\alpha < \epsilon\}\) and if \((\beta, \alpha)\) has property B they are the sets of the form \(\{T \in \Delta^{(1)}_{\pi_1}; \lambda_V < \min\{\lambda_L, \epsilon\}\}\), in both cases the \((d - 1)\)-volume of these sets is obviously proportional to \(\epsilon\). In this paragraph we define two families of subsets of the targets above, one for each case, whose \((d - 1)\)-volume is still proportional to \(\epsilon\). In paragraph 4.4.1 we show that they belong to the sigma-algebra generated by the connected components of \(F_\eta\) (see notion of \(F_\eta\)-measurability).

4.3.1. General case. Let \(W\) be any letter in the alphabet \(A\) and let us denote with \(A_W\) the sub-alphabet \(A \setminus \{W\}\).

**Definition 4.13.** Let us consider any \(\pi \in \mathcal{R}\), any letter \(W \in A\) and any \(\epsilon > 0\).

Let \(E(\pi, W, \epsilon)\) be the set of those \(A_W\)-colored paths \(\gamma\) starting at \(\pi\) which satisfy \(q_\gamma^W > 1/\epsilon\) and are minimal with these properties with respect to the ordering \(\prec\). Then we define
\[
\Delta^{(1)}_{E(\pi, W, \epsilon)} := \bigsqcup_{\gamma \in E(\pi, W, \epsilon)} \Delta^{(1)}_{\gamma}.
\]

Let \(N(\pi, W, \epsilon)\) be the set of those paths \(\nu\) starting at \(\pi\) which satisfy \(q_\nu^W < 1/\epsilon\), end with an arrow with winner \(W\) and are minimal with these two properties with respect to \(\prec\). Then we define
\[
\Delta^{(1)}_{N(\pi, W, \epsilon)} := \bigsqcup_{\nu \in N(\pi, W, \epsilon)} \Delta^{(1)}_{\nu}.
\]

The unions in the definition are disjoint because of the minimality of paths. It is easy to see from the definition that we have \(\Delta^{(1)}_{\pi} = \Delta^{(1)}_{E(\pi, W, \epsilon)} \sqcup \Delta^{(1)}_{N(\pi, W, \epsilon)} \mod 0\).

**Lemma 4.14.** For any letter \(W \in A\), any \(\pi \in \mathcal{R}\) and any \(\epsilon > 0\) we have
\[
\Delta^{(1)}_{E(\pi, W, \epsilon)} \subset \{\lambda \in \Delta^{(1)}_{\pi}; \lambda_W < \epsilon\}.
\]

**Proof:** It is enough to prove that for any \(\gamma \in E(\pi, W, \epsilon)\) and for any \(T \in \Delta^{(1)}_{\gamma}\) we have \(\lambda_W < \epsilon\), where we write \(T = (\pi, \lambda)\). We observe that the length datum \(\lambda\) of any \(T\) in \(\Delta^{(1)}_{\gamma}\) is a convex combination of the vectors \(v_X := (1/q_X^\gamma)^t B_\gamma e_X\) for \(X \in A\). Any of these vectors has \(W\)-coordinate equal to
\[
\langle v_X, e_W \rangle = \frac{(B_\gamma e_X, e_W)}{q_X^\gamma} = \frac{(e_X, B_\gamma e_W)}{q_X^\gamma}.
\]
Since $v_W$ is the highest vertex of the simplex $\Delta_1(\gamma)$ in the $W$ direction, the scalar product above reaches its maximum for $X = W$, with value $\langle B_\gamma e_W, e_W \rangle(q_W^-)^{-1}$. By definition of $E(\pi, W, \epsilon)$, the letter $W$ wins in the path $\gamma$, so $B_\gamma e_W = e_W$ and the maximum value of the coordinate $W$ is $(q_W^-)^{-1}$, which is smaller than $\epsilon$ by definition of $E(\pi, W, \epsilon)$. The lemma is proved. \hfill $\Box$

According to theorem 5.4 in paragraph 5.2 there exists a positive constant $C$ depending only on the number of intervals $d$ such that for any $\epsilon > 0$

\begin{equation}
(4.12) \quad \text{Leb}_{d-1}(\Delta_1 E(\pi, W, \epsilon)) \geq C\epsilon.
\end{equation}

4.3.2. Pairs with property $B$. In this paragraph we consider the combinatorial datum $\pi_0$ of Theorem 4.3 and we fix a pair $(\beta, \alpha)$ with property $B$ and the associated letters $V$ and $L$ as in Definition 4.1. Then we chose a pre-reference path $\eta$ for $(\beta, \alpha)$, that is a neat path starting in $\pi_0$ and as in Lemma 4.4. We call $\pi_1$ the ending point of $\eta$. Lemma 4.4 says that the element $\pi(\beta, \alpha)$ associated to $(\beta, \alpha)$ in Definition 4.1 is in second to last position in $\eta$ and $\alpha$ wins in the last arrow of $\eta$. It follows that $\pi_1$ satisfies $\{\xi \in A; \pi_1(\xi) < \pi_1(\alpha)\} \cup \{V\} = \{\xi \in A; \pi_1(\xi) < \pi_1(\beta)\}$ and $\pi_1(V) = \pi_1(\beta)$, that is

$$\pi_1 = \left( \begin{array}{cccc}
\cdots & L & \cdots & \alpha \\
\cdots & V & \cdots & L & \beta & \cdots & \alpha
\end{array} \right).$$

Let us define the set $A' := \{\xi \in A; \pi_1(\xi) < \pi_1(\alpha)\}$ and let us call $a$ the number of elements of $A'$. Let us consider the essential $(A \setminus A')$-decorated Rauzy class $R_* = R_* (\pi_1)$ which contains $\pi_1$ and the associated decorated Rauzy class $R^{\text{red}} = R^{\text{red}} (\pi_1)$. Since $\pi_1$ is an essential element of $R_*$ then there exists a good letter for $\pi_1$, furthermore there exists only one good letter and it is evident that it is $V$ (the notions in the formalism for the reduction of Rauzy classes are described in paragraph 2.2).

Remark 4.15. Let us consider any path $\gamma \in E(\pi_1, V, \epsilon)$ (Definition 4.13). Since by definition $V$ never wins in $\gamma$, then $\gamma$ is $A'$-separated.

Lemma 4.16. Let $\hat{\gamma} : \pi_1 \rightarrow \hat{\pi}$ be any $A'$-separated path starting at $\pi_1$ and ending in $\hat{\pi}$. Then for any letter $x \in A' \cup \{\alpha\}$ we have:

\begin{equation}
(4.13) \quad \hat{\pi}^t(\xi) = \pi_1(\xi).
\end{equation}

Moreover if $\hat{\gamma} : \pi_1 \rightarrow \hat{\pi}$ is maximal $A'$-separated, then its ending point $\hat{\pi}$ satisfies $\hat{\pi}^b(L) = d$ and we have:

\begin{equation}
(4.14) \quad B_{\hat{\gamma} e V} = \sum_{\xi \in A \setminus A'} e_{\xi}.
\end{equation}

Proof: If the first statement of the lemma is not true there exists an $A'$-separated path $\hat{\gamma} : \pi_1 \rightarrow \hat{\pi}$ starting at $\pi_1$ and ending in $\hat{\pi}$ such that equation (4.13) does not holds. We can suppose that $\hat{\gamma}$ is minimal with this property, that is $\hat{\pi}$ is the first element in $\hat{\gamma}$ where condition (4.13) does not hold, therefore there exists a letter $\xi \in A' \cup \{\alpha\}$ such that $\hat{\pi}^t(\xi) = \pi_1(\xi) + 1$. Let $\gamma_{\text{last}} : \pi \rightarrow \hat{\pi}$ be the last arrow in $\hat{\gamma}$ and let $W$ be its winner. $\gamma_{\text{last}}$ has to be a bottom arrow and its starting point $\pi$ has to satisfy $\pi^t(W) < \pi^t(\xi)$. Since $\hat{\gamma}$ is $A'$-separated then $W \in A \setminus A'$, therefore $\pi$ still doesn’t satisfy condition (4.13), which is absurd by minimality of $\hat{\gamma}$.

Now let us consider a maximal $A'$-separated path $\hat{\gamma} : \pi_1 \rightarrow \hat{\pi}$ starting at $\pi_1$ and ending in $\hat{\pi}$. By maximality of $\hat{\gamma}$ there exists a letter $\xi \in A'$ such that $\hat{\pi}^t(\xi) = d$
or $\hat{\pi}^b(\xi) = d$. By the first part of the lemma the only possibility is $\hat{\pi}^b(\xi) = d$. Moreover $L$ is the rightmost letter of $A'$ in the permutation $\pi_1$ and in order to invert its position with respect to any other letter of $\xi \in A'$ it has to arrive in last position in the bottom line at least one time. Since $\hat{\gamma}$ is $A'$-separated this can happen only at its ending point $\hat{\pi}$, therefore we have $\hat{\pi}^b(L) = d$.

To prove equation (4.14) let us decompose $\hat{\gamma}$ as $\hat{\gamma} = \gamma^{(1)} \gamma^{(2)} \cdots \gamma^{(m)} \gamma m$, where $m = d - a - 1$ and for any $i = 1, \ldots, m$ the sub-path $\gamma^{(i)}$ is not drifting and $\gamma_i$ is a drifting arrow. Let us write $\hat{\gamma}^{(i)} := \gamma^{(1)} \gamma^{(2)} \cdots \gamma^{(i)}$. For any $i = 1, \ldots, m$ let us also call $\alpha_i$ and $\beta_i$ respectively the winner and the loser of the arrow $\gamma_i$ and $\pi^{(i)}_s$ and $\pi^{(i)}_l$ respectively the starting and ending point of $\gamma_i$. Since $V$ is the only good letter for $\pi$ we have $\alpha_1 = V$. Then we have

$$B_{\pi^{(i)}_s} e_V = e_V \text{ and } B_{\pi^{(i)}_l} e_V = e_V + e_{\beta_i}$$

and the only good letters for $\pi^{(i)}_s$ are $V$ and $\beta_1$. Let us put $I_0 := \{V\}$. Let us fix $k \leq m$ and suppose by induction that for any $1 \leq i < k$ we have that there exists a subset $I_i \subset A \setminus A'$ with $i + 1$ elements and such that

$$B_{\pi^{(i)}_s} (e_V) = \sum_{\xi \in I_{i-1}} e_{\xi} \text{ and } B_{\pi^{(i)}_l} (e_V) = \sum_{\xi \in I_i} e_{\xi}$$

and such that the good letters for $\pi^{(i)}_s$ are exactly the letters of $I_i$. We observe that the base of the induction hypothesis is satisfied by $I_0$. Let us consider the path $\hat{\gamma}^{(k)} = \hat{\gamma}^{(k-1)} \gamma_{k-1} \gamma^{(k)}$. By the induction hypothesis $B_{\gamma^{(k-1)} \gamma_{k-1}} (e_V) = \sum_{\xi \in I_{k-1}} e_{\xi}$ and the good letters for $\pi^{(k-1)}_s$ are exactly the letters of $I_{k-1}$. Since there is no drift for any arrow in $\gamma^{(k)}$ then the winner of such arrows is never in $I_{k-1}$, therefore we have $B_{\gamma^{(k)}} (e_V) = \sum_{\xi \in I_{k-1}} e_{\xi}$. Now let us consider the $k$-th drifting arrow $\gamma_k$, its winner $\alpha_k$ and its loser $\beta_k$. The first part of the lemma says that all the drifting arrows of $\hat{\gamma}$ are top arrows, therefore $\pi^{(k)}_s(\beta_k) = d$ and $\beta_k$ is not an element of $I_{k-1}$. Moreover since $\gamma_k$ is drifting we have $\pi^{(k)}_s(\alpha_k) = d$ and $\pi^{(k)}_s(\alpha_k) < d_k(\pi^{(k)}_s)$. By the first part of the lemma $\pi^{(k)}_s(\alpha_k)$ is drifting we have $\pi^{(k)}_s(\alpha_k) = d$ and $\pi^{(k)}_s(\alpha_k) < d_k(\pi^{(k)}_s)$ (see paragraph 4.1.1 for the notation), therefore $\pi^{(k)}_s(\beta_k) = \pi^{(k)}_s(\alpha_k) + 1$, that is $\beta_k$ moves in good position for $\pi^{(k)}_s$. Putting $I_k := I_{k-1} \cup \{\beta_k\}$ the inductive step follows. The lemma is proved.

We fix a path $\gamma \in E(\pi_1, V, \epsilon)$, according to remark 4.1.13 is $A'$-separated. We introduce the set $\Gamma^\alpha_{\pi_1}$ of those maximal $A'$-separated paths $\hat{\gamma}$ which start at $\pi_1$, end in $\hat{\pi}$ and such that $\gamma \prec \hat{\gamma}$ (that is $\hat{\gamma}$ begins with $\gamma$). For any $\hat{\gamma} \in \Gamma^\alpha_{\pi_1}$ the ending point $\hat{\pi}$ of $\hat{\gamma}$ belongs to $\mathcal{R}_\alpha$ and thanks to lemma 4.1.10 it satisfies $\hat{\pi}^b(L) = d$, therefore there is a bottom arrow starting at $\hat{\pi}$ which has $L$ as winner. Let us call $\nu(L, a)$ the path starting at $\hat{\pi}$ where $L$ wins exactly $d - a$ times. Let us call $\Gamma(L, a)$ the set of path starting at $\hat{\pi}$ where $L$ wins less than $d - a$ times and then it loses one time.

**Definition 4.17.** For any $\epsilon > 0$ we define

$$\mathcal{E}(\pi_1, V, \epsilon) := \bigcup_{\gamma \in E(\pi_1, V, \epsilon)} \bigcup_{\hat{\gamma} \in \Gamma^\alpha_{\pi_1}} \hat{\gamma} \nu(L, a),$$

$$\mathcal{N}(\pi_1, V, \epsilon) := N(\pi_1, V, \epsilon) \cup \left( \bigcup_{\gamma \in E(\pi_1, V, \epsilon)} \bigcup_{\hat{\gamma} \in \Gamma^\alpha_{\pi_1}} \bigcup_{\hat{\zeta} \in \nu(L, a)} \hat{\gamma} \hat{\zeta} \right)$$.
Then we set
\[ \Delta^{(1)}_{E(\pi_1, V, \epsilon)} := \bigcup_{\gamma' \in E(\pi_1, V, \epsilon)} \Delta^{(1)}_{\gamma'} \quad \text{and} \quad \Delta^{(1)}_{N(\pi_1, V, \epsilon)} := \bigcup_{\nu' \in N(\pi_1, V, \epsilon)} \Delta^{(1)}_{\nu'} . \]

**Lemma 4.18.** Let \((\beta, \alpha)\) be any pair satisfying property B, let \(V \) and \(L\) be the associated letters as in definition \[4.1\] and let \(\eta : \pi_0 \to \pi_1\) be a pre-reference path for \((\beta, \alpha)\) given by lemma \[4.4\]. For any \(\epsilon > 0\)
\[ \Delta^{(1)}_{E(\pi_1, V, \epsilon)} \subset \{ \lambda \in \Delta^{(1)}_\pi ; \lambda_V < \min\{\lambda_L, \epsilon\} \} . \]

**Proof:** It is enough to prove that for any \(\gamma' \in E(\pi_1, V, \epsilon)\) the length datum \(\lambda\) of any \(T \in \Delta^{(1)}_\gamma\) satisfies \(\lambda_V < \min\{\lambda_L, \epsilon\}\). Since any \(\gamma' \in E(\pi_1, V, \epsilon)\) begins with a path \(\gamma \in E(\pi, V, \epsilon)\) then lemma \[4.14\] implies that for such \(\lambda\) we have \(\lambda_V < \epsilon\), therefore it is enough to prove that \(\lambda_V < \lambda_L\).

Let us consider any \(\gamma' \in E(\pi_1, V, \epsilon)\) and decompose it as \(\gamma' = \hat{\gamma} \nu(L, a)\), where \(\hat{\gamma} \in \Gamma^{d}_\gamma\) for some \(\gamma \in E(\pi_1, V, \epsilon)\). We have
\[ B_{\gamma'} e_V = B_{\nu(L, a)} (\sum_{\xi \in A \setminus A'} e_\xi) = \sum_{\xi \in A \setminus A'} e_\xi , \]
where the first equality follows from equation \[4.14\] and the second from the fact that the winner in \(\nu(L, a)\) is always \(L\) and this letter is not contained in \(A \setminus A'\). On the other hand we have
\[ B_{\gamma'} e_L = B_{\nu(L, a)} e_L = e_L + (\sum_{\xi \in A \setminus A'} e_\xi) . \]

Here the first equality follows since \(\hat{\gamma}\) is \(A'\)-separated, therefore \(L\) newer wins in it, the second follows since equation \[4.13\] implies that the ending point \(\hat{\pi}\) of \(\hat{\gamma}\) satisfies \(\{ \xi \in A ; \hat{\pi}(\xi) > a \} = A \setminus A'\) and \(\nu(L, a)\) is the concatenation of exactly \(d-a\) bottom arrows with winner \(L\). Therefore we have \(B_{\gamma'}(e_L - e_W) = e_L\). The lemma follows from the the following general argument:

For any pair of letters \(W\) and \(L\) in \(A\) and any \(\pi \in R\), if we have a path \(\gamma\) starting at \(\pi\) and satisfying \(B_{\gamma'}(e_L - e_W) \in \mathbb{N}^d\), then for any \((\pi, \lambda) \in \Delta^{(1)}_\gamma \subset \Delta^{(1)}_\pi\) we have \(\lambda_W \leq \lambda_L\).

Reasoning in the same way as in lemma \[4.14\] in order to have that \(\lambda_W \leq \lambda_L\) for any \(\lambda \in \Delta^{(1)}_\gamma\) it is equivalent to have that the condition is true for the vertices \(v_X\) of the simplex \(\Delta^{(1)}_\gamma\). The condition is \(\langle v_X, e_L - e_W \rangle \geq 0\) for any \(X \in A\), that is
\[ \langle ^t B_{\gamma} e_X, e_L - e_W \rangle = \langle e_X, ^t B_{\gamma} (e_L - e_W) \rangle \geq 0 \]
for any \(X \in A\), that is \(B_{\gamma}(e_L - e_W) \in \mathbb{N}^d\). The proof of the argument is complete and therefore lemma \[4.18\] is proved. \(\square\)

**Proposition 4.19.** There exists a positive constant \(C' > 0\) depending only from \(d\) such that for any pair of letters \((\beta, \alpha)\) satisfying property B and for any \(\epsilon > 0\) the family of paths \(E(\pi_1, V, \epsilon)\) introduced in definition \[4.17\] satisfies
\[ \text{Leb}(\Delta^{(1)}_{E(\pi_1, V, \epsilon)}) \geq C' \epsilon . \]

**Proof:** Let us consider any \(\gamma \in E(\pi_1, V, \epsilon)\) and the set \(E(\pi_1, V, \epsilon; \gamma)\) of those paths \(\gamma' \in E(\pi_1, V, \epsilon)\) which begins with \(\gamma\). We prove that
\[ P_\gamma (E(\pi_1, V, \epsilon; \gamma)) \geq \frac{1}{2^{d-a}}. \]
then, combining this estimate with equation (4.12) in paragraph 2.3 we get the proposition with $C' := C2^{-(d-a)}$, where $C$ is the constant appearing in equation (4.12). We recall that for a fixed $\gamma \in E(\pi_1, V, \epsilon)$ any $\gamma' \in E(\pi_1, V, \epsilon|\gamma)$ is decomposed as $\gamma' = \tilde{\gamma}\nu(L, a)$, where $\tilde{\gamma} \in \Gamma_{\hat{\gamma}}^\prime$. We have

$$P_\gamma(E(\pi_1, V, \epsilon|\gamma)) = \sum_{\hat{\gamma} \in \Gamma_{\hat{\gamma}}^\prime} P_\gamma(\Delta_{\hat{\gamma}\nu(L, a)}^{(1)}) =$$

$$\sum_{\hat{\gamma} \in \Gamma_{\hat{\gamma}}^\prime} P_\gamma(\Delta_{\hat{\gamma}}^{(1)})P_\gamma(\Delta_{\hat{\gamma}\nu(L, a)}^{(1)}) \geq \inf_{\hat{\gamma} \in \Gamma_{\hat{\gamma}}^\prime} P_\gamma(\Delta_{\hat{\gamma}\nu(L, a)}^{(1)})$$

since $\{\Delta_{\hat{\gamma}}^{(1)}; \hat{\gamma} \in \Gamma_{\hat{\gamma}}^\prime\}$ form a partition mod 0 of $\Delta_{\hat{\gamma}}^{(1)}$. For any $\hat{\gamma} \in \Gamma_{\hat{\gamma}}^\prime$ the path $\nu(L, a)$ is concatenation of $d-a$ bottom arrows starting at the ending point $\hat{\pi}$ of $\hat{\gamma}$. Any of the $d-a$ letters in $A \setminus A'$ is the loser of exactly one of the arrows in $\nu(L, a)$, on the other hand the winner is always the letter $L$. It follows that for the path $\gamma' = \tilde{\gamma}\nu(L, a)$ we have

$$q_\xi' = q_\xi + q_{\hat{\pi}} \text{ for } \xi \in A \setminus A' \text{ and } q_\xi' = q_{\hat{\pi}} \text{ for } \xi \in A'.$$

Since the trajectories $\hat{\gamma} \in \Gamma_{\hat{\gamma}}^\prime$ are $\{L\}$-separated we have $q_{\hat{\pi}} = 1$, thus it follows that $q_\xi' = q_{\hat{\pi}} + 1 \leq 2q_{\hat{\pi}}$ for any $\xi \in A \setminus A'$. From equation (2.3) in paragraph 2.3 we have

$$P_\gamma(\Delta_{\nu(L, a)}^{(1)}) = \prod_{\xi \in A} \frac{q_\xi}{2q_{\hat{\pi}}},$$

so we get $P_\gamma(\Delta_{\nu(L, a)}^{(1)}) > 2^{-(d-a)}$, which implies $P_\gamma(E(\pi_1, V, \epsilon|\gamma)) \geq 2^{-(d-a)}$. The proposition is proved. 

4.4. Borel-Cantelli formulation and end of the proof. In this paragraph we complete the proof of proposition 4.3. We recall that proposition 4.12 gives a sufficient shrinking target criterion in terms of the map $F_{\eta_\gamma} : \Delta_{\gamma_\eta}^{(1)} \to \Delta_{\gamma_\eta}^{(1)}$, where $\eta_\gamma$ is a pre-reference path starting at $\pi_0$, ending in $\pi_1$ and chosen accordingly to definition 4.10. In this paragraph we state and prove a stronger sufficient criterion, namely proposition 4.24 which implies that the sufficient condition in proposition 4.12 is satisfied and therefore completes the proof of proposition 4.3. Proposition 4.24 corresponds to the second half of the Borel-Cantelli lemma (the divergent part), its proof requires some more properties of $F_{\eta_\gamma}$, or equivalently of the pre-reference path $\eta_\gamma$. We first state the properties that we need, then we prove that they are compatible with the properties of a pre-reference path described in definition 4.10. This will lead to the notion of reference path (definition 4.22). Finally we state and prove the sufficient Borel-Cantelli-type criterion.

4.4.1. $F_{\eta_\gamma}$-measurability. Let $\gamma$ be any neat path ending in $\pi_1$ and let us consider the map $F_{\eta_\gamma}$ defined in equation (4.15). We say that a finite path $\nu$ starting at $\pi_1$ is $F_{\eta_\gamma}$-measurable if the simplex $\Delta_{\gamma_\eta}^{(1)}$ is measurable with respect to the $\sigma$-algebra generated by the connected components of the domain of the map $F_{\eta_\gamma}$, that is the $\sigma$-algebra whose atoms are the simplices $\Delta_{\gamma_\eta}^{(1)}$ for $\gamma \in \Gamma_{\eta_\gamma}$. It is easy to see that a finite path $\nu$ ending in $\pi_1$ is $F_{\eta_\gamma}$-measurable if and only if it does not contain $\eta$ as a sub-path. If $\nu$ is $F_{\eta_\gamma}$-measurable we set

$$\Gamma_{\nu}^\eta := \{\gamma \in \Gamma_{\eta}; \nu \prec \gamma\}.$$
A path $\gamma$ belongs to $\Gamma_\eta^\alpha$ if and only if $\Delta_{\nu|\Delta_\eta}^{(1)}$ is a connected component of the restriction of $\mathcal{F}_{\eta|\Delta_\eta}$. More generally, if $\nu$ is a path starting at $\pi_1$ which does not contain $k$ distinct copies of the path $\eta$, then we write

$$\Gamma_{\nu|\Delta_\eta}^{(k)} := \{ \gamma_k \in \Gamma_\eta; \nu \prec \gamma_k \}.$$ 

A path $\gamma_k$ belongs to $\Gamma_{\nu|\Delta_\eta}^{(k)}$ if and only if $\Delta_{\nu\gamma_k}^{(1)}$ is a connected component of the restriction of $\mathcal{F}_{\eta|\Delta_\eta}^{(1)}$.

**Lemma 4.20.** If $(\beta, \alpha)$ has property A a pre-reference path $\eta$ can be chosen as in lemma 4.3 and in a way such that for any $\epsilon > 0$ all the paths in the families $\mathcal{E}(\pi_1, \alpha, \epsilon)$ and $\mathcal{N}(\pi_1, \alpha, \epsilon)$ are $\mathcal{F}_{\eta|\Delta_\eta}$-measurable.

If $(\beta, \alpha)$ has property B a pre-reference path $\eta$ can be chosen as in lemma 4.3 and in a way such that for any $\epsilon > 0$ all the paths in the families $\mathcal{E}(\pi_1, V, \epsilon)$ and $\mathcal{N}(\pi_1, V, \epsilon)$ are $\mathcal{F}_{\eta|\Delta_\eta}$-measurable.

**Proof:** For any $\epsilon > 0$ the letter $\alpha$ never wins in paths $\gamma \in E(\pi_1, \alpha, \epsilon)$ and wins just one time in paths $\nu \in E(\pi_1, \alpha, \epsilon)$. We have the required property if and only if any of these $\gamma$ and $\nu$ does not contain $\eta$, therefore it is enough to choose a pre-reference path $\eta : \pi_0 \to \pi_1$ which contains at least two arrows whose winner is $\alpha$. This is possible as observed in remark 4.11.

Assume now that the pair $(\beta, \alpha)$ has the property B. Let us recall the definition of the sub-alphabet $A'$ of $A$ given in paragraph 4.3.2 and let us denote with $a$ the number of its elements. In order to have the required property it is enough to choose a pre-reference path $\eta : \pi_0 \to \pi_1$ containing at least $d - a + 2$ arrows with winner $V$ and this is possible as observed in remark 4.11. Such an $\eta$ cannot be a sub-path of any path $\gamma \in \mathcal{E}(\pi_1, V, \epsilon)$ or of any $\nu \in \mathcal{N}(\pi_1, \alpha, \epsilon)$, this because the letter $V$ wins at most $d - a + 1$ times in paths in these sets. The lemma is proved.

\(\square\)

4.4.2. Equilibrated paths. For any real number $M$ with $M > 1$ we say that a vector $q \in \mathbb{R}^A_+$ is $M$-equilibrated if $q_X < Mq_Y$ for any $X, Y \in A$. We say that a finite Rauzy path $\eta$ is a $M$-equilibrated path if $B_\eta q$ is $M$-equilibrated vector for any $q \in \mathbb{R}^A_+$.

**Lemma 4.21.** For any $M > 1$, if $\eta$ is a $M$-equilibrated path and $\gamma$ is any finite Rauzy path ending with $\eta$, then we have

\[ \| \frac{dP_{\gamma}}{d\text{Leb}_{d-1}} \| \leq M^d \quad \text{and} \quad \| \frac{d\text{Leb}_{d-1}}{dP_{\gamma}} \| \leq M^d. \]

**Proof:** Let $\gamma$ be any finite path in the Rauzy diagram ending at the element $\pi$ and consider the probability measure $P_{\gamma}$. Equation (2.5) in paragraph 2.3 implies that for any Rauzy path $\nu$ starting at the element $\pi$ where $\gamma$ ends we have

\[ \frac{P_{\gamma}(\Delta_{\nu|\Delta_\eta}^{(1)})}{\text{Leb}_{d-1}(\Delta_{\nu|\Delta_\eta}^{(1)})} = \frac{\Pi_{\alpha \in A} q_{\alpha}^{\gamma} q_{\alpha}^{\nu}}{\Pi_{\alpha \in A} q_{\alpha}^{\gamma}}. \]

Since $\gamma$ ends with $\eta$ we can write $\gamma = \gamma' \eta$ for some $\gamma'$ and therefore $q_{\gamma} = B_\eta q_{\gamma'}$. Since $\eta$ is an $M$-equilibrated path it follows that $q_{\gamma}$ is an $M$-equilibrated vector. Moreover, since $q_{\gamma'} = B_{\eta} q_{\gamma}$, for any $\alpha \in A$ we have $M^{-1} q_{\alpha} \leq q_{\alpha}^{\gamma'} \leq Mq_{\alpha}$, and therefore

\[ \frac{1}{M^d} \leq \frac{P_{\gamma}(\Delta_{\nu|\Delta_\eta}^{(1)})}{\text{Leb}_{d-1}(\Delta_{\nu|\Delta_\eta}^{(1)})} \leq M^d. \]
When $\nu$ varies among all the finite Rauzy paths starting at $\pi$, the sub-simplices $\Delta^{(1)}_{\nu}$ form a base of the Borel $\sigma$-algebra of $\Delta^{(1)}_{\pi}$, therefore the required estimation follows. The lemma is proved. □

4.4.3. Choice of a reference path.

**Definition 4.22.** Let $(\beta, \alpha)$ be a pair with $\pi^0_0(\alpha) > 1$ and $\pi^0_0(\beta) > 1$. A reference path for $(\beta, \alpha)$ is a pre-reference path $\eta : \pi_0 \to \pi_1$ as in definition 4.11 which satisfies the supplementary conditions in lemma 4.30 and which is $M$-equilibrated for some $M > 1$.

**Lemma 4.23.** For any pair $(\beta, \alpha)$ there exists a reference path $\eta$ for $(\beta, \alpha)$ as in definition 4.22.

**Proof:** We consider first a pre-reference path $\eta$ having the property prescribed by lemma 4.20 according to the two cases of a pair $(\beta, \alpha)$ with property A or property B. Then we require the extra condition that $\eta$ contains all the arrows of the Rauzy diagram $D$ of $R$. According to remark 4.11 this is compatible both with the definition of pre-reference path and with lemma 4.20. Since in any Rauzy diagram any letter wins against any other, then for such a $\eta$ all the entries of the matrix $B_\eta$ are positive. Finally we observe that a sufficient condition on $\eta$ for being $M$-equilibrated is that all the entries of the matrix $B_\eta$ are positive and that its norm $\|B_\eta\|$ is less that $M$. Then for the value $M := \|B_\eta\|$ the path $\eta$ is trivially $M$-equilibrated too. The lemma is proved □

4.4.4. Borel-Cantelli formulation. Let $\{A_k\}_{k \in \mathbb{N}}$ be any countable family of sub-sets of some set $A$. We put

$$\limsup_k A_k := \bigcap_{k \geq 0} \bigcup_{i \geq k} A_i.$$ 

Let us consider the map $F_\eta : \Delta^{(1)}_{\pi_1} \to \Delta^{(1)}_{\pi_1}$ defined by equation (4.15) and the sequence $\psi_k$ introduced in equation (4.10).

Let $(\beta, \alpha)$ be a pair with property A and let $\eta$ be a reference path for $(\beta, \alpha)$ as in definition 4.22 chosen accordingly to lemma 4.3. For any $k \in \mathbb{N}$ let us consider the set $\Delta^{(1)}_{E(\pi_1, \alpha, \psi_k)}$ introduced in definition 4.13. Proposition 4.12 combined with lemma 4.14 implies that it is enough to prove that for almost any $T \in \Delta^{(1)}_{\pi_1}$ the $k$-th iterated $F_\eta^k(T)$ belongs to $\Delta^{(1)}_{E(\pi_1, \alpha, \psi_k)}$ for infinitely many $k$, which is equivalent to

$$T \in \limsup_k F_{\eta}^{-k} \Delta^{(1)}_{E(\pi_1, \alpha, \psi_k)}.$$ 

Similarly let $(\beta, \alpha)$ be a pair with property B, let $V$ be the associated letter and let $\eta$ be a reference path for $(\beta, \alpha)$ as in definition 4.22 chosen accordingly to lemma 4.4. For any $k \in \mathbb{N}$ we consider the set $\Delta^{(1)}_{E(\pi_1, V, \psi_k)}$ introduced in definition 4.17. Proposition 4.12 combined with lemma 4.18 implies that it is enough to prove that for almost any $T \in \Delta^{(1)}_{\pi_1}$ the $k$-th iterated $F_\eta^k(T)$ belongs to $\Delta^{(1)}_{E(\pi_1, V, \psi_k)}$ for infinitely many $k$, which is equivalent to

$$T \in \limsup_k F_{\eta}^{-k} \Delta^{(1)}_{E(\pi_1, V, \psi_k)}.$$ 

It follows that in order to prove proposition 1.5 it is enough to prove:
 Proposition 4.24. If the pair \((\beta, \alpha)\) has property \(A\) and \(\eta\) is a reference path for 
\((\beta, \alpha)\) chosen accordingly to lemma 4.3 then 
\[
\text{Leb}_{d-1} \left( \limsup_{k} \sum_{\pi} F_{\eta}^{-k} (\Delta(1))_{E(\pi, \alpha, \psi_k)} \right) = 1.
\]

If the pair \((\beta, \alpha)\) has property \(B\), if \(V\) is the associated letter and \(\eta\) is a reference path for 
\((\beta, \alpha)\) chosen accordingly to lemma 4.4 then 
\[
\text{Leb}_{d-1} \left( \limsup_{k} \sum_{\pi} F_{\eta}^{-k} (\Delta(1))_{E(\pi, V, \psi_k)} \right) = 1.
\]

4.4.5. Proof of proposition 4.24. From now on the proof is the same both for pairs 
with property \(A\) and pairs with property \(B\). Therefore, in order to simplify our 
notation, for any \(k \in \mathbb{N}\) we just write \(E_k\) instead of \(E(\pi_1, \alpha, \psi_k)\) 
(if the pair \((\beta, \alpha)\) has property \(A\)) or instead of \(E(\pi_1, V, \psi_k)\) 
(if the pair \((\beta, \alpha)\) has property \(B\)). Similarly 
we write \(N_k\) instead of \(N(\pi_1, \alpha, \psi_k)\) (if the pair \((\beta, \alpha)\) property \(B\)) or instead of \(N(\pi_1, V, \psi_k)\) 
(if the pair \((\beta, \alpha)\) has property \(B\)). We also introduce the notation: 
\[
\Delta^{(1)}(E_k) := \bigcup_{\gamma \in E_k} \Delta^{(1)} \quad \text{and} \quad \Delta^{(1)}(N_k) := \bigcup_{\gamma \in N_k} \Delta^{(1)}.
\]

For any \(k \in \mathbb{N}\) we have \(\Delta^{(1)}_{\gamma_k} = \Delta^{(1)}(E_k) \cup \Delta^{(1)}(N_k) \mod 0\), 
which is equivalent to \(\Delta^{(1)}_{\gamma_k} = F_{\eta}^{-k} \Delta^{(1)}(E_k) \cup F_{\eta}^{-k} \Delta^{(1)}(N_k) \mod 0\).

Lemma 4.25. There exists a constant \(C > 0\), depending only on the number of 
intervals \(d\), such that for any \(k \in \mathbb{N}\) and for any \(\gamma_k \in \Gamma(1)\) we have 
\[
P_{\gamma_k}(\Delta^{(1)}(N_k)) \leq (1 - C\psi_k).
\]

Proof: Since \(\Delta^{(1)}_{\gamma_k} = \Delta^{(1)}(E_k) \cup \Delta^{(1)}(N_k) \mod 0\) we first observe 
that the statement is equivalent to 
\[
P_{\gamma_k}(\Delta^{(1)}(E_k)) \geq C\psi_k.
\]

Let us consider \(M > 1\) such that the reference path \(\eta\) is \(M\)-equilibrated. By the 
definition of the map \(F_{\eta}\) (in equation 4.3) any path \(\gamma_k \in \Gamma(1)\) ends with \(\eta\), 
therefore lemma 4.21 applies and we get \(dP_{\gamma_k} \leq M^d\) and \(d\text{Leb}_{d-1} \leq M^d\).

If we are in the case of a pair \((\beta, \alpha)\) with property \(A\), then it follows directly from 
equation 4.1.12 that \(\text{Leb}_{d-1}(\Delta^{(1)}(E_k)) \geq C\psi_k\), where \(C\) is the constant appearing in 
the equation. Therefore combining this estimate with the estimate on the distortion 
of \(P_{\gamma_k}\) we get 
\[
P_{\gamma_k}(\Delta^{(1)}(E_k)) \geq CM^d\psi_k.
\]

On the other hand, if we are in the case of a pair \((\beta, \alpha)\) with property \(B\), then 
we apply proposition 4.19 which implies that \(\text{Leb}_{d-1}(\Delta^{(1)}(E_k)) \geq C'\psi_k\), where \(C'\) 
is the constant appearing in the proposition. We combine again this estimate with 
the estimate on the distortion of \(P_{\gamma_k}\) we get 
\[
P_{\gamma_k}(\Delta^{(1)}(E_k)) \geq C'M^d\psi_k.
\]

Then we re-define \(C\) as \(CM^d\) or \(C'M^d\) according to the two cases. The lemma is 
proved.

Let us define \(C_k := F_{\eta}^{-k} \Delta^{(1)}(N_k)\). The statement in proposition 4.24 can 
be written as \(\text{Leb}_{d-1}(\limsup_{k} F_{\eta}^{-k} E_k) = 1\), which is equivalent to 
\[
\text{Leb}_{d-1}\left(\bigcup_{k \geq 0} \bigcap_{i \geq k} C_k\right) = 0.
\]
Lemma 4.26. Let $C$ be the constant appearing in Lemma 4.25. For any pair of integers $m, n$ with $m \geq n$ we have:

$$\text{Leb}_{d-1}(\bigcap_{k=n}^{m} C_k) \leq \prod_{k=n}^{m} (1 - C\psi_k).$$

Proof: During the proof we write simply Leb instead of Leb$_{d-1}$. For any $n \in \mathbb{N}$ we have

$$C_n = \bigsqcup_{\gamma_n \in \Gamma^{(n)}, \nu_n \in \mathcal{N}_n} \Delta^{(1)}_{\gamma_n \nu_n},$$

therefore

$$\text{Leb}(C_n) = \sum_{\gamma_n \in \Gamma^{(n)}, \nu_n \in \mathcal{N}_n} \text{Leb}(\Delta^{(1)}_{\gamma_n \nu_n}) = \sum_{\gamma_n \in \Gamma^{(n)}, \nu_n \in \mathcal{N}_n} \sum \text{Leb}(\Delta^{(1)}_{\gamma_n}) P_{\gamma_n}(\Delta^{(1)}_{\nu_n}) = \sum_{\gamma_n \in \Gamma^{(n)}, \eta} \text{Leb}(\Delta^{(1)}_{\gamma_n}) P_{\gamma_n}(\Delta^{(1)}(\mathcal{N}_n)) \leq (1 - C\psi_n) \sum_{\gamma_n \in \Gamma^{(n)}, \eta} \text{Leb}(\Delta^{(1)}_{\gamma_n}) = (1 - C\psi_n)$$

thanks to Lemma 4.25. According to Lemma 4.20 any $\nu_n \in \mathcal{N}_n$ is $\mathcal{F}_\eta$-measurable, this means that for any $\nu_n \in \mathcal{N}_n$ we have

$$\Delta^{(1)}_{\gamma_n \nu_n} = \bigcup_{\gamma_{n+1} \in \Gamma_{\nu_n}^{(n+1), \eta}} \Delta^{(1)}_{\gamma_{n+1}}.$$  

where we use the notation of paragraph 4.4.1. For any $\gamma_n \in \Gamma^{(n), \eta}$, any $\nu_n \in \mathcal{N}_n$ and any $\gamma_1 \in \Gamma_{\nu_n}^\eta$ we define the path $\gamma_{n+1} = \gamma_n \nu_n \gamma_1$, which corresponds to a connected component $\Delta^{(1)}_{\gamma_{n+1}}$ of the domain of $\mathcal{F}_\eta^{(n+1)} \Delta^{(1)}_{\gamma_n \nu_n}$. The decomposition above is equivalent to

$$\Delta^{(1)}_{\gamma_n \nu_{n+1}} = \bigcup_{\gamma_{n+1} \in \Gamma_{\nu_n}^{(n+1), \eta}} \Delta^{(1)}_{\gamma_{n+1}}.$$  

We proved the lemma for any $n \in \mathbb{N}$ and for $m = n$. The proof goes on by induction over $m \geq n$. Let us suppose that for $m > n$ we have $\text{Leb}(\bigcap_{k=n}^{m} C_k) \leq \prod_{k=n}^{m} (1 - C\psi_k)$ and moreover that the intersection can be written as

$$\bigcap_{k=n}^{m} C_k = \bigcup_{\gamma_n \in \Gamma^{(n)}, \nu_n \in \mathcal{N}_n} \cdots \bigcup_{\gamma_{m-1} \in \Gamma^{(m-1)}, \nu_{m-1} \in \mathcal{N}_{m-1}} \bigcup_{\nu_m \in \mathcal{N}_m} \Delta^{(1)}_{\gamma_m \nu_m}.$$  

We observe that we yet proved equation (1.15) for any $n \in \mathbb{N}$ and $m = n$. Then we argue again that Lemma 4.20 implies that any $\nu_m \in \mathcal{N}_m$ is $\mathcal{F}_\eta$-measurable, therefore for any such $\nu_m$ we have

$$\Delta^{(1)}_{\gamma_m \nu_m} = \bigcup_{\gamma_{m+1} \in \Gamma_{\nu_m}^{(m+1), \eta}} \Delta^{(1)}_{\gamma_{m+1}}.$$  

For any $\gamma_m$ in the union in equation (4.15), any $\nu_m \in \mathcal{N}_m$ and any $\gamma_1 \in \Gamma_{\nu_m}^\eta$, we define the path $\gamma_{m+1} = \gamma_m \nu_m \gamma_1$, which corresponds to a connected component $\Delta^{(1)}_{\gamma_{m+1}}$ of the domain of $\mathcal{F}_\eta^{(m+1)} \Delta^{(1)}_{\gamma_m \nu_m}$. The decomposition of $\Delta^{(1)}_{\gamma_m \nu_m}$ into connected component of the domain of $\mathcal{F}_\eta$ is equivalent to

$$\Delta^{(1)}_{\gamma_m \nu_m} = \bigcup_{\gamma_{m+1} \in \Gamma_{\nu_m}^{(m+1), \eta}} \Delta^{(1)}_{\gamma_{m+1}}.$$  

For any $\gamma_m$ in the union in equation (4.15), any $\nu_m \in \mathcal{N}_m$ and any $\gamma_1 \in \Gamma_{\nu_m}^\eta$, we define the path $\gamma_{m+1} = \gamma_m \nu_m \gamma_1$, which corresponds to a connected component $\Delta^{(1)}_{\gamma_{m+1}}$ of the domain of $\mathcal{F}_\eta^{(m+1)} \Delta^{(1)}_{\gamma_m \nu_m}$.
Equations (4.15) and (4.16) imply that
\[ \bigcap_{k=n}^{m+1} C_k = \bigcup_{\gamma_n \in \Gamma^{(n)}}, \nu_n \in N_n \bigcup_{\gamma_{m+1} \in \Gamma^{(m+1)}}, \nu_{m+1} \in N_{m+1} \Delta^{(1)}_{\gamma_{m+1}, \nu_{m+1}}, \]
that is equation (4.15) holds for \( m + 1 \) and therefore it holds by induction for all \( m > n \). Using the identity \( \text{Leb}(\Delta^{(1)}_{\gamma_{m+1}, \nu_{m+1}}) = \text{Leb}(\Delta^{(1)}_{\gamma_{m+1}}) P_{\gamma_{m+1}}(\Delta^{(1)}_{\nu_{m+1}}) \) and recalling the estimate in lemma 4.25 we have
\[ \text{Leb}(\bigcap_{k=n}^{m+1} C_k) = \sum_{\gamma_n \in \Gamma^{(n)}}, \nu_n \in N_n \sum_{\gamma_{m+1} \in \Gamma^{(m+1)}}, \nu_{m+1} \text{Leb}(\Delta^{(1)}_{\gamma_{m+1}}) P_{\gamma_{m+1}}(\Delta^{(1)}_{\nu_{m+1}}) \leq (1 - C \psi_{m+1}) \sum_{\gamma_n \in \Gamma^{(n)}}, \nu_n \in N_n \sum_{\gamma_{m+1} \in \Gamma^{(m+1)}}, \nu_{m+1} \text{Leb}(\Delta^{(1)}_{\gamma_{m+1}}). \]
Equation (4.15) says that the paths \( \Delta^{(1)}_{\gamma_{m+1}} \) for \( \gamma_{m+1} \in \Gamma^{(m+1)}, \nu_{m+1} \) form a partition mod 0 of the simplex \( \Delta^{(1)}_{\gamma_{m}, \nu_{m}} \), therefore together with equation (4.16) it implies that
\[ \bigcup_{\gamma_n \in \Gamma^{(n)}}, \nu_n \in N_n \bigcup_{\gamma_{m+1} \in \Gamma^{(m+1)}}, \nu_{m+1} \Delta^{(1)}_{\gamma_{m+1}} = \bigcup_{\gamma_n \in \Gamma^{(n)}}, \nu_n \in N_n \bigcup_{\gamma_{m+1} \in \Gamma^{(m+1)}}, \nu_{m+1} \Delta^{(1)}_{\gamma_{m}, \nu_{m}} = \bigcap_{k=n}^{m+1} C_k, \]
that is
\[ \text{Leb}(\bigcap_{k=n}^{m+1} C_k) \leq (1 - C \psi_{m+1}) \text{Leb}(\bigcap_{k=n}^{m} C_k) \leq \prod_{k=n}^{m+1} (1 - C \psi_k) \]
and the proof of the inductive step is complete. The lemma is proved. \( \square \)

We recall our assumption \( \sum_{n=1}^{\infty} \varphi(n) = \infty \) in proposition. For the sequence \( \psi_k = \theta^k \varphi(\theta^k) \) introduced in equation (4.10) the assumption implies that \( \sum_{k=1}^{\infty} \psi_k = \infty \). Then we recall a classical fact in elementary calculus, that is for any sequence of positive numbers \( a_n \), if \( \sum_{n=1}^{\infty} a_n = \infty \) then \( \prod_{n=1}^{\infty} (1 - a_n) = 0 \). Lemma 4.26 therefore implies that for any \( k \geq 0 \) we have
\[ \text{Leb}_d(\bigcap_{i \geq k} C_i) = 0, \]
thus the countable union \( \bigcup_{k=0}^{\infty} \bigcap_{i \geq k} C_i \) is a set of measure zero and proposition (4.24) follows. Therefore part b) of theorem (4.3) follows. This complete the discussion of the divergent case.

5. Main technical results.

5.1. Main combinatorial property. In this paragraph we state and prove a combinatorial property of Rauzy classes which implies that any pair \( (\beta, \alpha) \) as in definition (1.1) has either property A or property B (or both). In order to simplify notation, for a Rauzy class \( \mathcal{R} \) over an alphabet \( \mathcal{A} \), we call \( X = X(\mathcal{R}) \) and \( Y = Y(\mathcal{R}) \) the two letters in \( \mathcal{A} \) such that respectively \( \pi^\ell(X) = 1 \) and \( \pi^b(Y) = 1 \) for all \( \pi \in \mathcal{R} \).
Theorem 5.1. Let $\mathcal{R}$ be any Rauzy class with alphabet $\mathcal{A}$ and $(\beta, \alpha)$ be any ordered pair of letters with $\beta \neq Y$ and $\alpha \neq X$. Then at least one of the following two statements is true:

a: There exists an element $\pi \in \mathcal{R}$ such that

\begin{equation}
\pi^t(\alpha) = \pi^b(\beta) = d
\end{equation}

b: There exist two (different) elements $\pi, \pi' \in \mathcal{R}$ and two letters $V, V' \in \mathcal{A}$ such that

\begin{equation}
\{ \xi \in \mathcal{A} : \pi^t(\xi) < \pi^t(\alpha) \} \cup \{ V \} = \{ \xi \in \mathcal{A} : \pi^b(\xi) < \pi^b(\beta) \}
\end{equation}

\begin{equation}
\pi^t(V) = \pi^b(\alpha) = d
\end{equation}

\begin{equation}
\{ \xi \in \mathcal{A} : \pi^t(\xi) < \pi^t(\alpha) \} = \{ \xi \in \mathcal{A} : \pi^b(\xi) < \pi^b(\beta) \} \cup \{ V' \}
\end{equation}

\begin{equation}
\pi^b(V') = \pi^t(\beta) = d
\end{equation}

Note: Observe that the case a is compatible just with pair of different letters. In case b, when $\beta = \alpha$ equation (5.2) implies $\pi^t(\alpha) = d - 1$ and $\pi^b(\alpha) = d$ and on the other hand equation (5.3) implies $\pi^t(\alpha) = d$ and $\pi^b(\alpha) = d - 1$.

Proof: In $[\mathcal{A}, \mathcal{V}]$ it is proven that $\mathcal{R}$ always contains a standard $\tilde{\pi}$, that is a combinatorial datum such that $\tilde{\pi}^t(X) = \tilde{\pi}^b(Y) = 1$ and $\tilde{\pi}^t(Y) = \tilde{\pi}^b(X) = d$. Let us display also the second letters in the top and bottom lines of $\tilde{\pi}$, that is we write

$$
\tilde{\pi} = \left( \begin{array}{ccc}
X & A & \ldots & Y \\
Y & B & \ldots & X 
\end{array} \right).
$$

Lemma 5.2. Equation (5.1) holds for all pairs $(X, \alpha)$ with $\alpha \neq X$ and all pairs $(\beta, Y)$ with $\beta \neq Y$.

Proof: Since $\tilde{\pi}$ is standard, then it is the base point of two loops in $\Pi(\mathcal{R})$ of length $d-1$. One of these two loops is concatenation of $d-1$ bottom arrows with winner $X$. Any letter $\alpha \neq X$ loses against $X$ in some arrow in this loop, therefore the lemma follows for the corresponding pair $(X, \alpha)$. The other loop is the concatenation of $d-1$ top arrows with winner $Y$ and with the symmetric argument we get the statement for the pairs $(\beta, Y)$ with $\beta \neq Y$. The lemma is proved.

The proof continues by induction on the number of letters $d$. All Rauzy classes with $d \leq 4$ are easily computable and for these classes the statement in theorem 5.1 is just matter of checking a small number of conditions. Therefore we consider a Rauzy class $\mathcal{R}$ on an alphabet $\mathcal{A}$ with $d \geq 5$ letters and we suppose that the lemma is true for any Rauzy class $\mathcal{R}'$ on an alphabet $\mathcal{A}'$ with $d' < d$ letters.

5.1.1. Pairs $(\beta, \alpha)$ with $\alpha \neq A, X$ and $\beta \neq Y, X$. We consider the alphabet $\mathcal{A}_X := \mathcal{A} \setminus \{ X \}$ and the essential $\mathcal{A}_X$-decorated Rauzy class $\mathcal{R}_X$ which contains $\tilde{\pi}$ (the only non-essential element of $\mathcal{R}_X$ is $\tilde{\pi}$). Let $\mathcal{R}_X^{es} \subset \mathcal{R}_X$ be the subset of essential elements and let $\mathcal{R}_X^{red}$ be the associated reduced Rauzy class. Since the letter $Y$ is first in the bottom line and last in the top line of $\tilde{\pi}$, when we delete the letter $X$ from any $\pi$ in $\mathcal{R}_X$ we get an irreducible permutation, therefore the alphabet of $\mathcal{R}_X^{red}$ is $\mathcal{A}_X$.

Lemma 5.3. If the statement in theorem 5.1 holds for $\mathcal{R}_X^{red}$, then it holds for $\mathcal{R}$ for all the pairs $(\beta, \alpha)$ with $\alpha \neq A, X$ and $\beta \neq Y, X$. 
Proof: We observe that any element \( \hat{\pi} \) in \( R_X^{red} \) satisfies \( \hat{\pi}^t(A) = \hat{\pi}^h(Y) = 1 \), therefore for a pair \((\beta, \alpha)\) with \( \alpha \neq A, X \) and \( \beta \neq Y, X \) we can apply the induction hypothesis with respect to \( R_X^{red} \). If there exists \( \hat{\pi} \in R_X^{red} \) satisfying equation \( (5.1) \), then its (unique) essential pre-image \( \pi = \text{red}^{-1}(\hat{\pi}) \in R_X^{cas} \) is a solution of equation \( (5.1) \) with respect to the class \( \mathcal{R} \) and the statement in theorem \( 5.1 \) holds for the pair \((\beta, \alpha)\). On the other hand for \((\beta, \alpha)\) we may have both \( \hat{\pi} \in R_X^{red} \) and \( V \in A_X \) satisfying equation \( (5.2) \) and \( \hat{\pi}' \in R_X^{red} \) and \( V' \in A_X \) satisfying equation \( (5.3) \). This case is more complicated to discuss.

We first consider \( \hat{\pi} \in R_X^{red} \) and \( V \in A_X \) satisfying equation \( (5.2) \). Let \( \pi \in R_X^{cas} \) be the (unique) essential pre-image of \( \hat{\pi} \). We have \( \pi^t(X) = 1 \) and according to the position of \( X \) in the bottom line of \( \pi \) we consider two cases.

1. If \( \pi^h(X) < \pi^h(\beta) \) then \( \pi \) is a solution of equation \( (5.2) \).
2. If \( \pi^h(X) > \pi^h(\beta) \) all the pairs \((\beta, \alpha)\) with \( \alpha = \beta \) are automatically excluded. In this case we have

\[
\pi = \begin{pmatrix} X & A & \ldots & \alpha & \ldots & V \\ Y & \ldots & V & \ldots & \alpha & \ldots & \beta & \ldots & X \end{pmatrix}
\]

with \( \pi^t(\alpha) = \pi^h(\beta) \), therefore \( \pi \) is not solution of equation \( (5.2) \), anyway the following three Zorich steps

\[
\pi \mapsto \begin{pmatrix} X & A & \ldots & \alpha & \ldots & V \\ Y & \ldots & V & \ldots & \alpha & \ldots & \beta & \ldots & X \end{pmatrix}
\]

\[
\mapsto \begin{pmatrix} X & \ldots & V & A & \ldots & \alpha \\ Y & \ldots & V & \ldots & \alpha & \ldots & \beta & \ldots & X \end{pmatrix}
\]

\[
\mapsto \begin{pmatrix} X & \ldots & V & A & \ldots & \alpha \\ Y & \ldots & V & \ldots & \alpha & \ldots & \beta & \ldots \end{pmatrix}
\]

give a solution of \( (5.1) \). (Note that the argument works in both cases \( V \neq Y \) and \( V = Y \).)

Now we consider \( \hat{\pi}' \in R_X^{red} \) and \( V' \in A_X \) satisfying equation \( (5.3) \). Note that we have \( Y \neq A, V' \), therefore the general form of \( \hat{\pi}' \) is

\[
\hat{\pi}' = \begin{pmatrix} A & Y & \ldots & \alpha & \ldots & \beta \\ Y & \ldots & \beta & \ldots & V' \end{pmatrix}
\]

with \( \hat{\pi}'^t(\alpha) = \pi'^h(\beta) + 1 \).

Let \( \pi' \in R_X^{cas} \) be the (unique) essential pre-image of \( \hat{\pi}' \). We have \( \pi'^t(X) = 1 \) and we consider three cases for the position of \( X \) in the bottom line of \( \pi' \).

1. If \( \pi'^h(X) < \pi'^h(\beta) \) then \( \pi' \) is a solution of equation \( (5.3) \).
2. If \( \pi'^h(\beta) < \pi'^h(X) < \pi'^h(\alpha) \) all the pairs \((\beta, \alpha)\) with \( \beta = \alpha \) are automatically excluded. We also observe that we cannot have \( \pi'^t(V') = \pi'^t(\alpha) - 1 \), since in this case \( \pi' \) would not be admissible, so let us call \( W \neq V' \) the letter which appears just before \( \alpha \) in the top line. The general form of \( \pi' \) therefore is

\[
\pi' = \begin{pmatrix} X & A & \ldots & V' & \ldots & W & \alpha & \ldots & \beta \\ Y & \ldots & W & \ldots & \beta & \ldots & X & \ldots & \alpha & \ldots & V' \end{pmatrix}
\]

with \( \pi'^t(\alpha) = \pi'^h(\beta) + 2 \), which is not a solution of \( (5.3) \). We apply the following Zorich steps

\[
\pi' \mapsto \begin{pmatrix} X & A & \ldots & V' & \alpha & \ldots & \beta & \ldots & W \\ Y & \ldots & W & \ldots & \beta & \ldots & X & \ldots & \alpha & \ldots \end{pmatrix}
\]
and we get a solution of equation (5.1). Note that this sequence of steps works in both cases $A = V'$ and $A \neq V'$.

(3) If $\pi^b(\alpha) < \pi^b(X)$ both cases $\alpha = \beta$ and $\alpha \neq \beta$ are possible. If $\beta = \alpha$ then

$$\pi' = \begin{pmatrix} X & A & \ldots & V' & \alpha & \ldots & \beta & \ldots & W \\ Y & \ldots & W & \ldots & \alpha & \ldots & V' & \ldots & \beta & \ldots & X \end{pmatrix}$$

and letting $\alpha$ win one time we get a solution of equation (5.3). It rests to consider the case $\beta \neq \alpha$, which we separate into two sub-cases: $A = V'$ and $A \neq V'$. In the sub-case $A = V'$ the general form of $\pi'$ is

$$\pi' = \begin{pmatrix} X & A & \ldots & \beta & \ldots & \alpha & \ldots & X & \ldots & A \\ Y & \ldots & \alpha & \ldots & X & \ldots & \beta & \ldots & A \end{pmatrix}$$

where the asterisks denote the letters $\{\xi \in A; \pi^n(\xi) < \pi^n(\alpha)\} \setminus \{A\}$, which coincide with the letters $\{\xi \in A; \pi^b(\xi) < \pi^b(\beta)\}$, and where $\pi^n(\alpha) = \pi^b(\beta) + 2$. We get a solution of equation (5.3) applying the following Zorich steps

$$\pi' \mapsto \begin{pmatrix} X & A & \ldots & \beta & \ldots & \alpha & \ldots & X & \ldots & A \\ Y & \ldots & \alpha & \ldots & X & \ldots & \beta & \ldots & A \end{pmatrix}$$

$$\mapsto \begin{pmatrix} X & A & \ldots & \beta & \ldots & \alpha & \ldots & X & \ldots & A \\ Y & \ldots & \alpha & \ldots & X & \ldots & \beta & \ldots & A \end{pmatrix}$$

In the sub-case $A \neq V'$ the general form of $\pi'$ is

$$\pi' = \begin{pmatrix} X & A & \ldots & V' & \ldots & \alpha & \ldots & \beta & \ldots & \alpha & \ldots & X & \ldots & V' \\ Y & \ldots & A & \ldots & \beta & \ldots & \alpha & \ldots & X & \ldots & V' \\ Y & \ldots & \alpha & \ldots & X & \ldots & \beta & \ldots & A \end{pmatrix}$$

with $\pi^n(\alpha) = \pi^b(\beta) + 2$. It is not a solution of (5.3). We get a solution of equation (5.1) applying the following Zorich steps

$$\pi' \mapsto \begin{pmatrix} X & A & \ldots & V' & \ldots & \beta & \ldots & \alpha \\ Y & \ldots & A & \ldots & \beta & \ldots & \alpha & \ldots & X & \ldots & V' \\ Y & \ldots & \alpha & \ldots & X & \ldots & \beta & \ldots & A \end{pmatrix}$$

$$\mapsto \begin{pmatrix} X & A & \ldots & V' & \ldots & \beta & \ldots & \alpha & \ldots & V' & \ldots & X \\ Y & \ldots & A & \ldots & \beta & \ldots & \alpha & \ldots & V' & \ldots & X \\ Y & \ldots & \alpha & \ldots & V' & \ldots & \beta & \ldots & A \end{pmatrix}$$

$$\mapsto \begin{pmatrix} X & A & \ldots & V' & \ldots & \beta & \ldots & \alpha \\ Y & \ldots & A & \ldots & \beta & \ldots & \alpha & \ldots & V' & \ldots & X \\ Y & \ldots & \alpha & \ldots & V' & \ldots & \beta & \ldots & A \end{pmatrix}$$

$$\mapsto \begin{pmatrix} X & A & \ldots & V' & \ldots & \beta & \ldots & \alpha \\ Y & \ldots & A & \ldots & \beta & \ldots & \alpha & \ldots & V' & \ldots & X \\ Y & \ldots & \alpha & \ldots & V' & \ldots & \beta & \ldots & A \end{pmatrix}$$

$$\mapsto \begin{pmatrix} X & A & \ldots & V' & \ldots & \beta & \ldots & \alpha \\ Y & \ldots & A & \ldots & \beta & \ldots & \alpha & \ldots & V' & \ldots & X \\ Y & \ldots & \alpha & \ldots & V' & \ldots & \beta & \ldots & A \end{pmatrix}$$

□
5.1.2. *Pairs (β, α) with α ≠ X, Y and β ≠ B, Y.* We consider the alphabet \( \mathcal{A}_Y := \mathcal{A} \setminus \{Y\} \) and the essential \( \mathcal{A}_Y \)-decorated Rauzy class \( \mathcal{R}_Y \) which contains \( \tilde{\pi} \) (the only non-essential element of \( \mathcal{R}_Y \) is \( \tilde{\pi} \)). Let \( \mathcal{R}_Y^{red} \subset \mathcal{R}_Y \) be the subset of essential elements and let \( \mathcal{R}_Y^{red} \) be the associated reduced Rauzy class. Arguing as in paragraph 5.1.2 we get that the alphabet of \( \mathcal{R}_Y^{red} \) is the entire \( \mathcal{A}_Y \) and any element \( \hat{\pi} \) in \( \mathcal{R}_Y^{red} \) satisfies \( \hat{\pi}^b(B) = \hat{\pi}^t(X) = 1 \).

The letters \( X \) and \( A \) play a symmetric role with respect to the letters \( Y \) and \( B \), therefore changing respectively \( A \) with \( B \), \( X \) with \( Y \) and the role of bottom line with the role of the top one, the symmetric argument of lemma proves that if the statement in theorem 5.1 holds for \( \mathcal{R}_Y^{red} \), then it holds for \( \mathcal{R} \) for all the pairs \((β, α)\) with \( α \neq X, Y \) and \( β \neq B, Y \).

5.1.3. *The pair (B, A).* Combining the results in lemma 5.2 and in paragraphs 5.1.1 and 5.1.2 we get the inductive proof of the statement in theorem 5.1 for the class \( \mathcal{R} \) and for any pair of letters except that for the pair \((β, α) = (B, A)\). To complete the induction we provide a solution for this pair. We come back to the standard \( \tilde{\pi} \) in \( \mathcal{R} \). As it is shown in \([A, V]\), it is possible to find a standard \( \hat{\pi} \) which is good or degenerate, where a standard permutation is said good if the permutation that we get deleting the letters \( X \) and \( Y \) from \( \tilde{\pi} \) is still admissible and is said degenerate if there exists a letter \( C \in \mathcal{A} \) different from \( X \) and \( Y \) that is second or second to last in both the top and bottom lines. We consider separately the two cases:

If \( \tilde{\pi} \) is good then \( A \neq B \). Let \( \pi \) be the element obtained from \( \tilde{\pi} \) letting \( Y \) win one time, that is,

\[
\pi = \begin{pmatrix}
X & A & \ldots & Y \\
Y & X & B & \ldots
\end{pmatrix}
\]

We consider the alphabet \( \mathcal{A}_Y = \mathcal{A} \setminus \{Y\} \) and the \( \mathcal{A}_Y \)-decorated Rauzy class \( \mathcal{R}_Y \) which contains \( \pi \). We note that \( \mathcal{R}_Y \) is an essential decorated Rauzy class and we call \( \mathcal{R}_Y^{red} \) its reduction. Since \( \tilde{\pi} \) is good then the alphabet of \( \mathcal{R}_Y^{red} \) is \( \mathcal{A}_Y := \mathcal{A} \setminus \{X, Y\} \). Let \( \hat{\pi}_{st} \) be a standard element in \( \mathcal{R}_Y^{red} \), that is,

\[
\hat{\pi}_{st} = \begin{pmatrix}
A & \ldots & B \\
B & \ldots & A
\end{pmatrix}.
\]

An essential pre-image of \( \hat{\pi}_{st} \) in \( \mathcal{R}_Y \) is an element of the form

\[
\begin{pmatrix}
X & A & \ldots & Y & \ldots \\
Y & X & B & \ldots & A
\end{pmatrix}.
\]

Letting \( A \) win the correct number of times we get

\[
\begin{pmatrix}
X & A & \ldots & Y \\
Y & X & B & \ldots & A
\end{pmatrix},
\]

which is a solution of equation (5.2). Since the argument is symmetric changing the top line with the bottom one, we can get also a solution of equation (5.3).

If \( \tilde{\pi} \) is degenerate both cases \( B = A \) and \( B \neq A \) are possible. We consider them separately. If \( A = B \) we call \( W \) the last letter in the bottom line before \( X \), that is the letter such that \( \tilde{\pi}^b(W) = d - 1 \). We apply the sequence of Zorich steps

\[
\begin{align*}
\tilde{\pi} & = \begin{pmatrix}
X & A & \ldots & W & \ldots & Y \\
Y & A & \ldots & W & X
\end{pmatrix} \implies \begin{pmatrix}
X & \ldots & W & \ldots & Y & A \\
Y & A & \ldots & W & X
\end{pmatrix} \\
& \implies \begin{pmatrix}
X & \ldots & W & \ldots & Y & A \\
Y & A & X & \ldots & W
\end{pmatrix} \implies \begin{pmatrix}
X & \ldots & W & A & \ldots & Y \\
Y & A & X & \ldots & W
\end{pmatrix}
\end{align*}
\]
For any Rauzy path $\gamma$ we get a solution of equation $(5.3)$. Thanks to the symmetry of $\tilde{\pi}$ we can get also a solution of equation $(5.2)$. If $A \neq B$, since $d \geq 5$, there exists a letter $C \neq X,Y,A,B$ which is second to last both in top and bottom lines and the general form of $\tilde{\pi}$ is

$$
\tilde{\pi} = \begin{pmatrix}
X & A & \ldots & B & \ldots & C & Y \\
Y & B & \ldots & A & \ldots & C & X
\end{pmatrix}.
$$

We get a solution of equation $(5.1)$ for $(B,A)$ applying the following sequence of Zorich steps:

$$
\pi \mapsto \begin{pmatrix}
X & A & \ldots & B & \ldots & C & Y \\
Y & B & \ldots & A & \ldots & C & X
\end{pmatrix} \mapsto \begin{pmatrix}
X & A & Y & \ldots & B & \ldots & C \\
Y & X & B & \ldots & A & \ldots & C
\end{pmatrix} \mapsto \begin{pmatrix}
X & A & Y & \ldots & B & \ldots & C \\
Y & X & B & \ldots & A & \ldots & C
\end{pmatrix} \mapsto \begin{pmatrix}
X & Y & \ldots & B & \ldots & C & A \\
Y & A & \ldots & C & B & \ldots & X
\end{pmatrix} \mapsto \begin{pmatrix}
X & Y & \ldots & B & \ldots & C & A \\
Y & A & \ldots & C & B & \ldots & X
\end{pmatrix}.
$$

\[\square\]

Note. It would have been natural to claim that for all pairs $(\beta,\alpha)$ with $\alpha \neq \beta$ (and $\beta \neq Y, \alpha \neq X$) we always have a solution of $(5.1)$ and we need to consider equations $(5.2)$ or $(5.3)$ just for the pairs with $\beta = \alpha$. Unfortunately this is not true, for example it can be proved that if $\mathcal{R}$ in an hyperelliptic Rauzy class, then for the pair $(B,A)$ considered in paragraph 5.1.3 there is no solution of equation $(5.1)$.

5.2. Main estimate. Let us fix any $\pi \in \mathcal{R}$, any letter $W$ in $\mathcal{A}$ and any $\epsilon > 0$. Let us recall the definition of the set $E(\pi,W,\epsilon)$ of those $\mathcal{A}_W$-colored paths $\gamma$ starting at $\pi$ which satisfy $q_W^\pi > 1/\epsilon$ and are minimal with these properties with respect to $\prec$. Let us denote $\Delta_{E(\pi,W,\epsilon)}^{(1)} := \bigcup_{\gamma \in E(\pi,W,\epsilon)} \Delta_{\gamma}^{(1)}$.

Theorem 5.4. There exists a positive constant $C$, depending only from the number $d$ of letters of $\mathcal{A}$, such that for any $\pi \in \mathcal{R}$, any $W \in \mathcal{A}$ and any $\epsilon > 0$ we have

$$
\text{Leb}_{d-1}(\Delta_{E(\pi,W,\epsilon)}^{(1)}) \geq C\epsilon.
$$

5.2.1. Preliminary facts. Let $\pi$ be an element of some Rauzy class $\mathcal{R}$. Recall from paragraph 2.3 that for a proper sub-alphabet $\mathcal{A}'$ of $\mathcal{A}$ with $d'$ letters we denote $\tilde{\Delta}_{\mathcal{R}',\mathcal{A}'}^{(1)}$, the $(d'-1)$--hyper-face of $\Delta_{\mathcal{R}}^{(1)}$ whose extremal points are the vectors $e_\alpha$ with $\alpha \in \mathcal{A}'$. For any Rauzy path $\gamma$ starting at $\pi$ we also denote $\tilde{\Delta}_{\mathcal{R},\mathcal{A}'}^{(1)}$ the $(d'-1)$--hyper-face of $\Delta_{\gamma}^{(1)}$ spanned by the vectors $\{(1/q_\alpha^\gamma)^{1/2}B_\gamma e_\alpha\}_{\alpha \in \mathcal{A}'}$.

Let us fix a letter $W$ in $\mathcal{A}$ and consider the sub-alphabet $\mathcal{A}_W = \mathcal{A} \setminus \{W\}$. In this case we simply write $\tilde{\Delta}_{\mathcal{R}}^{(1)}$ instead of $\tilde{\Delta}_{\mathcal{R},\mathcal{A}_W}^{(1)}$. Similarly for any $\mathcal{A}_W$-colored path $\gamma$ starting at $\pi$ we write $\tilde{\Delta}_{\mathcal{R}}^{(1)}$ instead of $\Delta_{\gamma,\mathcal{A}_W}^{(1)}$. They are all $(d-2)$--simplices. Observe that if $\gamma$ is $\mathcal{A}_W$-colored then the set $\{e_\alpha\}_{\alpha \neq W}$ is $B_\gamma$--invariant, therefore $\tilde{\Delta}_{\mathcal{R}}^{(1)}$ is a sub-simplex of $\Delta_{\mathcal{R}}^{(1)}$. By our choice of the normalization of the Lebesgue measure Leb$_{d-2}$ on $\Delta_{\gamma}^{(1)}$, for any such $\gamma$ we have

$$
\text{Leb}_{d-1}(\Delta_{\gamma}^{(1)}) = \frac{1}{q_W^\gamma}\text{Leb}_{d-2}(\tilde{\Delta}_{\gamma}^{(1)}).
$$
Let us consider the $A_W$-decorated Rauzy class $R^{col}_W$ which contains $\pi$. Here we suppose that $R^{col}_W$ is essential. In this case we can associated to it a reduced Rauzy class $R^{red}_W$ on some sub-alphabet $A^{red}_W \subset A_W$. The notion of decorated, essential decorated and reduced Rauzy class are introduced in paragraph 2.2.1 in the background, exactly as the rest of the formalism we are going to use. To simplify notation we write $R^{col}$ instead of $R^{col}_W$, $R^{red}$ instead of $R^{red}_W$ and $A^{red}$ instead of $A^{red}_W$.

We have a reduction map $\mathit{red}: R^{col} \to R^{red}$, which extends to a map $\mathit{red}: \Pi^{col}(R^{col}) \to \Pi(R^{red})$. The inclusions $A^{red} \subset A_W \subset A$ induce naturally a decomposition

$$\mathbb{R}^d = \mathbb{R}^d_{A^{red}} \oplus \mathbb{R}^{d-A_W\backslash A^{red}} \oplus \mathbb{R}^{d-A_W}.$$  

We consider the canonical projections $P_{A^{red}}: \mathbb{R}^d \to \mathbb{R}^{d-A^{red}}$ and $P_{A_W \backslash A^{red}}: \mathbb{R}^d \to \mathbb{R}^{d-A_W \backslash A^{red}}$ respectively on the first and on the second factors in equation (5.5). Any path $\nu$ in $\Pi^{col}(R^{col})$ is $(A_W \backslash A^{red})$—separated, therefore for any such $\nu$ we have to commutative diagrams

$$
\begin{array}{cccc}
\mathbb{R}^d_+ & \to & B_\nu & \to & \mathbb{R}^d_+ \\
\downarrow & & \downarrow & & \downarrow \\
\mathbb{R}^{d-A^{red}}_+ & \to & B_{\nu,A^{red}} & \to & \mathbb{R}^{d-A^{red}}_+ \\
\downarrow & & \downarrow & & \downarrow \\
\mathbb{R}^{d-A_W \backslash A^{red}}_+ & \to & B_{\nu,A_W \backslash A^{red}} & \to & \mathbb{R}^{d-A_W \backslash A^{red}}_+.
\end{array}
$$

For any vector $\nu \in \mathbb{R}^d_+$ we introduce the simplified notation $\hat{\nu} := P_{A^{red}}(\nu)$. Similarly for paths $\nu \in \Pi^{col}(R^{col})$ we write $\hat{\nu} := \mathit{red}(\nu)$. The associated matrix $B_\nu$ acts on the first factor in equation (5.5). We proved that for any $\nu \in \Pi^{col}(R^{col})$ the vector $q^\nu = B_\nu \hat{\nu}$ satisfies

$$\hat{q}^\nu = P_{A^{red}}(q^\nu) \text{ and } P_{A_W \backslash A^{red}}(q^\nu) = \hat{1}.$$  

Let $a$ be the number of letters of $A^{red}$. Let us consider the element $\tilde{\pi} := \mathit{red}(\pi)$ of $R^{red}$ and the corresponding $(a-1)$-simplex $\Delta^{(1)}_{\tilde{\pi}}$. For any $\nu \in \Pi^{col}(R^{col})$ let us consider the reduced path $\tilde{\nu} = \mathit{red}(\nu)$ in $\Pi(R^{red})$ and the $(a-1)$-sub-simplex $\Delta^{(1)}_{\tilde{\nu}}$. Equation (5.6) implies

$$\text{Leb}_{d-2}(\Delta^{(1)}_{\tilde{\nu}}) = \text{Leb}_{a-1}(\Delta^{(1)}_{\tilde{\nu}}).$$

5.2.2. Proof of the estimate. By definition paths in $E(\pi,W,\epsilon)$ are $A_W$-colored. For any $\gamma \in E(\pi,W,\epsilon)$ we consider the $(d-1)$-simplex $\Delta^{(1)}_{\gamma}$ and its base $\Delta^{(1)}_{\gamma}$ defined in paragraph 5.2.1. According to equation (5.3) it is sufficient to prove that there exist a positive integer $N$ and a constant $C$ with $0 < C < 1$ we have

$$\text{Leb}_{d-2}(\{\gamma \in E(\pi,W,\epsilon): q^\gamma_W < 2^N / \epsilon \}) \geq C,$$

then the theorem follows redefining $C$ by $C = C2^{-N}$. Let us consider the decorated Rauzy class $R^{col}$ which contains $\pi$. If $R^{col}$ is not essential then there exists an arrow $\gamma_W$ both starting and ending in $\pi$ such that $\Pi^{col}(R^{col}) = \{(\gamma_W)^r\}_{r \in \mathbb{N}}$, where $(\gamma_W)^r$ is the concatenation of $r$ copies of the arrow $\gamma_W$. For any $r$ the vector $q^{(r)}$ associated to $(\gamma_W)^r$ satisfies $q^{(r+1)}_W = q^r_W + 1$, therefore equation (5.8) follows immediately. It follows that from now on and during all the proof of the theorem we can suppose that $R^{col}$ is essential. Let us call $R^{red}$ the associated reduced Rauzy class and $A^{red} \subset A_W$ the corresponding sub-alphabet.
We consider $\mathcal{A}_W$-colored paths $\nu$ starting at $\pi$ and such that $q^*_W < 1/\epsilon$. Any $\gamma \in E(\pi, W, \epsilon)$ begins with such a $\nu$. For any such $\nu$ let us call $E(\pi, W, \epsilon[\nu])$ the set of those $\gamma \in E(\pi, W, \epsilon)$ which begin with $\nu$. Let $\pi' \in \mathcal{R}^{\text{col}}$ be the point where $\nu$ ends. We also call $S(\pi, W, \epsilon[\nu])$ the set of paths $\eta \in \Pi(\mathcal{R}^{\text{col}})$ starting at $\pi'$ and such that the composed path $\gamma = \nu \eta$ is in $E(\pi, W, \epsilon[\nu])$. For $\nu$ as before and $\eta \in S(\pi, W, \epsilon[\nu])$ the composed path $\gamma = \nu \eta$ satisfies $q^* = B_{\eta} q^*$.

**Definition 5.5.** An intermediate path is an $\mathcal{A}_W$-colored path $\nu$ starting at $\pi$ which satisfies $q^*_W < 1/\epsilon$ and the following extra property: if $\pi' \in \mathcal{R}^{\text{col}}$ is the ending point of $\nu$, then for any $\mathcal{A}_W$-colored path $\eta$ starting at $\pi'$ and containing at least one arrow where $W$ loses, we have $(B_{\eta} q^*)_{W} \geq 1/\epsilon$.

We introduce the set $I(\pi, W, \epsilon)$ of the intermediate paths starting at $\pi$ minimal with respect to this ordering $\prec$.

**Lemma 5.6.** For any $\gamma \in E(\pi, W, \epsilon)$ there exists an unique path $\nu$ in $I(\pi, W, \epsilon)$ such that $\gamma \in E(\pi, W, \epsilon[\nu])$. On the other hand for any $\nu \in I(\pi, W, \epsilon)$ the set $E(\pi, W, \epsilon[\nu])$ is not empty.

**Proof:** Let us consider any $\gamma \in E(\pi, W, \epsilon)$ and let us decompose it as $\gamma = \gamma' \gamma_{\text{last}}$, where $\gamma_{\text{last}}$ is the last arrow of $\gamma$. By minimality of paths in $E(\pi, W, \epsilon)$ the arrow $\gamma_{\text{last}}$ has $W$ as loser. The path $\gamma'$ is of course $\mathcal{A}_W$-colored and satisfies $q^*_W < 1/\epsilon$. Let us call $\pi'$ the ending point of $\gamma'$. Any $\mathcal{A}_W$-colored path $\eta$ starting at $\pi'$ has $\gamma_{\text{last}}$ as first arrow, since the other one starting at $\pi'$ has $W$ as winner. It follows that we can decompose any such $\eta$ as $\eta = \gamma_{\text{last}} \eta'$ since $B_{\eta} q^* = B_{\eta'} q^*$, then $(B_{\eta} q^*)_{W} > q^*_W > 1/\epsilon$, thus $\gamma'$ is intermediate. For any $\gamma \in E(\pi, W, \epsilon)$, among the intermediate paths $\gamma'$ with $\gamma' \prec \gamma$ there exist a minimal one, and this one is of course unique by minimality. The second statement is evident. The lemma is proved. □

**Remark 5.7.** Let us decompose any $\gamma \in E(\pi, W, \epsilon)$ as $\gamma = \nu \eta$, where $\nu = \nu(\gamma) \in I(\pi, W, \epsilon)$ is given by lemma 5.6 and $\eta \in S(\pi, W, \epsilon[\nu])$. Let us also consider the sub-path $\gamma_{\text{2nd-to-last}} \prec \gamma$ which ends with the arrow where the letter $W$ loses for the second to last time in $\gamma$. The letter $W$ loses exactly one time in $\eta$ (at the end of it), therefore $\gamma_{\text{2nd-to-last}} \prec \nu$, but in general $\gamma_{\text{2nd-to-last}}$ is not intermediate and it does not coincide with $\nu$.

**Corollary 5.8.** We have a mod 0 partition

$$\hat{\Delta}^{(1)} = \bigcup_{\nu \in I(\pi, W, \epsilon)} \hat{\Delta}^{(1)}_{\nu}.$$

**Proof:** The family of simplices $\{\hat{\Delta}^{(1)}_{\nu}, \gamma \in E(\pi, W, \epsilon)\}$ form a partition mod 0 of the base $\hat{\Delta}^{(1)}_{\pi}$. On the other hand any $\gamma \in E(\pi, W, \epsilon)$ starts with the (unique) minimal intermediate path $\nu = \nu(\gamma)$ given by lemma 5.6 and for the associated simplices we have $\hat{\Delta}^{(1)}_{\nu} \subset \hat{\Delta}^{(1)}_{\pi}$. This means that the family of all simplices $\hat{\Delta}^{(1)}_{\nu}$ with $\nu \in I(\pi, W, \epsilon)$ covers ( mod 0) the base $\hat{\Delta}^{(1)}_{\pi}$. By minimality of paths $\nu \in I(\pi, W, \epsilon)$ the simplices $\hat{\Delta}^{(1)}_{\nu}$ are all disjoint, therefore they form a partition mod 0. The corollary is proved. □

For any integer $k \in \mathbb{N}$ let us consider the set $I(\pi, W, \epsilon[k])$ of paths $\nu \in I(\pi, W, \epsilon)$ with $M(q^*) \geq 2^k/\epsilon$. We have $I(\pi, W, \epsilon) = \bigcup_{k=1}^{\infty} I(\pi, W, \epsilon[k])$ (the union is not disjoint).
Lemma 5.9. There exist two positive constants $C$ and $\theta$, depending only from the number of intervals $d$, such that for any $k \in \mathbb{N}^*$ we have

$$\text{Leb}_{d-2}(I(\pi, W, \epsilon|k)) \leq Ck^\theta 2^{-(k-1)}.$$  

Proof: We decompose any $\nu \in I(\pi, W, \epsilon)$ as $\nu = \nu'\nu_{\text{last}}$, where $\nu_{\text{last}}$ is the last arrow in $\nu$, and we consider the family

$$I'(\pi, W, \epsilon) := \{\nu' ; \nu \in I(\pi, W, \epsilon)\}.$$  

We note that the map $I(\pi, W, \epsilon) \to I'(\pi, W, \epsilon); \nu \mapsto \nu'$ is a bijection. This because if for some $\nu' \in I'(\pi, W, \epsilon)$ there exist two paths $\nu_1$ and $\nu_2$ in $I(\pi, W, \epsilon)$ such that $\nu' = \nu'_1 = \nu'_2$, then $\nu'$ would be intermediate, which is absurd by minimality of paths in $I(\pi, W, \epsilon)$. For any $k \in \mathbb{N}$ we define the set $I'(\pi, W, \epsilon|k)$ of paths $\nu'$ such that the associated $\nu$ is in $I(\pi, W, \epsilon|k)$.

Let us fix any $k \in \mathbb{N}$ and consider any $\nu'$ in $I'(\pi, W, \epsilon|k)$. By minimality of paths in $I(\pi, W, \epsilon)$ the sub-path $\nu'$ is not intermediate. Therefore, if $\pi' \in \mathcal{R}^{\text{col}}$ is the element where $\nu'$ ends, there exists an $\mathcal{A}_W$-colored path $\eta'$ starting at $\pi'$ which contains one arrow where $W$ loses and such that $(B_{\eta'}q''')_W < 1/\epsilon$. Let us call $X \in \mathcal{A}_W^{\text{red}}$ the letter which wins against $W$ in $\eta'$. We obviously have $q''_X < 1/\epsilon$.

In terms of the reduced path $\hat{\nu}'$ and the reduced vector $\hat{q}''$, according to equation (5.6), we have

$$\hat{q}''_X < 1/\epsilon.$$  

Let us fix $X \in \mathcal{A}_W^{\text{red}}$ and define the set $I(\pi, W, \epsilon|k, X)$ of those paths $\nu \in I(\pi, W, \epsilon|k)$ such that the associated $\nu' \in I'(\pi, W, \epsilon|k)$ satisfies $q''_X < 1/\epsilon$. We also define the set $I'(\pi, W, \epsilon|k, X)$ of all the paths $\nu'$ obtained from $\nu$ in $I(\pi, W, \epsilon|k, X)$. We have

$$I(\pi, W, \epsilon|k) = \bigcup_{X \in \mathcal{A}_W^{\text{red}}} I(\pi, W, \epsilon|k, X),$$  

where the union is not necessarily disjoint. We observe that for any $\nu \in I(\pi, W, \epsilon|k, X)$ we have $M(q''_X) < 2M(q''')$, thus $M(q''') > 2^{k-1}/\epsilon$. Recalling equation (5.6) again we have $M(\hat{q}'') = M(q''')$ ($= M_{\mathcal{A}_W^{\text{col}}}(q'''))$, that is

$$M(\hat{q}'') > 2^{k-1}/\epsilon.$$  

It follows that the set $\text{Red}(I'(\pi, W, \epsilon|k, X))$ of the reduced paths $\hat{\nu}'$ obtained from $\nu'$ in $I'(\pi, W, \epsilon|X, k)$ is contained in

$$\{\hat{\gamma} \in \Pi(\mathcal{R}^{\text{red}}); \hat{q}''_X < 1/\epsilon, M(\hat{q}'') \geq 2^{k-1}/\epsilon\}.$$  

Let us call $a$ the cardinality of the reduced alphabet $\mathcal{A}_W^{\text{red}}$. Equation (2.6) in paragraph 2.3 implies that there exist two positive constants $C$ and $\theta$ depending only from $a$ such that for any $k \in \mathbb{N}$ we have

$$\text{Leb}_{a-1}(\hat{\gamma} \in \Pi(\mathcal{R}^{\text{red}}); \hat{q}''_X < 1/\epsilon, M(\hat{q}'') \geq 2^{k-1}/\epsilon) \leq Ck^\theta 2^{k-1}.$$  

By inclusion it follows that $\text{Leb}_{a-1}(\text{Red}(I'(\pi, W, \epsilon|k, X))) \leq Ck^\theta 2^{k-1}$, then equation (5.7) implies $\text{Leb}_{d-2}(I'(\pi, W, \epsilon|k, X)) \leq Ck^\theta 2^{k-1}$ and from this last one it follows trivially that $\text{Leb}_{d-2}(I(\pi, W, \epsilon|k, X)) \leq Ck^\theta 2^{k-1}$. We sum over all $X \in \mathcal{A}_W^{\text{red}}$ and modulo changing $C$ with $ac$ we get the required estimate. The number of choices of $\pi$ in $\mathcal{R}$ and $W$ in $\mathcal{A}$ is finite and depends only from $d$, therefore we can chose a pair of constants $C$ and $\theta$ working for any choice, that is depending only from $d$. The lemma is proved. \qed
For any \( \nu \in I(\pi, W, \epsilon) \) and any integer \( m \geq 1 \) let us consider the set \( S(\pi, W, \epsilon|\nu, m) \) of paths \( \eta \in S(\pi, W, \epsilon|\nu) \) such that we have \( (B_{\nu}q^{\nu})_{W} \geq 2^m M(q^{\nu}) \).

**Lemma 5.10.** There exist two positive constant \( C \) and \( \theta \), depending only from the number \( d \) of letters of \( A \), such that for any \( \nu \in I(\pi, W, \epsilon) \), for the reduced path \( \hat{\nu} \in \Pi(\mathcal{R}^{\text{red}}) \) and for any integer \( m \geq 1 \) we have
\[
P_{\hat{\nu}}(\text{Red}(S(\pi, W, \epsilon|\nu, m))) \leq Cm^{\theta}2^{-(m-1)}.
\]

**Proof:** Let us fix \( \nu \in I(\pi, W, \epsilon) \) and a positive integer \( m \). Let us consider any \( \eta \in S(\pi, W, \epsilon|\nu) \) and let us decompose it as \( \eta = \eta' \eta^{\text{last}} \), where \( \eta^{\text{last}} \) is its last arrow. Since the composed path \( \gamma = \nu \eta \) is in \( E(\pi, W, \epsilon) \) then the arrow \( \eta^{\text{last}} \) has the letter \( W \) as loser. Moreover, since \( \nu \) is minimal intermediate, the sub-path \( \eta' \) is \( \{W\} \)-separated. Let \( Y \in \mathcal{A}^{\text{red}} \) be the letter which wins against \( W \) in the arrow \( \eta^{\text{last}} \). Since \( \eta' \) is \( \{W\} \)-separated and obviously \( q_{\nu}^{W} \leq M(q^{\nu}) \), then
\[
M(B_{\nu}q^{\nu}) \geq 2^{m-1}M(q^{\nu}),
\]
and for any \( \nu \in I(\pi, W, \epsilon|\nu, m) \) \( \eta^{\text{last}} \) is \( \{W\} \)-separated and obviously \( q^{\nu} \leq M(q^{\nu}) \), then
\[
M(B_{\nu}q^{\nu}) \geq 2^{m-1}M(q^{\nu}).
\]

Let us consider the ending point \( \nu' \) of the fixed path \( \nu \) and for \( M \geq m - 1 \) let \( \Gamma(\nu|M) \) be the set of \( \{W\} \)-separated paths \( \eta' \) in \( \Pi(\mathcal{R}^{\text{red}}) \) starting at \( \nu' \) such that
\[
2^mM(q^{\nu}) \leq M(B_{\nu'}q^{\nu'}) < 2^{M+1}M(q^{\nu}).
\]

Let us denote \( \tilde{\Gamma}(\nu|M) := \text{Red}(\Gamma(\nu|M)) \). Since any path \( \eta' \in \Gamma(\nu|M) \) is \( \{W\} \)-separated, remark \[2.3\] in paragraph \[2.2.4\] implies that there exists a positive integer \( s \) with \( s \leq 2(d-1) \) and \( \pi \) non-complete paths \( \tilde{\eta}_{\nu} \) in \( \Pi(\mathcal{R}^{\text{red}}) \) such that the reduced \( \tilde{\eta}' \) of the path \( \eta' \) is a concatenation
\[
\tilde{\eta} = \tilde{\eta}_{1}...\tilde{\eta}_{s}
\]
(non-completeness of the paths \( \tilde{\eta}_{\nu} \) is referred to the reduced alphabet \( \mathcal{A}^{\text{red}} \)). We put \( q^{(0)} := q^{\nu} \) and \( q^{0} := \nu \) and for any \( i = 1, s \) we define inductively \( \tilde{\eta}^{i} := \tilde{\eta}_{i-1}^{i-1} \tilde{\eta}_{i}^{i} \) and \( q^{(i)} := B_{\tilde{\eta}^{i}}q^{(i-1)} \). We can find \( s \) non-negative integers \( m_{1},...,m_{s} \) such that for any \( i \in \{1,...,s\} \) we have:
\[
2^{m_{i}}M(q^{(i-1)}) \leq M(B_{\tilde{\eta}^{i}}q^{(i-1)}) < 2^{m_{i}+1}M(q^{(i-1)}).
\]

It turns out that \( m_{1},...,m_{s} \) satisfy the relation:
\[
M - s - 1 \leq m_{1} + .. + m_{s} \leq M.
\]

Let us fix a positive integer \( s \) with \( s \leq 2(d-1) \) and \( s \) non-negative integers \( m_{1},...,m_{s} \) satisfying equation \[5.11\] and let us define \( \Gamma(\nu|m_{1},...,m_{s}) \) as the set of \( \eta' \in \Gamma(\nu|M) \) such that the corresponding reduced path \( \tilde{\eta}' \), decomposed as in equation \[5.9\], satisfies the \( s \) conditions in equation \[5.10\] for the values \( m_{1},...,m_{s} \). For any \( k \in \{0,...,s-1\} \) we also define the set \( \tilde{\Gamma}(\nu|m_{1},...,m_{k}) \) of those \( \tilde{\eta}^{k} \) which satisfy the first \( k \) conditions in equation \[5.10\] for the first \( k \) integers \( m_{1},...,m_{k} \). For any \( \tilde{\eta}^{k} \in \tilde{\Gamma}(\nu|m_{1},...,m_{k}) \) we define the set \( \tilde{\Gamma}(\nu|m_{1},...,m_{k}|\tilde{\eta}^{k}) \) of those \( \tilde{\eta}^{k+1} \in \tilde{\Gamma}(\nu|m_{1},...,m_{k+1}) \) which begin with \( \tilde{\eta}^{k} \).

Let us fix any \( s \) with \( s \leq 2(d-1) \), any set of \( s \) integers \( m_{1},...,m_{s} \) satisfying equation \[5.11\], any \( k \in \{0,...,s-1\} \) and any \( \tilde{\eta}^{k} \in \tilde{\Gamma}(\nu|m_{1},...,m_{k}) \). Equation \[2.7\] implies that there exist two positive constants \( C \) and \( \theta \), depending only from the cardinality of \( \mathcal{A}^{\text{red}} \), such that
\[
P_{\tilde{\eta}^{k}}(\tilde{\Gamma}(\nu|m_{1},...,m_{k+1})) \leq C(m_{k+1} + 1)^{\theta}2^{-m_{k+1}}.
\]
Applying this last equation \( s \) times we get

\[
P_p(\hat{\Gamma}(\nu|m_1,\ldots,m_s)) \leq \prod_{i=1}^{s} C(m_i+1)^{\theta}2^{-m_i} \leq C^s M^{\theta}2^{-M+s+1}.
\]

For any \( s \) with \( s \leq 2(d-1) \) the number of possible vectors \((m_1,\ldots,m_s) \in \mathbb{N}^s\) satisfying equation (5.11) is proportional to \( M^{s-1} \), therefore summing over all the possible \((m_1,\ldots,m_s) \in \mathbb{N}^s\) and all the \( s \in \{1,\ldots,2(d-1)\} \), modulo changing the constants \( C \) and \( \theta \), we get

\[
P_p(\hat{\Gamma}(\nu|M)) \leq C(M+1)^{\theta}2^{-M}.
\]

Since \( \{\tilde{\gamma}; \eta \in S(\pi,W,\epsilon|\nu,m)\} \subset \bigcup_{M \geq m-1} \hat{\Gamma}(\nu|M) \), summing over all \( M \geq m-1 \) we get

\[
P_p(\{\tilde{\gamma}; \eta \in S(\pi,W,\epsilon|\nu,m)\} \leq Cm^{\theta}2^{-(m-1)}
\]

which implies trivially \( P_p(\text{Red}(S(\pi,W,\epsilon|\nu,m))) \leq Cm^{\theta}2^{-(m-1)} \). We argue as in the end of the proof of lemma 5.10 that the pair of constant \( C \) and \( \theta \) can be chosen in order to work for all \( \pi \) and \( W \), that is they depend only from \( d \). The lemma is proved.

Here we finish the proof of theorem 5.4. For any \( k \in \mathbb{N} \) let us define

\[
\mathcal{I}(\pi,W,\epsilon|k) := I(\pi,W,\epsilon) \setminus I(\pi,W,\epsilon|k).
\]

For any \( \nu \in I(\pi,W,\epsilon) \) and for any integer \( m \geq 1 \) let us define

\[
\mathcal{G}(\pi,W,\epsilon|\nu,m) := S(\pi,W,\epsilon|\nu) \setminus S(\pi,W,\epsilon|\nu,m).
\]

Let us fix any pair of positive integers \((k,m)\). For any \( \nu \in \mathcal{I}(\pi,W,\epsilon|k) \) we have \( M(\tilde{\gamma}^\nu) < 2^k/\epsilon \). For any \( \nu \in \mathcal{I}(\pi,W,\epsilon|k) \) and for any \( \eta \in \mathcal{G}(\pi,W,\epsilon|\nu,m) \) we have

\[
(B_\eta q^\nu)_W < 2^m M(\tilde{\gamma}^\nu) < 2^{k+m}/\epsilon.
\]

Let \( C \) and \( \theta \) be the constant appearing in lemmas 5.9 and 5.10. We take \( N \in \mathbb{N} \) such that \( CN^{2^{-N-1}} \leq 1 \) and we put \( c := 1 - CN^{2^{-N-1}} \). We proved that the set \( \{\gamma \in E(\pi,W,\epsilon)\setminus \tilde{q}_W < 2^{2N}/\epsilon\} \) contains

\[
\bigcup_{\nu \in \mathcal{I}(\pi,W,\epsilon|N)} \mathcal{G}(\pi,W,\epsilon|\nu,N).
\]

For any \( \nu \in I(\pi,W,\epsilon) \) we have

\[
\text{Leb}_{d-2}(\mathcal{G}(\pi,W,\epsilon|\nu,N)) = \text{Leb}_{d-2}(\tilde{\Delta}^{(1)}_\nu)P_p(\text{Red}(\mathcal{G}(\pi,W,\epsilon|\nu,N)))
\]

therefore we get

\[
\text{Leb}_{d-2}(\gamma \in E(\pi,W,\epsilon)\setminus \tilde{q}_W < 2^{2N}/\epsilon) \geq \sum_{\nu \in \mathcal{I}(\pi,W,\epsilon|N)} \text{Leb}_{d-2}(\mathcal{G}(\pi,W,\epsilon|\nu,N)) \geq c \text{Leb}_{d-2}(\mathcal{I}(\pi,W,\epsilon|N)) \geq c^2
\]

from the results in lemmas 5.9 and 5.10. Equation (5.8) therefore follows with \( C = c^2 \). Theorem 5.2 is proved.
References


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