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Reference
CLIFFORD LINKS ARE THE ONLY MINIMIZERS OF THE ZONE MODULUS AMONG NON-SPLIT LINKS

GRÉGOIRE-THOMAS MONIOT

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0. Introduction

Langevin and O’Hara introduced in [1] a conformally invariant functional for knots, called the measure of acyclicity. It is the volume (with respect to a conformally invariant measure on the space of all round spheres) of the set of spheres that cut the knot in at least four points. There exists a constant $C$ such that a curve with measure of acyclicity below $C$ is the unknot. To prove this, they introduced a knot modulus called the zone modulus.

This work comes after O’Hara’s definition in [3] of the concept of a knot energy. Roughly, a functional on the space of knots is an energy when it blows up near a self-intersection. An energy is also expected to possess thresholds such that a curve with energy lower than a particular threshold must belong to a particular knot type. A knot representative in a knot class that realizes the minimum energy provides the best shaped knot of its class.

One of the most famous knot energy functionals, introduced by O’Hara in [3], is

$$E(\gamma) = \iint \left\{ \frac{1}{|\gamma(v) - \gamma(u)|^2} - \frac{1}{D(\gamma(u), \gamma(v))^2} \right\} \|\gamma'(u)||\gamma'(v)| \, du \, dv,$$

where $\gamma$ is an embedded curve and $D(\gamma(u), \gamma(v))$ denotes the length of the shortest path from $\gamma(u)$ to $\gamma(v)$ on $\gamma$. In [4] Freedman, He and Wang proved

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the conformal invariance of $E$ and called $E$ the Möbius energy. In the same paper they showed that the energy of a closed curve is always greater than or equal to 4 and that equality holds only for circles. They proved also that each prime knot class has an energy-minimizing representative, and that, given $m > 0$, there are finitely many knot types such that $E \leq m$. In [5], Kim and Kusner constructed explicit examples of knotted curves which are critical for $E$.

In [2], Langevin and the author proved that the minimum of the zone modulus over all non-split two-component links is $(\sqrt{2} + 1)^2$. This minimum is attained by a special configuration of two circles called a Clifford link, defined as follows:

**Definition 1.** We say that a link is a Clifford link when it consists of two circles such that each sphere containing one of the circles is perpendicular to each of the spheres containing the other circle. Equivalently, a Clifford link is a conformal image of the standard geometric Hopf link.

In [4], Freedman, He and Wang defined the mutual Möbius energy of two curves as

$$E(\gamma_1, \gamma_2) = \iint \frac{|\gamma_1'(u)||\gamma_2'(v)|}{|\gamma_1(u) - \gamma_2(v)|^2} \, du \, dv.$$  

Kim and Kusner showed in [5] that the standard geometric Hopf link is critical for $E$. In [7], He gave a geometric interpretation of the Euler-Lagrange equation for any $E$-critical pair of curves. He showed that there exists a pair of curves that minimizes $E$ over all linked pairs of loops and that every such pair is ambiently isotopic to the Hopf link. As far as the author knows, it is still a conjecture that Clifford links are the only configurations that minimize the Möbius energy among two-component non-split links.

The purpose of the present paper is to solve the analogous conjecture for the zone modulus. We will show:

**Theorem 1.** The two-component links that realize the minimum zone modulus among all non-split two-component links are the Clifford links.

It should be noted that the standard geometric Hopf link or its conformal class, the Clifford links, seems to be a recurrent minimizer or maximizer of various functionals. For example, Kusner proved in [6] that the thickness of a non-split two-component link in $S^3$ cannot exceed that of the standard geometric Hopf link, which equals $\pi/4$. In [2], we proved that the standard geometric Hopf is the only non-split two-component link with thickness $\pi/4$.

1. Preliminary definitions and known facts

We will recall in this section the definition of the zone modulus of a two-component link and some results of [2].
1.1. The modulus of a zone between two spheres. We first define the modulus of a zone between two disjoint spheres, which we call for simplicity the modulus of two spheres.

**Definition 2.** Given two disjoint spheres $S_1$ and $S_2$ in $\mathbb{R}^3$, let us choose a conformal transformation that makes the two spheres concentric with radii $R_2 > R_1$. Then the modulus $\mu(S_1, S_2)$ of the two spheres is the ratio $R_2/R_1 > 1$.

We can express the modulus in terms of the cross-ratio. Recall that the cross-ratio of four collinear points is defined as

$$
Cr(x_1, x_2, x_3, x_4) = (x_1 - x_3)(x_2 - x_4)/(x_2 - x_3)(x_1 - x_4).
$$

The cross-ratio is invariant by any homography of the line. We can extend its definition to four concircular points as follows: The cross-ratio of four points on a circle is the cross-ratio of the four image points by a stereographic projection of the circle onto a line.

Two disjoint spheres $S_1$ and $S_2$ generate a pencil of spheres with limit points. It is the set of spheres perpendicular to all the circles perpendicular to both $S_1$ and $S_2$. The limit points are the two points of intersection of these circles. Consider a circle perpendicular both to $S_1$ and $S_2$ as in Figure 1. It contains the limit points $l_1$ and $l_2$ of the pencil generated by $S_1$ and $S_2$ and intersects each $S_i$ in two points. Let us take two of these points, $p_1$ and $p_2$, such that $l_1, p_1, p_2, l_2$ are in this order on the circle.

Let $I$ be a Möbius transformation that sends $l_2$ to infinity. The spheres $I(S_1)$ and $I(S_2)$ are now concentric and we have

$$
Cr(I(p_1), I(p_2), I(l_2), I(l_1)) = R_2/R_1,
$$

where $R_1$ and $R_2$ are the radii of $I(S_1)$ and $I(S_2)$. By definition, we have $\mu(S_1, S_2) = R_2/R_1$. Thus

$$
\mu(S_1, S_2) = Cr(p_1, p_2, l_2, l_1).
$$

![Figure 1. Modulus in term of cross-ratio.](image)
Remark 1. Let $P$ be a plane and $S$ a sphere disjoint from $P$ as in Figure 2. The abscissa $\lambda$ of the limit point of the pencil generated by $S$ and $P$ is $\sqrt{ab}$. Then,

$$\mu(P, S) = \text{Cr}(0, a, \lambda, -\lambda) = \frac{\sqrt{ab} + a}{\sqrt{ab} - a}.$$

![Figure 2. Modulus of a sphere and a plane.](image)

Remark 2. As a consequence, if $P$ is a plane and $S_1$ and $S_2$ are two spheres with the same radius and if $S_1$ is closer to the plane than $S_2$, then we have $\mu(P, S_1) < \mu(P, S_2)$.

Remark 3. As another consequence, if a sphere $S$ of constant radius approaches a plane $P$, without intersecting it, then the modulus of $P$ and $S$ tends to 1. Indeed, if $b - a$ is constant and $a$ tends to 0, then $\mu(P, S)$ tends to 1.

Remark 4. Let $S_1$, $S_2$ and $S_3$ be three disjoint spheres. Suppose the open 3-ball bounded by $S_2$ contains $S_3$, but is disjoint from $S_1$. Then $\mu(S_1, S_2) < \mu(S_1, S_3)$.

This can be proved by performing a conformal transformation that turns $S_1$ into a plane and computing the two cross-ratios.

1.2. The zone modulus of a link. Let $K_1$ and $K_2$ be two embedded curves in $S^3$.

Definition 3. A pair $(S_1, S_2)$ of spheres is said to be non-trivial for $K_1$ and $K_2$ if they are disjoint and if, for each sphere, there is at least one point of $K_1$ and one point of $K_2$ on it.

Definition 4. The zone modulus of $K_1$ and $K_2$ is the supremum of the moduli of all non-trivial pairs of spheres for $K_1$ and $K_2$.

The main result of [2] is the following:
Theorem 2. Two linked curves have a zone modulus greater than or equal to \((1 + \sqrt{2})^2\).

1.3. Trisecants. The following lemma is a concise rewriting of results of [2].

![Figure 3. A trisecant.](image)

Lemma 1. Let \(K_1\) and \(K_2\) be two linked curves such that \(K_1\) goes through infinity and let \(x\) be a point of \(K_2\). There exists a straight line \(L\) through \(x\) that cuts \(K_1\) in \(y \neq \infty\) and \(K_2\) again in \(z\) (see Figure 3). We call such a line a trisecant through \(x\). If the zone modulus of \(K_1\) and \(K_2\) equals \((1 + \sqrt{2})^2\), then \(y\) is the midpoint between \(x\) and \(z\) and there is no other point of intersection between \(L\) and \(K_1\) or \(K_2\).

Trisecants may be seen as a conformal version of quadrisecants for two linked curves. This subject goes back to 1933 (see Pannwitz’s work in [8]). A more modern treatment appears in Kuperberg’s paper [9] and Denne’s thesis.

2. Proof of Theorem 1

Let \(K_1\) and \(K_2\) be two linked curves. Two cases may occur:

1. For every point \(x\) on each curve, the other curve is contained in a sphere perpendicular at \(x\) to the first curve.
2. On one of the curves, say \(K_1\), there exists a point \(x_1\) such that no sphere perpendicular at \(x_1\) to \(K_1\) contains \(K_2\).

If the first case occurs, there exist two points \(x_1\) and \(x_2\) on \(K_1\) and two distinct spheres \(S_1\) and \(S_2\) containing \(K_2\) and perpendicular at \(x_1\) and \(x_2\) to \(K_1\). Thus \(K_2\) is the round circle intersection of \(S_1\) and \(S_2\). For the same reasons, \(K_1\) is also a round circle. Since \(K_1\) is perpendicular to \(S_1\) and \(S_2\), it is perpendicular to each sphere going through \(S_1 \cap S_2 = K_2\). Thus each sphere containing \(K_1\) is perpendicular to each sphere containing \(K_2\), so according to Definition 1, \(K_1\) and \(K_2\) form a Clifford link and the theorem is proved in the first case.

To conclude the proof, it is enough to prove that the second case never occurs when \(\text{modulus}(K_1, K_2) = (1 + \sqrt{2})^2\). We will suppose the contrary and show in the remainder of this section that this is impossible.
From now on, we suppose that \( \text{modulus}(K_1, K_2) = (1 + \sqrt{2})^2 \) and that there exists a point \( x_1 \) on \( K_1 \) such that no sphere perpendicular at \( x_1 \) to \( K_1 \) contains \( K_2 \). By a suitable Möbius transformation, we send \( x_1 \) to infinity and the tangent at \( x_1 \) to a vertical line. The spheres perpendicular to \( K_1 \) at \( x_1 \) are now all the horizontal planes. Then there exist two distinct horizontal planes \( P_{\text{top}} \) and \( P_{\text{bottom}} \) tangent to \( K_2 \) such that \( K_2 \) lies between these planes.

Let \( K_1 \) denote \( K_1 \setminus \infty \). Let \( x_2 \in K_2 \). By Lemma 1, there exists a trisecant \( L \) through \( x_2 \) which cuts \( K_1 \) in a point \( x_3 \) and \( K_2 \) again in a point \( x_4 \). The point \( x_3 \) is the midpoint between \( x_2 \) and \( x_4 \). The following lemma shows that \( K_2 \) is trapped between spheres in particular position with \( L \).

**Lemma 2.** Let \( c \) be the midpoint between \( x_2 \) and \( x_3 \). Let \( \Sigma \) and \( S \) be the spheres centered at \( c \) with \( \Sigma \) going through \( x_4 \) and \( S \) going through \( x_2 \) and \( x_3 \) (see Figure 4). The curve \( K_2 \) lies between \( \Sigma \) and \( S \).

**Proof.** Suppose that there exists a point \( x \) on \( K_2 \) outside the zone bounded by \( S \) and \( \Sigma \). Then \( x \) is either outside \( \Sigma \) or inside \( S \); see Figure 5. We will show that there exists a non-trivial pair of spheres of modulus strictly greater than \((1 + \sqrt{2})^2\), contradicting our assumption that \( \text{modulus}(K_1, K_2) = (1 + \sqrt{2})^2 \).

When \( x \) is outside \( \Sigma \), consider the line \( L' \) through \( c \) and \( x \) and the plane \( P' \) through \( x \) that is perpendicular to \( L' \). Since \( P' \) contains \( x_1 \in K_1 \) and \( x \in K_2 \), the pair \((S, P')\) is non-trivial. Let \( a \) and \( b \) be the two points of intersection of \( S \) with \( L' \). By Remark 1, \( \mu(S, P') \) is a function of the abscissa of \( a \) and \( b \) on \( L' \) if \( x \) marks the origin. With \( x \) outside \( \Sigma \), we have \(|b - a| < |x - a|\). Therefore, \( \mu(S, P') > (1 + \sqrt{2})^2 \).

When \( x \) is inside \( S \), consider the sphere \( S' \) through \( x \) that is tangent to \( S \) at \( x_3 \) and the plane \( P \) through \( x_4 \) that is perpendicular to \( L \). Since \( S' \) contains \( x_3 \in K_1 \) and \( x \in K_2 \), the pair \((S', P)\) is non-trivial. By Remark 4, \( \mu(S', P) > \mu(S, P) = (1 + \sqrt{2})^2 \). \( \square \)

**Corollary 1.** The curves \( K_1 \) and \( K_2 \) are perpendicular to \( L \).
Figure 5. A point $x$ of $K_2$ outside $\Sigma$ or inside $S$ exhibits a non-trivial pair of spheres whose modulus is too large.

Proof. Let $c_1$ be the midpoint between $x_2$ and $x_3$ and let $c_2$ be the midpoint between $x_3$ and $x_4$. Let $\Sigma_1$ and $S_1$ be the spheres centered at $c_1$ such that $\Sigma_1$ goes through $x_4$ and $S_1$ goes through $x_2$ and $x_3$. Let $\Sigma_2$ and $S_2$ be the spheres centered at $c_2$ such that $\Sigma_2$ goes through $x_2$ and $S_2$ goes through $x_3$ and $x_4$ (see Figure 6).

By Lemma 2, $K_2$ must lie between $\Sigma_1$ and $S_1$ and between $\Sigma_2$ and $S_2$. Therefore $K_2$ must be tangent to $S_1$ and $\Sigma_2$ at $x_2$ and tangent to $S_2$ and $\Sigma_1$ at $x_4$. Therefore $K_2$ is perpendicular to $L$.

We can now choose a Möbius transformation that keeps $L$ fixed and that exchanges $x_1$ with $x_2$. The same argument with $K_1$ and $K_2$ interchanged shows that $K_1$ is also perpendicular to $L$. □

Corollary 2. The trisecant $L$ through $x_2$ is unique.

Proof. Suppose, to the contrary, that there exists another trisecant $\tilde{L}$ through $x_2$ which cuts $\tilde{K}_1$ in $\tilde{x}_3$ and $K_2$ again in $\tilde{x}_4$. For convenience, let us work in the plane that contains $L$ and $L'$ (see Figure 7). Let $c$ be the midpoint between $x_2$ and $x_3$ and let $C$ be the circle through $x_4$ centered at $c$. By Lemma 2, $\tilde{x}_4$ lies in the interior of $C$. Therefore we have $|x_2 - \tilde{x}_4| < |x_2 - x_4|$. Analogously, if we consider $\tilde{c}$ the midpoint between $x_2$ and $x_3$ and let $\tilde{C}$ be
As a corollary, by moving the point $x_2$ on $K_2$, we can define a map $F: K_2 \to K_1$ that sends $x_2$ to $x_3$ and a map $G: K_2 \to K_2$ that sends $x_2$ to $x_4$. More precisely:

**Definition 5.** Let $x$ be any point of $K_2$. There exists a unique trisecant $L$ through $x$ that cuts $\tilde{K}_1$ and $K_2$ again. We define $F(x)$ to be the point where $\tilde{K}_1$ intersects $L$ and $G(x)$ to be the point other than $x$ where $K_2$ intersects $L$.

**Lemma 3.** The maps $F$ and $G$ are continuous.

**Proof.** Let $x \in K_2$ and let $x_n$ be a sequence of points of $K_2$, which converges to $x$. The curve $K_2$ is compact, so the sequence $y_n = G(x_n)$ has at least one point of accumulation $a$ in $K_2$. Let $y_n$ be a subsequence converging to $a$ and let $L_n$ denote the trisecant through $x_n$. These lines cut $\tilde{K}_1$ in a sequence of points $z_n = F(x_n)$. Since $z_n = (x_n + y_n)/2$, the sequence $z_n$ converges to a point $z = (x + a)/2$ of $\tilde{K}_1$. Hence there exists a line $L$ that cuts $\tilde{K}_1$ in $z$ and $K_2$ in $x$ and $a$ and that is therefore the unique trisecant through $x$. Thus, there exists only one accumulation point of the sequence $y_n$, which converges to $y = G(x)$. Therefore $G$ is continuous. Since $x_n$ and $y_n$ are both convergent, $z_n$ converges to the point $z = F(x)$, and therefore $F$ is continuous. □

**Lemma 4.** The map $G$ is a homeomorphism of $K_2$ with no fixed points such that $G \circ G(x) = x$.

**Proof.** Let $x$ and $y$ be two points of $K_2$ such that $G(x) = G(y) = z$. This means that there exists a trisecant $L$ through $x, F(x)$ and $z$, and another trisecant $L'$ through $y, F(y)$ and $z$. Since there exists only one trisecant through $z$, we must have $L = L'$. By Lemma 1, $K_2$ intersects $L$ in exactly
two distinct points. Since \( x \neq z \), we must have \( x = y \). The map \( G \) is therefore one-to-one.

Let \( x \) be a point of \( K_2 \) and \( y = G(x) \). The line through \( x \) and \( y \) is the unique trisecant through \( y \). Hence \( G(y) = x \). \( \square \)

**Lemma 5.** The curve \( K_2 \) is symmetric about a vertical line. The image \( F(K_2) \) is a segment of this line.

**Proof.** Recall that \( P_{top} \) and \( P_{bottom} \) are distinct horizontal planes that are tangent to \( K_2 \), such that \( K_2 \) lies between \( P_{top} \) and \( P_{bottom} \). Let \( t_2 \) be a point of \( K_2 \cap P_{top} \) and \( t_4 = G(t_2) \). Let \( b_2 \) be a point of \( K_2 \cap P_{bottom} \) and \( b_4 = G(b_2) \). Choose an orientation on \( K_2 \) such that \( t_2, b_2 \) and \( t_4 \) are in this order on \( K_2 \). The image by \( F \) of the arc joining \( t_2 \) to \( t_4 \) is a continuous path \( \delta \) of \( K_1 \) that contains \( F(b_2) = b_3 \). Thus \( \delta \) joins \( F(t_2) = t_3 \) to \( F(t_4) = t_3 \) through \( b_3 \). But since \( K_1 \) is a simple curve through infinity, \( \delta \) is described twice. Thus for every point \( z \in K_1 \) between \( t_3 \) and \( b_3 \) there exist at least two distinct points \( x \) and \( y \) on the arc of \( K_2 \) joining \( t_2 \) to \( t_4 \) such that \( F(x) = F(y) = z \).

Since \( G \) is orientation preserving, \( G(x) \) is on the arc of \( K_2 \) joining \( G(t_2) = t_4 \) to \( G(t_4) = t_2 \). Thus \( G(x) \neq y \). The trisecants \( L \) through \( x \) and \( z \) and \( L' \) through \( y \) and \( z \) are distinct. By Corollary 1, \( L \) and \( L' \) are perpendicular to \( K_1 \). Since the tangent to \( K_1 \) at \( x_1 \) has been chosen to be a vertical line, \( L \) and \( L' \) are horizontal. The plane containing \( L \) and \( L' \) is therefore horizontal and perpendicular to \( K_1 \) at \( z \). Thus, the tangent to \( K_1 \) at \( z \) is vertical. The arc of \( K_1 \) between \( t_3 \) and \( b_3 \) is therefore a segment of a vertical line. For any \( x \in K_2 \), the points \( x \) and \( G(x) \) are symmetric about this line since \( F(x) \) is the midpoint of \( x \) and \( G(x) \). \( \square \)

**Lemma 6.** The length between a point of \( K_2 \) and its image under \( F \) is constant.

**Proof.** Let \( \gamma(t) \) be a parametrization of \( K_2 \). We have:

\[
\frac{d}{dt}|F(\gamma(t)) - \gamma(t)|^2 = 2((F \circ \gamma)'(t) - \gamma'(t), F(\gamma(t)) - \gamma(t))
\]

By Corollary 1, \( F(\gamma(t)) - \gamma(t) \) is perpendicular to \( K_1 \) and \( K_2 \). Since \( (F \circ \gamma)'(t) \) is the tangent to \( K_1 \) and \( \gamma'(t) \) the tangent to \( K_2 \), we have

\[
\frac{d}{dt}|F(\gamma(t)) - \gamma(t)|^2 = 0.
\] \( \square \)

Let us summarize the situation: \( K_2 \) lies between two horizontal planes on a cylinder whose axis is a vertical line which coincides with \( K_1 \) in the region between the two planes (see Figure 8).

This configuration is in contradiction with Lemma 2. Indeed, the component \( K_2 \) is not contained in the interior of the sphere going through \( t_4 \) centered at the midpoint of \( t_2 \) and \( t_3 \).
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References

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