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WITH JUMPS
BY SHORT TERM ASYMPTOTICS

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1 Introduction

Option pricing models are typically tested by their ability to explain cross-section of option prices. It has long been noted that implied Black-Scholes volatilities of actual option prices vary with the strike price forming "smiles" or "smirks". This observation has led researchers and practitioners to seeking after models consistent with observed cross-sectional patterns. Stochastic volatility models that allow volatility to diffuse are able to generate a smile in implied volatility. However, one has often to assume unrealistic model parameters to be able to match actual prices of short term options.

In fact pure stochastic volatility models are restricted by the assumption that the price and its volatility follow diffusion processes only. Over a short horizon, the volatility "does not have time to change much", so, not surprisingly, the stochastic volatility does not provide enough improvement (skew) over fixed volatility in that case. An introduction of jumps is the natural way to increase flexibility of stochastic volatility models over short horizons. This is exactly the route pursued by most of the empirical literature. Stochastic volatility jump-diffusion models have been shown to be able to fit option data very well at short maturities, and can be considered as the workhorse of the recent empirical option pricing (see e.g. Andersen et al. (2002), Bates (2000), Bates (2003), Duffie et al. (2000), Pan (2002), Eraker et al. (2003), Eraker (2003)). Nevertheless calibration of a theoretical option pricing model to actual data remains a challenging task for researchers. On the technical side, for example, the inverse problem that arises in the calibration process is typically ill-posed leading to numerical instability of the algorithm (Cont and Tankov (2002)).

In this paper we propose the use of the short term asymptotics of call option prices to yield a simple calibration procedure of these option pricing models. We consider a general type of stochastic volatility jump-diffusion model with jumps only in returns (SVJ) nesting most theoretical models actually used in applications. The asymptotic approximation takes a simple analytical form and does not create numerical problems of calibration. Eraker et al. (2003) advocate an introduction of jumps in volatility. According to their empirical findings a model with contemporaneous jumps in returns and volatility (SVCJ) corrects the misspecification of the SVJ model at long maturities. Jumps in volatility serve to account for the persistent effect of rare events on stock returns, which is the scope of a long term analysis. We do not consider these more sophisticated models because jumps in volatility do not impact the short term asymptotics (see a note in the proof of Proposition 3).
The asymptotic formula can be used in several ways. First, it can be directly applied to inferring key model parameters from cross sections of option prices. These estimates can be used, for instance, to reduce the range of parameters to be inferred by the standard approach to those that affect option prices only at long maturities. The numerical analysis based on realistic model parameters shows that the asymptotic formula provides a very nice fit for the most liquid one month options. Second, inferred parameters can be used to select a proper parametric option pricing model. Their daily estimates can be plotted against daily estimates of spot volatility to visualize the underlying dependency (see the empirical section of this paper). Third, the calibration consistent with the historical dynamics of stock returns can be more easily implemented. For example, the IS-GMM approach by Pan (2002) can be reduced to simple GMM by making use of the asymptotic formula to infer spot volatilities. Fourth, the asymptotic formula can be used to test for the presence of jumps in returns. Carr and Wu (2002) show that the speed of convergence of out-of-the-money option prices to zero as the option approaches its maturity depends on the presence of a discontinuous component in the dynamics of returns. Our asymptotic results suggest that one can also deduce the type of process governing stock returns from the short term behavior of at-the-money implied volatilities. Finally the asymptotic formula can be used to fine tune or check the implementation of delicate and often unstable numerical routines (ill-posed inversion of Fourier transforms) through the comparison of approximates and true model prices.

This paper is closely related to Lewis (2000), Lee (2001), Fouque et al. (2000) where different types of asymptotics of implied volatilities are derived for the case of stochastic volatility models. The conceptual difference is that these papers deal with asymptotics with respect to some characteristics of the volatility process such as the volatility of volatility, the so-called vovol, or the time scale, whereas we study asymptotics with respect to maturity in the same spirit as Carr and Wu (2002). Lipton (2001), Berestycki et al. (2002) and Avellaneda et al. (2003) deal with asymptotics of local volatility models. We also take a slightly different approach from Carr and Wu (2002) by working directly with implied volatilities rather then option prices. We make direct use of the PDE for implied volatilities as the one derived in Ledoit et al. (2002) for the pure diffusion case. The advantage of our paper is that we are able to find a simple asymptotic formula for a very general stochastic volatility jump-diffusion specification, which seems to approximate well actual prices of short term options in practice.
The paper is organized as follows. Section 2 describes the model setup. In Section 3 we introduce a natural parameterization of option prices using the notion of normalized moneyness. In Section 4 we start developing the short term asymptotics for pure stochastic volatility models without jumps before examining the mixed jump-diffusion case. Section 5 is dedicated to a numerical study of the accuracy of the asymptotic approximation. We assess the relative and absolute performance of this approximation by comparing it with exact pricing in extensions of the Heston (1993) model with and without jumps that were calibrated in Bakshi et al. (1997), Duffie et al. (2002) and Pan (2002). An empirical application based on the asymptotic expansion is given in Section 6. Section 7 contains some concluding remarks. Appendices gather proofs and technical details.

2 Model setup

In this paper we consider a one factor jump-diffusion model. Under the risk-neutral measure, determined by market preferences, the joint dynamics of stock price and its volatility can be written as:

\[
\frac{dS_t}{S_t} = (r - \delta - \mu(\sigma_t))dt + \sigma_t dW_t^{(1)} + dJ_t, \tag{1}
\]

\[
d\sigma_t = a(\sigma_t)dt + b(\sigma_t)dW_t^{(2)},
\]

where \(W_t^{(1)}\) and \(W_t^{(2)}\) are two correlated standard Brownian motions and \(J_t\) is the Poisson jump process. The risk-free interest rate \(r\), the dividend yield \(\delta\) and the correlation \(\rho\) between the two Brownian motions are assumed constant. The expected jump size \(\mu(\sigma_t) = \lambda_t E_t(\Delta J)\) as well as the jump intensity \(\lambda_t = \lambda(\sigma_t)\) may depend on the volatility in a deterministic way but the occurrence of a jump is independent of the Brownian motions. The non-parametric time-homogeneous specification of Model (1) is general enough to host most parametric models actually used in practice.

The Black-Scholes implied volatility (or simply the implied volatility) \(I_t(K, T)\) of an option with maturity date \(T > t\) and strike price \(K > 0\) is defined as the value of the volatility parameter in the Black-Scholes formula such that the Black-Scholes price coincides
with the actual option price $C_t(K, T)$:

$$
C_t(K, T) = e^{-r(T-t)} \left[ F_t N \left( \frac{\log(F_t/K)}{I_t(K, T) \sqrt{T-t}} + \frac{I_t(K, T)}{2} \sqrt{T-t} \right) 
+ K N \left( \frac{\log(F_t/K)}{I_t(K, T) \sqrt{T-t}} - \frac{I_t(K, T)}{2} \sqrt{T-t} \right) \right],
$$

where $F_t = S_t e^{(r-\delta)(T-t)}$ denotes the forward price.

Now recall that the price of a European contingent claim is equal to the expectation of its final payoff under the risk-neutral probability measure discounted at the risk-free interest rate:

$$
C_t(K, T) = e^{-r(T-t)} E_t [S_T - K]_+ = Ke^{-r(T-t)} E_t \left[ \frac{F_T}{K} - 1 \right]_+.
$$

(2)

Since under Model (1) the dynamics of the log of the stock price (spot or forward) depends only on the volatility and the latter follows a Markov process, the expectation on the right hand side of (2) is a deterministic function of the moneyness $x_t = F_t/K$, time to maturity of the option $\tau = T-t$ and spot volatility $\sigma_t$. Hence, we can write:

$$
C_t(K, T) = Ke^{-r\tau} \varphi(x_t, \tau, \sigma_t),
$$

where $\varphi$ is a deterministic function.

As a special case, the Black-Scholes price can be also written in the form:

$$
C^{BS}(S_t, K, \sigma, T, t) = Ke^{-r\tau} \varphi^*(x_t, \tau, \sigma),
$$

where $\varphi^*$ can be easily found from the Black-Scholes formula.

Now using the definition of the implied volatility as the inverse of the Black-Scholes price viewed as a function of volatility only, we conclude that:

$$
I_t(K, T) = I(x_t, \tau; \sigma_t),
$$

where $I$ is a deterministic function.

It is important to note that the implied volatility function $I(x, \tau; \sigma)$ does not depend on
the risk-free rate and the dividend yield. Indeed, we have:

\[
\frac{F_t}{K} N \left( \frac{\log(F_t/K)}{I_t \sqrt{\tau}} + \frac{I_t}{2 \sqrt{\tau}} \right) - N \left( \frac{\log(F_t/K)}{I_t \sqrt{\tau}} - \frac{I_t}{2 \sqrt{\tau}} \right) = E \left[ \frac{F_T}{K} - 1 \right],
\]

where \( I_t \) is the implied volatility of an option under Model (1).

The result then follows by noting that the expectation on the right hand side of (3) does not depend on the risk-free rate and the dividend yield given \( x_t = F_t/K \).

## 3 Parameterization of implied volatilities

In the following we are concerned with the behavior of option prices near maturity. So before we start deriving asymptotic formulas we need to select a proper parameterization of implied volatilities. At each date, implied volatilities are generically characterized by two parameters. In the previous section we indexed option prices either by time-to-maturity and strike price or by time-to-maturity and moneyness. These two parameterizations are most frequently encountered in the empirical work on option pricing. Unfortunately, they are not convenient at all from a mathematical point of view if one studies the short term behavior of options. This point will become clearer in the next section but it is worth noting already now that implied volatilities of European options explode near maturity if the underlying model is not a pure diffusion one.

To deal with this problem, we allow moneyness to go to one, or, equivalently, the strike price to go to the spot price, as time-to-maturity approaches zero. This amounts to choosing a different parameterization of implied volatilities with one parameter being a function of both time-to-maturity and moneyness. Here we introduce a new parameter, which is called the moneyness degree and denoted by \( \theta \). As we will see shortly, there are good reasons to consider it as a more natural definition of moneyness than \( x_t = F_t/K \).

In fact, to get a workable definition of moneyness in our setting, we need to adjust the ratio of spot to strike for the time-to-maturity of an option. Let us return to the Black-Scholes model. We know that \( \log(S_T/K) \) in that model is normally distributed with mean \( \log F_t/K \) and standard deviation \( \sigma \sqrt{\tau} \) under the risk-neutral measure. Hence a natural
definition of moneyness would be:

$$\theta = \frac{\log(F_t/K)}{\sigma \sqrt{\tau}},$$

which reflects chances for the option to be of positive value at the expiration date.

Fixing $\theta$ implies that the strike price is forced to converge to the spot price as time-to-
maturity $\tau$ goes to zero. The rate of convergence is chosen in such a way that the options
indexed by $\tau$ in the sequence are similar in their "riskiness". Note, that, on the contrary,
options with the same strike price $K$ go infinitely deeper out-of- or in-the-money as maturity
approaches zero. In the rest of the paper we will use $I$ to denote the implied volatility function
corresponding to the $x$- or $\theta$-parameterizations depending on the context, hoping that this
will not cause any confusion.

Now if stochastic volatility and jumps enter the dynamics of stock returns, this de-
inition looks less appropriate. Indeed, spot volatility is no longer a meaningful characteristic of the
variance of the logarithm of the stock price over the entire lifetime of the option. Intuitively,
IMPLIED volatility should be a more reasonable candidate by the simple logic of its definition,
and one could think of replacing $\sigma$ by $I$ in (4) to get an even better definition of moneyness.
As it will become clear in the next section, these two measures of moneyness coincide asymptotically and there is no difficulty in obtaining one short term asymptotics from the other.
So we will proceed by deriving the short term asymptotic expansion of implied volatilities
corresponding to the moneyness degree defined by (4). Some simple adjustments will be
made afterwards to make it consistent with the other definition of moneyness based on $I$
instead of $\sigma$ (see Section 4.3).

4 Short term asymptotics

In this section we present the main results of the paper, namely the short term asymptotic
expansions of implied volatilities and options prices. We begin with a pure diffusion model
without jumps before turning our attention towards mixed jump-diffusion models.
4.1 Pure diffusion case

The next proposition contains our main result for implied volatilities in the pure diffusion case.

**Proposition 1** In Model (1) without jumps assume that the implied volatility as a function of log moneyness and time-to-maturity is infinitely continuously differentiable on its domain of definition, that is:

\[ I_t(\log x, \tau) \in C^\infty(-\infty, +\infty) \times [0, +\infty), \]  

then implied volatilities have the following short maturity asymptotics:

\[ I(\theta, \tau, \sigma) = \sigma + I_1(\theta, \sigma)\sqrt{\tau} + I_2(\theta, \sigma)\tau + O(\tau^{3/2}), \]

where \( I_1 \) and \( I_2 \) are functions of the moneyness degree \( \theta \) and the spot volatility \( \sigma \) only:

\[ I_1(\theta, \sigma) = \frac{\rho b \theta}{2}, \]

\[ I_2(\theta, \sigma) = \left( -\frac{5}{12} \frac{\rho^2 b^2}{\sigma} + \frac{1}{6} \frac{b^2}{\sigma} + \frac{1}{6} \rho^2 b b' \right) \theta^2 \]
\[ + \frac{a}{2} + \frac{\rho b \sigma}{4} + \frac{1}{24} \frac{\rho^2 b^2}{\sigma} + \frac{1}{12} \frac{b^2}{\sigma} - \frac{1}{6} \rho^2 b b', \]

with \( a = a(\sigma) \), \( b = b(\sigma) \) and \( b' = \partial b(\sigma)/\partial \sigma \).

**Proof.** See Appendix A. ■

In the statement of the proposition we have made the assumption that the implied volatility is "well-behaved" near maturity for any level of log moneyness \( \log x \) or, equivalently, any strike price \( K \). In particular, this induces that the implied volatility does not explode as the time-to-maturity shrinks to zero (\( K \) fixed), namely that there are no bubbles in the implied volatility. The absence of bubbles is typically assumed in the literature dealing with diffusion type market models of option prices: Schönbucher (1999) refers to it as the no-bubble constraint, Brace et al. (2001) and Brace et al. (2002) call it the feedback condition.

Proposition 1 states that the asymptotics are such that the implied volatility is equal to the spot volatility plus two correction factors whose forms are explicit functions of the moneyness degree \( \theta \) and the spot volatility \( \sigma \). The asymptotics further suggest that a stochastic
volatility model converges to the Black-Scholes model in the limit as time-to-maturity tends to zero (under proper parameterization). Besides if we limit ourselves to a first order approximation we can see that a non-zero volatility of volatility (vovol) induces a linear structure in the moneyness degree $\theta$. This structure is independent of the choice of risk-neutral measure since the volatility drift $\alpha$ does not turn up in $I_1$. This is quite intuitive. If time-to-maturity is small then the volatility "does not have time to change much", so, the volatility risk cannot have a first order effect on the option price. Note that $I_1 = 0$ when $\theta = 0$, which means that the order of convergence of at-the-money implied volatilities to the spot volatility is $\tau$ instead of $\sqrt{\tau}$ when $\theta \neq 0$.

Proposition 1 delivers asymptotics of implied volatilities, which are nothing else but prices of options quoted on the volatility scale. The next proposition shows how asymptotics of option prices themselves can be obtained from the asymptotics of implied volatilities.

**Proposition 2** Let us assume that we can write the implied volatility as:

$$I(\theta, \tau, \sigma) = \sigma + I_1(\theta, \sigma)\sqrt{\tau} + I_2(\theta, \sigma)\tau + I_3(\theta, \sigma)\tau\sqrt{\tau} + O(\tau^2)$$

for some functions $I_1$, $I_2$, $I_3$ of the moneyness degree $\theta$ and the spot volatility $\sigma$, then call option prices have the following short term asymptotics:

$$C(\theta, \sigma, \tau) = K\sigma [n + \theta N] \sqrt{\tau} +$$

$$+ K[\frac{\theta^2}{2} \sigma^2 + \frac{\theta n}{2} \sigma^2 + n I_1] \tau$$

$$+ K[\frac{\theta^3}{6} \sigma^3 + \frac{\theta^2 n}{6} \sigma^3 - \frac{\sigma^3 n}{24} \sigma^3 + \frac{n \theta^2 I_1^2}{2 \sigma} + \frac{\sigma}{2} \theta n I_1$$

$$+ n I_2 - r \sigma (n + \theta N)] \tau \sqrt{\tau} + O(\tau^2),$$

where we use $N = N(\theta)$ and $n = n(\theta)$ to abbreviate the Gaussian cdf and pdf evaluated at $\theta$.

**Proof.** See Appendix B. ■

The stated result is quite general and does not rely on any model assumption for the underlying process. The only assumption we need is that condition (7) holds, which one expects in most practical situations.
4.2 Mixed jump-diffusion case

Let us now derive the second order asymptotics for the mixed jump-diffusion model (1).

4.2.1 Constant intensity and independent jump size distribution

Let us first assume that the jump intensity is constant and the jump size distribution does not depend on \( \sigma \). In that framework it is easier to first characterize the short term asymptotics for option prices (see the proof of the next proposition for further explanations) before characterizing the short term asymptotics for implied volatilities.

**Proposition 3** Assume Model (1) with constant jump intensity and jump size distribution independent of spot volatility. If implied volatilities under the pure diffusion model (\( \mu \equiv 0 \)) satisfy (5), then option prices have the following short term asymptotics:

\[
C(\theta, \sigma, \tau) = K \sigma [n + \theta N] \sqrt{\tau} + \\
+ K \left[ \frac{\theta^2 N}{2} \sigma^2 + \frac{\theta n}{2} \sigma^2 - n I_1 + \eta \right] \tau \\
+ K \left[ \frac{\theta^3 N}{6} \sigma^3 + \frac{\theta^2 n}{6} \sigma^3 - \frac{\sigma^3 n}{24} + \frac{\mu^2 n}{2 \sigma} + \frac{\sigma n \mu}{2} + \frac{n \theta^2 I_2}{2 \sigma} \right] \tau \sqrt{\tau} \\
+ O(\tau^2),
\]

where \( \eta = \lambda E (\Delta J)_+ \) is the (unconditional) expected size of positive jump per unit of time, \( \chi = \lambda \Pr (\Delta J > 0) \) is the (unconditional) probability of positive jump per unit of time and \( I_1 \) and \( I_2 \) are the same as in Proposition 1.

**Proof.** See Appendix C. ■

Now let us compare Equation (8) with that of Proposition 2. By equalizing corresponding asymptotic terms we obtain the asymptotics of implied volatilities of the jump-diffusion model with constant intensity and jump size independent of the spot volatility. The result is summarized in Proposition 4.

**Proposition 4** Assume Model (1) with constant jump intensity and jump size distribution independent of spot volatility. If implied volatilities under the pure diffusion model (\( \mu \equiv 0 \))
satisfy (5), then the implied volatilities under the mixed model have the following short term asymptotics:

\[ I(\theta, \tau, \sigma) = \sigma + I_1(\theta, \sigma)\sqrt{\tau} + I_2(\theta, \sigma)\tau + O(\tau^{\sqrt{\tau}}), \]

where

\[ I_1 = -\frac{b\rho \theta}{2} - \mu g + \eta h, \]

\[ I_2 = -\frac{\mu^2 \theta^2}{2\sigma} g^2 - \frac{\eta^2 \theta^2}{2\sigma} h^2 + \frac{\mu \eta \theta^2}{\sigma} gh \]
\[ + \left[ -\frac{\mu b \rho \theta^3}{2\sigma} - \frac{\mu \theta \sigma}{2} - \sigma \theta \lambda \right] g \]
\[ + \left[ \frac{\eta b \rho \theta^3}{2\sigma} + \frac{\eta \theta \sigma}{2} + \sigma \theta \chi \right] h + P(\theta), \]

and \( P \) is a quadratic function of \( \theta \):

\[ P(\theta) = \left( -\frac{5}{12} \frac{\rho^2 b^2}{\sigma} + \frac{1}{6} \frac{b^2}{\sigma} + \frac{1}{6} \frac{\rho^2 b^2}{\sigma} - \frac{1}{2} \frac{\mu b \rho}{\sigma} \right) \theta^2 + \]
\[ + \frac{a}{2} + \frac{\rho b \sigma}{4} + \frac{\rho b \mu}{2\sigma} + \frac{1}{24} \frac{\rho^2 b^2}{\sigma} + \frac{1}{12} \frac{b^2}{\sigma} \]
\[ - \frac{1}{6} \frac{\rho^2 b^2}{\sigma} + \frac{\mu^2}{2\sigma} - \frac{\sigma \mu}{2} - \lambda \sigma, \]

with \( g = N(\theta)/n(\theta), h = 1/n(\theta) \).

The first order term of the asymptotic expansion suggests possible shapes of the implied volatility as a function of the moneyness near maturity. First, it is easy to check that the implied volatility is a convex function of the moneyness degree \( \left( \frac{\partial^2 I}{\partial \theta^2} \geq 0 \right) \). Further observe that \( I \) remains of order \( \sqrt{\tau} \) when \( \theta = 0 \) in the mixed jump-diffusion case, which is not the case in the pure diffusion case. The asymptotic formula suggests a simple test of the presence of jumps in returns similar to the one proposed in Carr and Wu (2002). At-the-money implied volatilities converge to the spot volatility with a speed of order \( \tau \) in the pure diffusion case (see previous section) and a speed of order \( \sqrt{\tau} \) in the mixed jump-diffusion case. Carr and Wu (2002) suggest investigating the short term behavior of at-the-money and out-of-the-money option prices. Since they study the asymptotic behavior of option prices by fixing
the strike price \(^3\), our results can be compared only for at-the-money options. Interestingly, the approach of Carr and Wu (2002) does not allow detecting the presence of jumps using at-the-money option prices (except for the pure jump case). On the contrary, we have just shown that this is in fact possible through the use of implied volatilities of options instead of their prices.

The shape of the short term implied volatilities also indicates the expected direction of jumps. Suppose that \(\mu < 0\), which means that under the risk-neutral measure a jump is expected to result in a negative shift in the price. If, in addition, a positive jump is highly unlikely (\(\eta\) is small) then the implied volatilities of short dated options should form a smirk (a decreasing convex function of moneyness) due to the term \(-\mu g(\theta)\). If, on the contrary, \(\eta\) is not negligible then the term \(\eta h(\theta)\) turns this smirk into a smile.

This observation means that if implied volatilities "smile" then the market expects that some positive news may arrive. The bigger is the smile the greater is the probability of positive news arrival. No smile (a smirk) means that no positive news are expected. This funny rule nicely summarizes the information content of the shape of implied volatilities at short maturities.

4.2.2 General case

In this section we extend the result obtained in the previous section to the general case of Model (1) when the intensity and the jump size distribution depend on the spot volatility.

**Proposition 5** In Model (1) if implied volatilities of option prices under its pure diffusion counterpart (\(\mu \equiv 0\)) satisfy (5), then the implied volatilities under the mixed model have the following short term asymptotics:

\[
I(\theta, \tau, \sigma) = \sigma + I_1(\theta, \sigma)\sqrt{\tau} + \tilde{I}_2(\theta, \sigma)\tau + O(\tau^{\sqrt{\tau}}),
\]

where

\[
\tilde{I}_2(\theta, \sigma) = I_2(\theta, \sigma) - \frac{1}{2}\rho b \sigma' + \frac{1}{2}\rho b \mu',
\]

while \(I_1\) and \(I_2\) are the same as in Proposition 3 and \(\mu' = \partial \mu(\sigma)/\partial \sigma\).

\(^3\)Recall that we fix \(\theta\), which is equivalent to forcing the strike price to converge to the spot price as time-to-maturity goes to zero.
The dependence of parameters of the jump process on volatility manifests itself only via
the derivative $\mu'$ of the correction term $\mu$. As it might have been expected, it has only the
second order effect as it was true for the volatility of volatility $b$.

4.3 Alternative parameterization of implied volatilities

As it was noted before, the definition of the moneyness degree (4) is not very well suited
to the general case of jump-diffusion process. A good candidate for a substitute for the
spot volatility is the implied volatility itself. By the logic of its definition, implied volatility
is indeed a better characterization of returns variation over the lifetime of an option than
the spot volatility. Hence, the following definition of the moneyness degree seems to be
intuitively more appealing:

$$\Theta = \log \left( \frac{F_t}{K} \right) I(K, T) \sqrt{\tau}. \quad (10)$$

Formally, we need to show that there is a one-to-one relationship with the former defini-
tion of the moneyness degree for this new parameterization to be correctly defined. From
(10), we have:

$$\Theta = \frac{\theta \sigma}{I(\theta, \tau; \sigma)} = \Phi(\theta, \tau, \sigma). \quad (11)$$

It is sufficient to show that $\Phi$ is a strictly increasing function in $\theta$. Taking the partial
derivative with respect to $\theta$, we obtain:

$$\frac{\partial \Phi}{\partial \theta} = \frac{\sigma}{I^2} \left( I - \theta \frac{\partial I}{\partial \theta} \right) = 1 + O \left( \sqrt{\tau} \right).$$

Hence we may conclude that, asymptotically for small maturities, $\Phi$ is a strictly increas-
ing function in $\theta$ and there is a one-to-one correspondence between the two measures of
moneyness.

The adoption of this new definition of the moneyness degree has unfortunately a draw-
back. We cannot use the asymptotic formula to price options since we have to know the
implied volatility before we calculate it. However, if we want to apply the asymptotics to
infer market parameters from observed option prices then this definition does not pose any
problem.

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The new characteristic of the moneyness is asymptotically equivalent to the previous one at the first order but not at the second order. Consequently, short term asymptotics corresponding to this new characteristic will be different also at the second order only. To derive the asymptotic formula under this new parameterization, we write a Taylor series expansion of the inverse function \( \theta = \Phi^{-1}(\Theta) \) (see (11)) and substitute it into the formula obtained in Proposition 4.

Taylor series expansion of \( \Phi \) for small \( \tau \) is:

\[
\Theta = \frac{\theta \sigma}{(\sigma + I_1(\Theta) \sqrt{\tau}) \sqrt{\tau}} + O(\tau) = \theta - \frac{\theta I_1(\theta)}{\sigma} \sqrt{\tau} + O(\tau)
\]

and that of the inverse function:

\[
\theta = \Theta + \frac{\Theta I_1(\Theta)}{\sigma} \sqrt{\tau} + O(\tau).
\] (12)

Substitution of (12) in the formula for the first order asymptotic term (see (9)) yields:

\[
I_1(\theta, \sigma) \sqrt{\tau} = \left( -\frac{b \rho}{2} - \mu g(\theta) - \eta h(\theta) \right) \sqrt{\tau} = I_1(\Theta, \sigma) \sqrt{\tau} + \left( -\frac{b \rho}{2} - \mu(1 + \Theta g(\Theta)) + \eta \Theta h(\Theta) \right) \frac{\Theta I_1(\Theta)}{\sigma} \sqrt{\tau} + O(\tau \sqrt{\tau})
\]

\[
= \left( -\frac{b \rho}{2} \Theta - \mu g + \eta h \right) \frac{\Theta}{\sigma} \tau + O(\tau \sqrt{\tau}),
\]

and finally,

\[
I_1(\theta, \sigma) \sqrt{\tau} = I_1(\Theta, \sigma) \sqrt{\tau} + \left( -\frac{\mu^2 \Theta^2}{\sigma} g^2 - 2 \frac{\mu \eta \Theta^2}{\sigma} g h + \frac{\eta^2 \Theta^2}{\sigma} h^2 + \frac{\mu b \rho \Theta^3}{2 \sigma} g + \frac{\mu b \rho \Theta g}{2 \sigma} + \frac{\mu^2 \Theta g}{\sigma} - \frac{\eta b \rho \Theta^3}{2 \sigma} h - \frac{\eta b \rho \Theta}{2 \sigma} h - \frac{\mu \eta \Theta}{\sigma} h + \frac{b^2 \rho^2 \Theta^2}{4 \sigma} \right) + O(\tau \sqrt{\tau}).
\] (13)

The expression in parentheses on the right hand side of (13) should now be added to the
second order term in (9) to obtain the desired expression. The result is summarized in the following Proposition.

Proposition 6 In Model (1) if implied volatilities of option prices under its pure diffusion counterpart \((\mu \equiv 0)\) satisfy (5), then the implied volatilities under the mixed model have the following short term asymptotics:

\[
I = \sigma + \left[ -\frac{b\rho \Theta}{2} - \mu g + \eta h \right] \sqrt{\tau} + \left[ \frac{\mu^2 \Theta^2}{2\sigma} g^2 + \frac{\eta^2 \Theta^2}{2\sigma} h^2 - \frac{\mu \eta \Theta}{\sigma} gh \right. \\
\left. \left( \frac{\mu \sigma}{2} - \sigma \lambda + \frac{\mu^2}{\sigma} + \frac{\mu \rho \mu}{2\sigma} \right) \Theta^2 g \right. \\
\left. + \left( \frac{\eta \sigma}{2} + \sigma \chi - \frac{\mu \eta}{\sigma} - \frac{\eta \rho}{2\sigma} \right) \Theta^2 h + P \right] \tau + O(\tau \sqrt{\tau}), \tag{14}
\]

where

\[
\Theta = \frac{\log (Se^{-(r-\delta)\tau}/K)}{I_t(K,T)\sqrt{\tau}},
\]

and

\[
P = \frac{1}{6} \left( -\frac{\rho^2 b^2}{\sigma} + \frac{b^2}{\sigma} + \rho^2 b b' \right) \Theta^2 \\
+ \frac{a}{2} + \frac{\rho b \sigma}{4} + \frac{\rho b \mu}{2\sigma} + \frac{1}{24} \frac{\rho^2 b^2}{\sigma} + \frac{1}{12} \frac{b^2}{\sigma} \\
- \frac{1}{6} \rho^2 b b' - \frac{1}{2} \rho b \mu' + \frac{\mu^2}{2\sigma} - \frac{\sigma \mu}{2} - \lambda \sigma,
\]

and \(g = N(\Theta)/n(\Theta), h = 1/n(\Theta)\).

Comparing (14) with (9) we see that these two asymptotic formulas differ significantly. To clarify this difference, let us assume that \(\eta = \chi = 0\) (jumps are always negative) and \(\mu < 0\). The leading second order term \(g^2\) determines the shape of the implied volatility approximation as a function of moneyness degree for deep in-the-money options \((\theta \text{ or } \Theta \gg 0)\). Expression (9) suggests an approximation, which is a decreasing function in \(\theta\) for \(\theta \gg 0\). On the contrary, (14) yields an approximation, which is an increasing convex function in \(\Theta\) for
Θ ≫ 0. Naturally, we expect the asymptotics based on the Θ-parameterization to exhibit better numerical accuracy.

5 Accuracy of asymptotics

In this section we examine how the results of the previous section can be applied to inferring market parameters from option prices. The success of this application depends on how well the asymptotic formula approximates theoretical prices of options under realistic model parameters and for "reasonably" short maturities. The approximation is interesting if it works well for most liquid options, that is, options with a maturity of about one month (Pan (2002)). Whether or not one month is a maturity sufficiently short for the asymptotics to be valid depends on the underlying model parameters. To study the practical accuracy of short term asymptotics, we borrow model parameters from several recent papers containing an empirical evidence on option pricing models. It means that we analyze for realistic model parameters how well the asymptotics based on the implied volatility expansion (14) approximate the implied volatility inferred from true model prices. We do not present the assessment of the accuracy based on the asymptotics corresponding to the θ-parameterization (9), since it appeared to be much worse in all cases.

We will use parameters of one-factor jump-diffusion models calibrated in Bakshi et al. (1997), Pan (2002) and Duffie et al. (2000). The latter is especially interesting since we will use the same database in the empirical section.

All three papers assume a Heston type specification of the variance process \( V_t = \sigma_t^2 \), namely:

\[
dV_t = k_t(V_t - \bar{v})dt + \sigma_v \sqrt{V_t} W_t^{(2)}. \tag{15}
\]

Using Ito’s lemma we can easily establish the relationship between specifications (15) and (1):

\[
a(\sigma) = \frac{k_t(\bar{v} - \sigma^2) - \sigma_v^2/4}{2\sigma}, \\
b(\sigma) = \frac{\sigma_v}{2}.
\]

Table 1 contains the parameter values of the stochastic volatility models both with and
without jumps (if available) calibrated in the papers mentioned above.

In Figure 1 we plot theoretical and asymptotic implied volatilities for one week and one month options based on two set of parameters obtained in Bakshi et al. (1997) and Duffie et al. (2000) for Heston model without jumps. The volatility is taken to be equal to the square root of the long-run mean of the variance, to which it reverts under the risk-neutral probability measure $\sqrt{\bar{\sigma}}$. In that figure the moneyness degree is expressed through implied standard deviations $\frac{\log(F/K)}{\sqrt{\tau}} = \Theta$. Given the natural interpretation of the moneyness degree as the number of implied standard deviations, we will consider a range between $-2$ and $2$ implied standard deviations.

In the figure we observe that the accuracy of the asymptotics depends on the model parameters. In particular, the steeper is the implied volatility smile the shorter is the range of moneyness degree where the asymptotics is accurate. Tables 2 and 3 provide a detailed information on the accuracy of the short term asymptotic approximation. For each set of parameter, there are two tables that contain errors of the approximation of model based implied volatilities and option prices. Two types of errors are reported: the absolute difference (AD) and the relative difference (RD). The latter is equal to the former divided by the actual level of approximated value.

The first observation is that relative errors in option prices for deep out-of-the money options ($\Theta \leq -1$) are quite high even for the maturity of one month (Tables 2b, 3b), whereas absolute errors are very small. This is not surprising since prices of deep out-of-the money options are close to zero, so even a small error can be huge in relative terms. The accuracy of the approximation for in-the-money option prices seems to be reasonably good for maturities up to one quarter and moneyness degree not greater than 2. The opposite pattern is observed if the accuracy of the approximation of implied volatilities is considered. The relative errors for out-of-the money options are small, whereas those for in-the-money options are rather large. Since prices of deep in-the-money options are relatively less sensitive to the implied volatility, these large errors appear to be not that large (see errors in prices). To get also an idea of the range of strike prices for which we have a reasonably good approximation, let us consider, for example, one month options. The range of moneyness degree between $-2$ and $2$ in the case of Bakshi et al. (1997) corresponds to the range of strike price (normalized by spot price) between 0.95 and 1.03. This range becomes wider as the maturity of options under consideration increases.
In Figures 2, 3, and 4 we plot theoretical and asymptotic implied volatilities for one week and one month options based on the three sets of parameters corresponding to the Heston type jump-diffusion models. In those figures the moneyness is also expressed in terms of implied standard deviations and the spot variance is taken equal to $\sqrt{\nu}$. The three sets of parameters provide an illustration of two common implied volatility patterns. Figure 2 presents an example of a "smile" in implied volatility, that is the case when the (risk-neutral) probability of a positive jump is significant. Figures 3 and 4 gather two examples of a "smirk". Tables 4, 5 and 6 provide a detailed information on the accuracy of the short term asymptotic approximation.

As it can be observed, the accuracy of this approximation crucially depends on the curvature of the implied volatility as a function of moneyness degree (compare Figures 2, 3 and 4). In the cases of Duffie et al. (2000) and Pan (2002) model parameters, the accuracy is relatively good for a broad range of moneyness degrees and option maturities. Remarkably, prices of one-month in-the-money options under Duffie et al. (2000) model parameters are approximated with the a relative error less then 1% even for the moneyness degree close to $-1.5$. In the case of Bakshi et al. (1997) model parameters, the accuracy of the approximation of in-the-money option prices is not so good at the first glance. It is poor for moneyness degree less then $-1$ even with a maturity of one month (see Figure 2 and also Table 4). This, however, appears to be not so unfortunate for practical situations since for one month options, this corresponds to strike prices smaller then 0.85 of the spot price.

6 Empirical application

In this section we provide an example of the use of short-term asymptotics to infer model parameters from a cross-section of option prices.

Our database contains implied volatilities of S&P500 index options from Ait-Sahalia and Lo (1998), covering a period of one year. The cross section of option prices for November 2, 1993 extracted from this database, was used in Duffie et al. (2000) to calibrate a stochastic volatility jump-diffusion model. Based on these estimates, we decided to reduce dimensionality of our calibration procedure by assuming that $\eta = \chi = 0$. Hopefully this will not affect estimates of the volatility of volatility, which is our main concern in this section.

In the numerical calibration we will use options with moneyness degree between $-1.5$
and 1.5 and time-to-maturity not exceeding one quarter. To reduce the noise, we skip days with less then 18 traded options satisfying above conditions or less then 3 observations per parameter. This means that we are left with 189 out of 251 days. The function to be calibrated to implied volatilities is:

\[ A(\Theta) = \alpha_0 + [\alpha_1 \Theta + \alpha_2 g(\Theta)] \sqrt{\tau} \\
+ \left[ \frac{\alpha_1^2 \Theta^2}{2\alpha_0} g^2(\Theta) + \left( \frac{\alpha_1 \alpha_0}{2} + \frac{\alpha_0 \alpha_3}{\alpha_0} + \frac{\alpha_1 \alpha_2}{2\alpha_0} \right) \theta g(\Theta) + \alpha_4 \Theta^2 + \alpha_5 \right] \tau. \]

It has six parameters and we have from (14):

\[ \alpha_0 = \sigma, \]

\[ \alpha_1 = -b\rho, \]

\[ \alpha_2 = -\mu, \]

\[ \alpha_3 = -\lambda. \]

The loss function to be minimized is the sum of squares of absolute errors in implied volatilities. Figure 5 presents plots of the four calibrated model parameters against calibrated spot volatilities. We excluded four observations with unrealistic parameter estimates (\( \lambda < 0 \) or \( \mu > 0 \)) and also two observations with extreme values of spot volatility (in excess of 0.015). Expected jump size was identified by dividing the estimate of the correction term \( \mu \) by the estimate of \( \lambda \) (\( E\Delta J \equiv \mu/\lambda \)). Each graph contains a non-parametric estimate and a quadratic fit of the underlying relationship with the volatility. Bandwidths for non-parametric regressions were chosen by cross-validation.

We observe that \( b\rho, \mu \) and \( \lambda \) clearly depend on the level of spot volatility. In particular, we see that the usual Heston specification of the vovol (\( b\rho \) is constant) is not consistent with the data. The presence of jumps in returns and a diffusion in volatility is confirmed by the strict negativity of the average values of \( \mu \) and \( b\rho \). Hence we arrive at the same conclusions as Carr and Wu (2002). The advantage of our procedure as compared to that of Carr and Wu (2002) is that we can infer much more information than just the presence of jumps, namely
the shape of jump characteristics (intensity and expected jump size). Table 7 provides the results of a quadratic fit of calibrated \( b \rho, \mu, \lambda \) and \( E \Delta J \) on calibrated spot volatility \( \sigma \). We conclude that the jump intensity can be specified as a linear function of the variance (the linear term is not statistically different from zero at the 5% level) and the expected jump size can be assumed to be independent of the volatility. This is exactly the specification adopted by Bates (2000) and Pan (2002). On the contrary, both linear and quadratic terms in the regression of \( b \rho \) on the volatility are significant, meaning that the affine specification of \( \nu \) is not justified at all.

To check the consistency of the estimates, let us compare our results on November 2, 1993 with those obtained in Duffie et al. (2000). Our estimates of \( b \rho, \mu \) and \( \lambda \) are \(-0.13, -0.011, 0.53\), respectively, whereas Duffie et al. (2000) obtained \(-0.11, -0.013, 0.11\). The first two estimated values are surprisingly very close, but the estimate of the jump intensity is notably biased upward. In fact, we expect the jump intensity to be overestimated since a high value of \( \lambda \) will somewhat compensate the excessive curvature of the asymptotic approximation (see the second order term in (14)).

Another way to check the consistency of our estimates is to infer the relationship between the calibrated \( \nu \) and the volatility of volatility induced by the dynamics of calibrated spot volatilities. First, to recover the \( \nu \) from \( b \rho \) we divide the estimates of \( b \rho \) by some assumed value of the correlation, for example \(-0.79\) \(^4\), which is the correlation parameter obtained in Duffie et al. (2000) for the Heston type jump-diffusion specification. Second, we estimate non-parametrically the diffusion parameter using the approach proposed by Stanton (1997) and the time series of spot volatility calibrated on the cross-section of option prices. In particular, we use Stanton’s second order approximation formulas for conditional mean and variance.

Figure 6 depicts the two estimates of the \( \nu \) function: the one obtained by non-parametric regression of \( \nu \) on volatility and that estimated from the volatility dynamics (diffusion estimate). A 95% pointwise confidence band for the latter was constructed using a block bootstrap with a size of blocks equal to 5. This corresponds to "the rule of thumb" proposed by Buhlmann (2002) that suggests the size of \( \sqrt{n}^{1/3} \).

As Figure 6 shows, the diffusion estimate of the \( \nu \) function appears to be consistent with the one obtained from regression if assumed correlation is not less than \(-0.9\). If we set

\(^4\)Note, that we cannot infer correlation without additional restrictions on the volatility process.
\( \rho = -0.79 \) (the estimate from Duffie et al. (2000)), the vovol inferred from option prices seems to be larger than the one inferred from the volatility dynamics. This observation is consistent with findings in Bakshi et al. (1997). Interestingly, the figure also suggests that choosing a correlation equal to its corner value \(-1\) results in a remarkable accordance between the two estimates.

7 Concluding remarks

In this paper we derived a short term asymptotic expansion of European type option prices. The asymptotic formula was shown to be reasonably accurate for a wide range of moneyness degrees and maturities of options. As an illustration of its practical relevance, we have calibrated model parameters from cross section of option prices and evaluated their dependence on spot volatility. The consistency test shows that when correlation is high, the vovol inferred from option prices accords with the one obtained from diffusion estimates using time series of spot volatility inferred from the same option prices.

The main contribution of the paper is that it provides an approximate analytical formula for option prices under a general stochastic volatility jump-diffusion specification. The formula can be applied to index options, which are typically of a European type. One of the area of future research would be to derive a similar formula based on long-term asymptotics of option prices with a view of its application to interest rate derivatives.
Appendix A. Proof of Proposition 1.

The steps of the proof are as follows. First we start with deriving a PDE for the implied volatility and then identify some functions $I_0$, $I_1$, $I_2$ in the generic asymptotic representation:

$$I(\theta, \tau, \sigma) = I_0(\theta, \sigma) + I_1(\theta, \sigma)\sqrt{\tau} + I_2(\theta, \sigma)\tau + O(\tau^{3/2}). \quad (16)$$

The coefficients $I_1$ and $I_2$ will be characterized through two second order ODEs, whose solutions are taken in the class of polynomials. Indeed, we have assumed in the statement of the proposition that implied volatility as a function of log moneyness $\log x$ and time-to-maturity $\tau$ belongs to the class of infinitely continuously differentiable functions on the domain of its definition ($I(\log x, \tau) \in C^\infty((0, +\infty) \times [0, +\infty)$). This implies that it can be represented as a Taylor series at $(0, 0)$:

$$I(\log x, \tau) = J_0 + J_1 \log x + J_2 \tau + J_{11} \log^2 x + J_{12} \log x + J_{22} \tau^2 + R, \quad (17)$$

where residual $R$ contains terms with $\tau^n \log^m x$ and $n + m \geq 3$.

To obtain short term asymptotics of the implied volatility for the $\theta$–parameterization, we need to replace $\log x = \theta\sigma\sqrt{\tau} - (r - \delta)\sigma\tau$ in (17), which yields:

$$I(\theta, \tau) = J_0 + \theta\sigma J_1 \sqrt{\tau} + (\theta^2\sigma^2 J_{11} + \theta\sigma J_{12} - (r - \delta)\sigma J_1) \tau + O(\tau^{3/2}). \quad (18)$$

Now if we compare (18) with (16), it is clear that $I_1$ should be a polynomial of degree 1 in $\theta$ (linear function) and $I_2$ a polynomial of degree 2 in $\theta$ (quadratic function). We can safely assume $r = \delta = 0$ since implied volatility as a function of moneyness, time-to-maturity and volatility is independent of the risk-free rate and the dividend yield. Ledoit et al. (2002) proved that implied volatility of at-the-money options converges to the spot volatility as time-to-maturity approaches zero. i.e. $J_0 = I_0 = \sigma$.

The PDE for the implied volatility can be obtained from the fundamental PDE for the call option price, which can be written under Model (1) with no jumps as:

$$\frac{\partial C}{\partial t} + \frac{1}{2} \frac{\partial^2 C}{\partial S^2} S^2 \sigma^2 + \frac{\partial C}{\partial S} rS + \frac{\partial C}{\partial \sigma} a(\sigma) + \frac{1}{2} \frac{\partial^2 C}{\partial \sigma^2} b^2(\sigma) + \frac{\partial^2 C}{\partial S \partial \sigma} S \sigma b(\sigma) \rho = 0. \quad (19)$$

By the definition of implied volatility, we have $C_t = C^{BS}(F_t, K, I(T - t, \sigma_t), T, t)$, where
\(C^{BS}\) denotes the Black-Scholes formula. After plugging this expression into Equation (19) the PDE for the implied volatility can be obtained after some simplifications. We skip these technical details since such a derivation has been made in Ledoit et al. (2002). Our model (without jumps in the price) is a particular case of their setting, and the PDE of the implied volatility is here:

\[
\frac{\rho \sigma}{I \sqrt{\tau}} \frac{\partial I}{\partial \sigma} d_2 - \frac{1}{2} \left( \frac{\partial I}{\partial \sigma} \right)^2 \frac{\partial^2 I}{\partial \log x \partial \sigma} b \sigma + \frac{I^2 - \sigma^2}{2 I \tau} + \frac{\partial I}{\partial \tau}
\]

\[
+ \frac{\sigma^2}{I \sqrt{\tau}} \frac{\partial I}{\partial \log x} d_2 - \frac{\rho \sigma}{I \sqrt{\tau}} \frac{\partial I}{\partial \log x} \frac{\partial I}{\partial \sigma} d_1 d_2 - \frac{1}{2} \left( \frac{\partial I}{\partial \log x} \right)^2 d_1 d_2
\]

\[
- \frac{1}{2} \sigma^2 \left( \frac{\partial^2 I}{\partial \log x^2} - \frac{\partial I}{\partial \log x} \right) - a \frac{\partial I}{\partial \sigma} - \frac{1}{2} \sigma^2 \left( \partial^2 I \right) = 0,
\]

where

\[
d_1 = \frac{\log x}{I \sqrt{\tau}} + \frac{I}{2 \sqrt{\tau}},
\]

\[
d_2 = \frac{\log x}{I \sqrt{\tau}} - \frac{I}{2 \sqrt{\tau}}.
\]

To simplify the derivation we split \(\theta\) into \(\theta = \vartheta/\sigma\) where \(\vartheta = \log x/\sqrt{\tau}\), and differentiate w.r.t \(\vartheta\) instead of \(\theta\) itself. We have successively:

\[
I = \sigma + I_1 \sqrt{\tau} + I_2 \tau + O(\tau \sqrt{\tau}),
\]

\[
I^2 = \sigma^2 + 2 \sigma I_1 \sqrt{\tau} + (I_1^2 + 2 \sigma I_2) \tau + O(\tau \sqrt{\tau}),
\]

\[
I^3 = \sigma^3 + 3 \sigma^2 I_1 \sqrt{\tau} + O(\tau),
\]

while derivatives of \(I\) are:
\[
\begin{align*}
\frac{\partial I}{\partial \log x} & = \frac{\partial I_1}{\partial \vartheta} + \frac{\partial I_2}{\partial \vartheta} \sqrt{\tau} + O(\tau), \\
\frac{\partial^2 I}{\partial \log x^2} & = \frac{\partial^2 I_1}{\partial \vartheta^2} \frac{1}{\sqrt{\tau}} + \frac{\partial^2 I_2}{\partial \vartheta^2} + O(\sqrt{\tau}), \\
\frac{\partial I}{\partial \sigma} & = 1 + \frac{\partial I_1}{\partial \sigma} \sqrt{\tau} + O(\sqrt{\tau}), \\
\frac{\partial^2 I}{\partial \log x \partial \sigma} & = \frac{\partial^2 I_1}{\partial \sigma \partial \vartheta} + O(\sqrt{\tau}), \\
\frac{\partial I}{\partial \tau} & = \left(\frac{1}{2} I_1 - \frac{1}{2} \frac{\partial I_1}{\partial \vartheta} \right) \frac{1}{\sqrt{\tau}} + I_2 - \frac{1}{2} \frac{\partial I_2}{\partial \vartheta} + O(\sqrt{\tau}).
\end{align*}
\]

Using these expressions and (20) we get after some algebra:

\[
A \frac{1}{\sqrt{\tau}} + B + O(\sqrt{\tau}) = 0,
\]

where

\[
A = \frac{\rho b \vartheta}{\sigma} + \frac{3}{2} I_1 + \frac{1}{2} \frac{\partial I_1}{\partial \vartheta} - \frac{1}{2} \sigma \frac{\partial I_2}{\partial \vartheta^2},
\]

and

\[
B = \frac{\rho b \vartheta}{\sigma} \frac{\partial I_1}{\partial \sigma} - \frac{2 \rho b \vartheta}{\sigma^2} I_1 - \frac{\rho b \sigma}{2} - \frac{b^2 \vartheta^2}{2 \sigma^3} - \rho b \frac{\partial^2 I_1}{\partial \sigma \partial \vartheta} - \frac{I_2^2}{2 \sigma} \\
- \frac{2 \vartheta I_1 \partial I_1}{\sigma \partial \vartheta} - \frac{\rho b \vartheta^2}{\sigma^2} \frac{\partial I_1}{\partial \vartheta} - \frac{\vartheta^2}{2 \sigma} \left( \frac{\partial I_1}{\partial \vartheta} \right)^2 \\
+ 2 I_2 + \frac{1}{2} \frac{\partial I_2}{\partial \vartheta} - \frac{1}{2} \sigma \frac{\partial^2 I_2}{\partial \vartheta^2} - \alpha.
\]

After setting \( A \) to zero, we arrive at the following ODE for \( I_1 \):

\[
-\frac{3}{2} I_1 - \frac{1}{2} \frac{\partial I_1}{\partial \vartheta} + \frac{1}{2} \sigma^2 \frac{\partial I_2^2}{\partial \vartheta^2} = \frac{\rho b \vartheta}{\sigma}.
\]

As already mentioned we select the linear solution:

\[
I_1(\vartheta, \sigma) = -\frac{\rho b \vartheta}{2 \sigma}.
\]

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Now setting the second asymptotic term $B$ equal to zero and taking into account (22), we get the ODE for $I_2$:

$$-2I_2 - \frac{1}{2} \vartheta \frac{\partial I_2}{\partial \vartheta} + \frac{1}{2} \sigma^2 \frac{\partial^2 I_2}{\partial \vartheta^2}$$

$$= \left( \frac{5}{4} \frac{\rho^2 b^2}{\sigma^3} - \frac{1}{2} \frac{b^2}{\sigma^3} - \frac{1}{2} \frac{\rho^2 b b'}{\sigma^2} \right) \vartheta^2 +$$

$$- a - \frac{\rho b \sigma}{2} - \frac{\rho^2 b^2}{2\sigma} + \frac{1}{2} \rho^2 b b'. $$

The quadratic solution to this ODE:

$$I_2 = \left( - \frac{5}{12} \frac{\rho^2 b^2}{\sigma^3} + \frac{1}{6} \frac{b^2}{\sigma^3} + \frac{1}{6} \frac{\rho^2 b b'}{\sigma^2} \right) \vartheta^2 +$$

$$+ \frac{a}{2} + \frac{\rho b \sigma}{4} + \frac{1}{24} \frac{\rho^2 b^2}{\sigma} + \frac{1}{12} \frac{b^2}{\sigma} - \frac{1}{6} \rho^2 b b'. $$

Finally we can use the relationship:

$$\vartheta = \sigma \theta,$$

to obtain the asymptotics in terms of $\theta$. 

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Appendix B. Proof of Proposition 2

The proof of this proposition uses the definition of implied volatility based on:

\[ C_t(\theta, T) = Ke^{-rT}(e^{\theta \sigma \sqrt{T}}N_1 - N_2), \]

where

\[ N_1 = N \left( \frac{\sigma \theta}{I} + \frac{I}{2} \sqrt{T} \right), \]

and

\[ N_2 = N \left( \frac{\sigma \theta}{I} - \frac{I}{2} \sqrt{T} \right). \]

Given the asymptotic expansion (7), we can write:

\[
\frac{\sigma \theta}{I} + \frac{I}{2} \sqrt{T} = \theta - \left( \frac{\theta I_1}{\sigma} - \frac{\sigma}{2} \right) \sqrt{T} - \left( \frac{\theta I_2}{\sigma} - \frac{I_1}{2} \right) \tau \\
- \left( \frac{\theta I_3}{\sigma} - \frac{I_2}{2} \right) \tau \sqrt{T} + O(\tau^2), \tag{23}
\]

and

\[
\frac{\sigma \theta}{I} - \frac{I}{2} \sqrt{T} = \theta - \left( \frac{\theta I_1}{\sigma} + \frac{\sigma}{2} \right) \sqrt{T} - \left( \frac{\theta I_2}{\sigma} + \frac{I_1}{2} \right) \tau \\
- \left( \frac{\theta I_3}{\sigma} + \frac{I_2}{2} \right) \tau \sqrt{T} + O(\tau^2). \]

Since the cdf of a standard normal random variable has a Taylor series expansion given by:

\[
N(y + \Delta y) = N(y) + n(y)\Delta y - \frac{1}{2} y m(y) (\Delta y)^2 + \frac{1}{6} (y^2 - 1)n(y) (\Delta y)^3 + O((\Delta y)^4).
\]

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we can write the following expansion of \( N \left( \frac{\theta}{\sqrt{\tau}} + \frac{1}{2} \sqrt{\tau} \right) \) around \( y = \theta \) based on (23):

\[
N_1 = N - n \left[ \frac{\theta I_1}{\sigma} - \frac{\sigma}{2} \sqrt{\tau} \right] \left[ \frac{\theta I_2}{\sigma} - \frac{I_1}{2} \right] + \frac{1}{2} \theta \left( \frac{\theta I_1}{\sigma} - \frac{\sigma}{2} \right)^2 \tau
\]

\[
N_2 = N - n \left[ \frac{\theta I_1}{\sigma} + \frac{\sigma}{2} \sqrt{\tau} \right] \left[ \frac{\theta I_2}{\sigma} + \frac{I_1}{2} \right] + \frac{1}{2} \theta \left( \frac{\theta I_1}{\sigma} + \frac{\sigma}{2} \right)^2 \tau
\]
where $N \equiv N(\theta)$, $n \equiv n(\theta)$. Using this result we find:

$$e^{\sigma \sqrt{\tau} N_1} = (1 + \sigma \theta \sqrt{\tau} + \frac{1}{2} \sigma^2 \theta^2 \tau + \frac{1}{6} \sigma^3 \theta^3 \tau \sqrt{\tau}) N \left( \frac{\sigma \theta}{I} + \frac{I}{2 \sqrt{\tau}} \right) + O(\tau^2) =$$

$$= N \left[ n \left( \frac{\theta I_1}{\sigma} - \frac{\sigma}{2} \right) - \sigma \theta N \right] \sqrt{\tau}$$

$$\quad - \left[ \theta n(\theta I_1 - \frac{\sigma^2}{2}) + n \left( \frac{\theta I_2}{\sigma} - \frac{I_1}{2} \right) \right] \tau$$

$$\quad + \frac{1}{2} \theta n \left( \frac{\theta I_1}{\sigma} - \frac{\sigma}{2} \right)^2 - \frac{1}{2} \sigma^2 \theta^2 N \right] \tau$$

$$\quad - \left[ \frac{1}{2} \theta^2 n(\theta I_1 \sigma - \frac{\sigma^3}{2}) + \theta n(\theta I_2 - \frac{I_1 \sigma}{2}) \right] \tau$$

$$\quad + \frac{1}{2} \theta^2 n(\theta I_1 - \frac{\sigma^2}{2})^2 + n \left( \frac{\theta I_3}{\sigma} - \frac{I_2}{2} \right)$$

$$\quad + \theta n \left( \frac{\theta I_1}{\sigma} - \frac{\sigma}{2} \right) \left( \frac{\theta I_2}{\sigma} - \frac{I_1}{2} \right)$$

$$\quad + \frac{1}{6} n(\theta^2 - 1) \left( \frac{\theta I_1}{\sigma} - \frac{\sigma}{2} \right)^3 - \frac{1}{6} \sigma^3 \theta^3 N \right] \tau \sqrt{\tau} + O(\tau^2).$$

Finally after some simple but tedious algebra we obtain:

$$CK^{-1} = e^{-r \tau} (e^{\sigma \sqrt{\tau} N_1} - N_2) =$$

$$\quad = \sigma \left[ n + \theta N \right] \sqrt{\tau} + \left[ \frac{\theta^2 N}{2} - \frac{\theta n \sigma^2 + n I_1}{\sigma} \right] \tau$$

$$\quad + \left[ \frac{\theta^3 N}{6} - \frac{\theta^2 n \sigma^3 + \frac{\sigma^3 n}{24}}{\sigma^3} \right.$$

$$\quad + \frac{n \theta^2 I_1^2}{2 \sigma^3} + \frac{\sigma}{2} \theta n I_1 + n I_2 - r \sigma (n + \theta N) \right] \tau \sqrt{\tau} + O(\tau^2).$$

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Appendix C. Proof of Proposition 3

Using the representation of the option price as an expectation of its discounted future payoff, we can write:

\[ C = e^{-r\tau} E_t (S_T - K)_+ = e^{-r\tau} \sum_{i=0}^{\infty} \Pr(i \text{ jumps}) E_t \{(S_T - K)_+ | i \text{ jumps}\}. \]

From the properties of the Poisson process we have:

\[
\begin{align*}
\Pr(\text{no jumps}) &= 1 - \lambda \tau + O(\tau^2), \\
\Pr(\text{one jump}) &= \lambda \tau + O(\tau^2), \\
\Pr(i \text{ jumps}) &= O(\tau^2) \quad \text{for } i \geq 2.
\end{align*}
\]

This means that we may ignore the possibility of multiple jumps during the lifetime of the option since we are looking for an asymptotic expansion of option prices up to \(O(\tau \sqrt{\tau})\). Using this we can write:

\[
C = e^{-r\tau} E_t (S_T - K)_+ = \lambda \tau e^{-r\tau} E_t \{(S_T - K)_+ | \text{jump}\} + (1 - \lambda \tau) e^{-r\tau} E_t \{(S_T - K)_+ | \text{no jump}\} + O(\tau^2)
\]

\[
= \lambda \tau E_t \{(S_T - K)_+ | \text{jump}\} + (1 - (\lambda + \mu) \tau) e^{-(r-\mu)\tau} E_t \{(S_T - K)_+ | \text{no jump}\} + O(\tau^2).
\]

Let us first evaluate the conditional expectation \( E_t \{(S_T - K)_+ | \text{jump}\} \) of the option payoff given that a jump occurs up to the term of order \(\sqrt{\tau}\). From (1), the log of the ratio of the price to the strike given that a jump occurs is equal to:

\[
\log \left( \frac{S_T}{K} \right) = \log \left( \frac{S_t}{K} \right) + (r - \delta - \mu) \tau + \int_t^T \sigma_s dW_s + \log(1 + \Delta J)
\]

\[
= \sigma_t \theta \sqrt{\tau} + \sigma_t (W_T - W_t) + \log(1 + \Delta J) + O(\tau).
\]
Note that \((W_T - W_t)\) is of order \(\sqrt{\tau}\). Hence:

\[
E_t \left\{ (S_T - K)_+ \mid \text{jump} \right\} = KE_t \left\{ \left( e^{\log S_T / K} - 1 \right)_+ \mid \text{jump} \right\} \\
= KE_t \left\{ \left[ (1 + \Delta J) \times \left( 1 + \sigma_t \theta \sqrt{\tau} + \sigma_t (W_T - W_t) \right) - 1 \right]_+ \mid \text{jump} \right\} + O(\tau) \\
= KE_t \left\{ \left[ \Delta J + (1 + \Delta J) \times \left( \sigma_t \theta \sqrt{\tau} + \sigma_t (W_T - W_t) \right) \right]_+ \mid \text{jump} \right\} + O(\tau) \\
= KE (\Delta J)_+ \\
+ K \sigma_t \theta (\Pr (\Delta J > 0) + E (\Delta J)_+) \sqrt{\tau} + O(\tau). \tag{25}
\]

The last equality is easy to understand intuitively: for small \(\tau\), the event \(\{(\sigma_t \theta \sqrt{\tau} + \sigma_t (W_T - W_t) + \Delta J) > 0\}\) happens "approximately" if and only if the event \(\{\Delta J > 0\}\) happens \(^5\).

The rigorous argument is the following.

\[
E_t \left\{ (S_T - K)_+ \mid \text{jump} \right\} = KE_t \left\{ \left( e^{\log S_T / K} - 1 \right)_+ \mid \text{jump} \right\} = \\
= KE_t \left\{ (\Delta J + \sqrt{\tau} (1 + \Delta J) \xi) \times 1_{\{(\Delta J + \sqrt{\tau} (1 + \Delta J) \xi) > 0\}} \mid \text{jump} \right\} + O(\tau).
\]

where \(\xi \sim N(\sigma \theta, \sigma)\) - independent of the jump. Now let us write the expectation explicitly

\(^5\) Note, that in the model with contemporaneous jump in returns and volatility there appears additional term \(\Delta \sigma (W_{t'} - W_t)\), where \(t'\) is the time of the jump and \(\Delta \sigma\) - the size of the jump in volatility. This term is also of order \(\sqrt{\tau}\) but has zero expectation (conditional on the jump occurrence), hence, using the same logic, we conclude that the introduction of a contemporaneous jump in volatility does not affect the analytical result.
\[
E \left\{ (\Delta J + \sqrt{\tau}(1 + \Delta J)\xi) \mathbf{1}_{\{\Delta J + \sqrt{\tau}(1 + \Delta J)\xi > 0\}} \right\} \\
= \int_{-1}^{\infty} f(x) \frac{1}{\sqrt{2\pi} (\sigma \sqrt{\tau})} \int_{-x/(1+x)}^{\infty} (x + (1 + x)y) e^{-\frac{1}{2} \left( \frac{(y - \sigma \sqrt{\tau})^2}{\sigma \sqrt{\tau}} \right)} dy \, dx \\
= \int_{-1}^{\infty} xf(x)N(\theta + \frac{x}{\sigma(1 + x)\sqrt{\tau}}) \, dx \\
+ \sqrt{\tau} \int_{-1}^{\infty} (1 + x)f(x)[\sigma n(\theta + \frac{x}{\sigma(1 + x)\sqrt{\tau}}) + \theta \sigma N(\theta + \frac{x}{\sigma(1 + x)\sqrt{\tau}})] \, dx, \\
\]
where \( f \) is the density of the jump size distribution defined on the domain of possible jump values between \(-1\) and \(+\infty\). \(^6\)

The first integral on the right hand side of (26) can be transformed using standard integration by parts with subsequent change of argument \( y = \frac{x}{\sigma \sqrt{\tau}} \). Denoting \( G(x) = \int_{0}^{x} sf(s)ds \) we obtain:

\[
\int_{-1}^{\infty} xf(x)N(\theta + \frac{x}{\sigma(1 + x)\sqrt{\tau}}) \, dx = G(\infty) - G(-1)N(-\infty) \\
- \frac{1}{\sigma \sqrt{\tau}} \int_{-1}^{\infty} G(x)n(\theta + \frac{x}{\sigma(1 + x)\sqrt{\tau}}) \, dx \\
= G(\infty) - \int_{-1/(\sigma \sqrt{\tau})}^{\infty} G(y\sigma \sqrt{\tau})n(\theta + \frac{y}{1 + y\sigma \sqrt{\tau}}) \, dy \\
= E(\Delta J)_+ + O(\tau).
\]

The last equality follows from the fact that \( G(0) = G'(0) = 0 \) and \( G(\infty) = E(\Delta J)_+ \). In a similar way we approximate the other two integrals on the right hand side of (26).

The other term \( e^{-(r-\mu)\tau} E_t \left\{ (S_T - K)_+ \mid \text{no jump} \right\} \) on the right hand side of (24) can be

\(^6\)Recall, that we deal with jumps in percentage points. Since stock price is always non-negative, jump cannot take value less than \(-1\).
evaluated using the asymptotics of call option price obtained for the pure diffusion case.
Indeed, conditional on no jump, we have a joint dynamics of price and volatility analogous
to the diffusion case except that \( r \) should be replaced by \( r - \mu \). We can use Propositions 1
and 2 to obtain asymptotics corresponding to a slightly different definition of moneyness:

\[
\tilde{\theta} = \frac{\log(S_t e^{(r-\delta-\mu)\tau}/K)}{\sigma \sqrt{\tau}}.
\]

That is, we have:

\[
e^{-{(r-\mu)\tau}} E_t \{(S_T - K)_+ \mid \text{no jump}\} = K \sigma \left[n + \tilde{\theta} N\right] \sqrt{\tau} + KA \tau + KB \tau \sqrt{\tau} + O(\tau^2), \tag{27}
\]

where

\[
A = \frac{\tilde{\theta}^2 \sigma^2}{2}\sigma^2 + \frac{\tilde{\theta} n}{2} \sigma^2 + n I_1,
\]

\[
B = \frac{\tilde{\theta}^3 N}{6} \sigma^3 + \frac{\tilde{\theta}^2 n}{6} \sigma^3 - \frac{\sigma^3 n}{24} + \frac{\tilde{\theta}^2 I_2^2}{2} \sigma^2 + \frac{\sigma}{2} n I_1 + n I_2 - (r - \mu) \sigma (n + \tilde{\theta} N),
\]

and \( I_1 = I_1(\tilde{\theta}) \) and \( I_2 = I_2(\tilde{\theta}) \) are the same as in Proposition 1, \( n = n(\tilde{\theta}) \), \( N = N(\tilde{\theta}) \). To
obtain the asymptotics corresponding to \( \theta \) we use the following relationship:

\[
\tilde{\theta} = \theta - \frac{\mu}{\sigma} \sqrt{\tau},
\]

which should be plugged in (27). Using a Taylor series expansion, we can write:

\[
n(\tilde{\theta}) + \tilde{\theta} N(\tilde{\theta}) = n(\theta) + \theta N(\theta) - N(\theta) \frac{\mu}{\sigma} \sqrt{\tau} + \frac{1}{2} n(\theta) \left( \frac{\mu}{\sigma} \right)^2 \tau + O(\tau \sqrt{\tau}),
\]

\[
N(\tilde{\theta}) = N(\theta) - n(\theta) \frac{\mu}{\sigma} \sqrt{\tau} + O(\tau),
\]

\[
n(\tilde{\theta}) = n(\theta) + \theta n(\theta) \frac{\mu}{\sigma} \sqrt{\tau} + O(\tau).
\]

After substitution of these expressions in (27) and collecting terms of same order, we
arrive at:

\[
e^{-\left(r-\mu\right)\tau}E_t \{ (S_T - K)_+ | \text{no jump} \}
= K\sigma \left[n + \theta N\right] \sqrt{\tau} + K A^* \tau + K B^* \tau \sqrt{\tau} + O(\tau^2),
\]  

whereby \(\eta = \lambda E(\Delta J)_+\)

\[
A^* = \frac{\theta^2 N}{2} \sigma^2 + \frac{\theta n}{2} \sigma^2 - \mu N + n I_1,
\]

\[
B^* = \frac{\theta^3 N}{6} \sigma^3 + \frac{\theta^2 n}{6} \sigma^3 - \frac{\sigma^3 n}{24} + \frac{\mu^2 n}{2\sigma} + \frac{\sigma \mu n}{2} + \frac{n \theta^2 I_1^2}{2\sigma} + \frac{\mu b\rho n}{2\sigma} + \left(\frac{\sigma}{2} + \frac{\mu}{\sigma}\right) n I_1 + n I_2 - \tau \sigma \left(n + \theta N\right),
\]

and \(I_1 = I_1(\theta), I_2 = I_2(\theta), n = n(\theta), N = N(\theta)\).

Let us now substitute (25) and (28) in (24), which yields after some reorganizing:

\[
C(\theta, \sigma, \tau) = K\sigma \left[n + \theta N\right] \sqrt{\tau}
+ K\left[\frac{\theta^2 N}{2} \sigma^2 + \frac{\theta n}{2} \sigma^2 - \mu N + n I_1 + \eta\right] \tau +
+ K \left[\frac{\theta^3 N}{6} \sigma^3 + \frac{\theta^2 n}{6} \sigma^3 - \frac{\sigma^3 n}{24} + \frac{\mu^2 n}{2\sigma} + \frac{\sigma \mu n}{2} + \frac{n \theta^2 I_1^2}{2\sigma} + \frac{\mu b\rho n}{2\sigma} \right] \tau \sqrt{\tau}
+ O(\tau^2),
\]

where we have denoted the (unconditionally) expected size of positive jump per unit of time by \(\eta = \lambda E(\Delta J)_+\) and the (unconditional) probability of positive jump per unit of time by \(\chi = \lambda \Pr(\Delta J > 0)\).
Appendix D. Proof of Proposition 5

We can safely assume \( r = \delta = 0 \) since the implied volatility as a function of moneyness, time-to-maturity and volatility is independent of the risk-free rate. The fundamental PDE for option price under the general setting of Model (1) is:

\[
\frac{\partial C}{\partial t} + \frac{1}{2} \frac{\partial^2 C}{\partial S^2} S^2 \sigma^2 + \frac{\partial C}{\partial S} r S + \frac{\partial C}{\partial \sigma} a(\sigma) + \frac{1}{2} \frac{\partial^2 C}{\partial \sigma^2} b^2(\sigma) + \frac{\partial^2 C}{\partial S \partial \sigma} S \sigma b(\sigma) \rho
\]

\[+ \lambda E [C(S + S \Delta J) - C(S)] - \frac{\partial C}{\partial S} \mu S = 0.\] (29)

Equation (29) differs from (19) by the last two terms on the left hand side. Similarly the PDE for the implied volatility in the general case differs from (20) in the following term on the left hand side:

\[
D = \left( -\lambda E [C(S + S \Delta J) - C(S)] + \frac{\partial C}{\partial S} \mu S \right) \left( \frac{\partial C_{BS}}{\partial \sigma} \right)^{-1},\] (30)

where

\[
\frac{\partial C_{BS}}{\partial \sigma} = \frac{\partial C_{BS}}{\partial \sigma} \left( F_t, K, I, \frac{F_t}{K}, T - t, \sigma, T, t \right).
\]

is the derivative of the Black-Scholes formula with respect to volatility evaluated at the corresponding implied volatility.

Let us now derive asymptotics of the additional term (30). We make use of the asymptotic expansion (6) but with \( \hat{I}_1 \) and \( \hat{I}_2 \) instead of \( I_1 \) and \( I_2 \).

First, we find that:

\[
C(S + \Delta S) = C_{BS}(S + \Delta S, K, I + \Delta I, T, t)
\]

\[= S(1 + \Delta J)N_1 - K N_2 = S \left[ (1 + \Delta J)N_1 - e^{-\theta \sigma \sqrt{T}} N_2 \right],\] (31)

where

\[
N_1 = N \left( \frac{\sigma \theta}{I + \Delta I} + \log(1 + \Delta J) \frac{1}{(I + \Delta I) \sqrt{T}} + \left( \frac{I + \Delta I}{2} \right) \right),
\]

\[
N_2 = N \left( \frac{\sigma \theta}{I + \Delta I} + \log(1 + \Delta J) \frac{1}{(I + \Delta I) \sqrt{T}} + \left( \frac{I + \Delta I}{2} \right) \right),
\]

and

\[
I + \Delta I = I \left( \theta + \frac{\log(1 + \Delta J)}{\sigma \sqrt{T}} \right).
\]

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If $\Delta J < 0$, respectfully $\Delta J > 0$, then both $N_1$ and $N_2$ converge exponentially to zero, respectfully to one. So, intuitively, when looking for asymptotics of the expectation, we may set $N_1$ and $N_2$ equal to their limits and, using (31), write:

$$\lambda E \{ C(S + \Delta S) \} = \lambda S \left[ E(\Delta J) + \Pr(\Delta J > 0) \theta \sigma \sqrt{\tau} + O(\tau) \right]$$

$$= S \left[ \eta + \chi \theta \sigma \sqrt{\tau} + O(\tau) \right].$$

(32)

This intuition is not entirely correct. However, it yields correct expression for the first order asymptotics (32). The rigorous argument is the following. Using integration by parts, we have

$$E(1 + \Delta J)N_1 = \int_{-1}^{\infty} (1 + x)f(x)N\left( \frac{\sigma \theta}{I} + \frac{\log(1 + x)}{I \sqrt{\tau}} + \frac{I \sqrt{\tau}}{2} \right) dx =$$

$$= G(\infty) - \int_{-1}^{\infty} G(x)n\left( \frac{\sigma \theta}{I} + \frac{\log(1 + x)}{I \sqrt{\tau}} + \frac{I \sqrt{\tau}}{2} \right)$$

$$\times \left( \frac{\sigma \theta \partial I}{I^2 \partial x} + \frac{\log(1 + x) \partial I}{I \sqrt{\tau} \partial x} + \frac{1}{(1 + x)I \sqrt{\tau}} + \frac{\partial I \sqrt{\tau}}{\partial x 2} \right) dx,$$

where

$$G(x) = \int_{0}^{x} (1 + s)f(s)ds,$$

$$I = I \left( \theta + \frac{\log(1 + x)}{\sigma \sqrt{\tau}} \right),$$

and $f(x)$ denotes the pdf of the jump-size distribution.

Now after the change of variables:

$$y = \frac{\log(1 + x)}{\sigma \sqrt{\tau}}$$
we arrive at

\[ E(1 + \Delta J)N_1 = G(\infty) - \int_{-1}^{\infty} G(e^{\gamma \sqrt{\tau}} - 1)n \left( \frac{\sigma \theta}{T} + \frac{\sigma y}{T} + \frac{I \sqrt{\tau}}{2} \right) \times \left( \frac{\sigma \theta}{T^2} \frac{\partial I}{\partial y} + \frac{\sigma y}{T} \frac{\partial I}{\partial y} + \frac{\sigma}{T} + \frac{\partial I}{\partial y} \frac{\sqrt{\tau}}{2} \right) dy \]

\[ = E(\Delta J)_+ + P(\Delta J > 0) - \sqrt{\tau f(0)\sigma} \int_{-1}^{\infty} yn (\theta + y) dy + O(\tau), \tag{33} \]

where we have used \( G(0) = 0 \), \( G'(0) = f(0) \) and \( \frac{\partial I}{\partial y} = \frac{\partial I}{\partial \theta} = O(\sqrt{\tau}) \).

In a similar way, we obtain:

\[ EN_2 = P(\Delta J > 0) - \sqrt{\tau f(0)\sigma} \int_{-1}^{\infty} yn (\theta + y) dy + O(\tau). \tag{34} \]

Now using (31) and expressions (33) and (34), we arrive at (32).

Proposition 2 suggests that:

\[ C(S) = K\sigma [n + \theta N] \sqrt{\tau} + O(\tau) = S\sigma [n(\theta) + \theta N(\theta)] \sqrt{\tau} + O(\tau). \tag{35} \]

The partial derivative of Black-Scholes function with respect to the volatility can be written as:

\[ \frac{\partial C_{BS}}{\partial \sigma} (I) = S\sqrt{\tau} n \left( \frac{\sigma \theta}{T} + \frac{I \sqrt{\tau}}{2} \right) \]

\[ = S\sqrt{\tau} n(\theta) \left[ 1 + \theta \left( \frac{\theta \hat{I}_1}{\sigma} - \frac{\sigma}{2} \right) \sqrt{\tau} + O(\tau) \right]. \tag{36} \]

Now putting (35), (36) and (32) together, we obtain:

\[ -\lambda E [C(S + S\Delta J) - C(S)] \left( \frac{\partial C_{BS}}{\partial \sigma} \right)^{-1} \]

\[ = -\frac{\eta}{n} \frac{1}{\sqrt{\tau}} + \frac{1}{n} \chi \theta \sigma - \theta \eta \left( \frac{\theta \hat{I}_1}{\sigma} - \frac{\sigma}{2} \right) - \sigma [n + \theta N] \right] + O(\sqrt{\tau}), \tag{37} \]
where \( n = n(\theta) \), \( N = N(\theta) \).

In a similar fashion we obtain:

\[
\left( \frac{\partial C}{\partial S} \mu S \right) \left( \frac{\partial C^{BS}}{\partial \sigma} \right)^{-1} = \mu \frac{N}{n} \frac{1}{\sqrt{\tau}} - \frac{1}{n} \left( \frac{\theta \hat{I}_1}{\sigma} - \frac{\sigma}{2} \right) (n + \theta N) + O(\sqrt{\tau}). \tag{38}
\]

Now substituting (37) and (38) in (30) yields:

\[
D = \left[ \mu \frac{N}{n} - \frac{\eta}{n} \right] \frac{1}{\sqrt{\tau}} + \frac{1}{n} \left[ \chi \theta \sigma - \theta \eta \left( \frac{\theta \hat{I}_1}{\sigma} - \frac{\sigma}{2} \right) - \sigma [n + \theta N] \right.
\]

\[
- \mu \left( \frac{\theta \hat{I}_1}{\sigma} - \frac{\sigma}{2} \right) (n + \theta N) \left. \right] + O(\sqrt{\tau}).
\]

After supplementing this term to (20) we deduce the following asymptotics of the PDE for the implied volatility:

\[
\hat{A} \frac{1}{\sqrt{\tau}} + \hat{B} + O(\sqrt{\tau}) = 0,
\]

with

\[
\hat{A} = \mu \frac{N}{n} - \frac{\eta}{n} + A,
\]

\[
\hat{B} = \frac{1}{n} \left[ (\chi + \eta/2) \theta \sigma - \eta \frac{\theta^2 \hat{I}_1}{\sigma} - \sigma [n + \theta N] - \mu \left( \frac{\theta \hat{I}_1}{\sigma} - \frac{\sigma}{2} \right) (n + \theta N) \right] + B,
\]

and where \( A, B \) are the same as in the proof of Proposition 1 except for \( \hat{I}_1 \) and \( \hat{I}_2 \) replacing \( I_1 \) and \( I_2 \).

Proceeding the same way as in the proof of Proposition 1, we derive the ODE for \( \hat{I}_1 \):

\[
- \frac{3}{2} \hat{I}_1 - \frac{1}{2} \theta \frac{\partial \hat{I}_1}{\partial \vartheta} + \sigma^2 \frac{1}{2} \frac{\partial^2 \hat{I}_1}{\partial \theta^2} = \frac{\rho \theta}{\sigma} + \frac{\mu N(\vartheta/\sigma)}{n(\vartheta/\sigma)} - \frac{\eta}{n(\vartheta/\sigma)} \tag{39}.
\]

We can easily verify that

\[
\hat{I}_1 = \overline{I}_1 = - \frac{\rho \theta}{2\sigma} - \mu \frac{N(\vartheta/\sigma)}{n(\vartheta/\sigma)} + \frac{\eta}{n(\vartheta/\sigma)} \tag{40}
\]

is the solution of (39) (see Proposition 3 and recall that \( \theta = \vartheta/\sigma \)).

The ODE for \( \hat{I}_2 \) has the same homogeneous part as in the pure diffusion case since \( \hat{I}_2 \) does
not enter $D$. However, the non-homogeneous part of this ODE in the jump-diffusion case is, of course, different. Let us denote it as $Q(\vartheta, \sigma)$ in the particular case of a jump process with time invariant parameters, the case considered in Proposition 4. The non-homogeneous part in the general case differs from $Q$ due to terms with partial derivative of $\widehat{I}_1$ with respect to $\sigma$ in $B$. Indeed, $\mu$ and $\eta$ depend on $\sigma$, so the partial derivative $\frac{\partial \mu}{\partial \sigma}$ will include $\mu' = \frac{\partial \mu}{\partial \sigma}$ and $\eta' = \frac{\partial \eta}{\partial \sigma}$.

Using the expression for $\widehat{I}_1$ (40) and the expression for $B$ (21), we obtain the ODE:

$$-2\widehat{I}_2 - \frac{1}{2} \vartheta \frac{\partial \widehat{I}_2}{\partial \vartheta} + \frac{1}{2} \sigma^2 \frac{\partial^2 \widehat{I}_2}{\partial \vartheta^2} = Q(\vartheta, \sigma) + \rho b \mu'. \quad (41)$$

From Proposition 4 we know that:

$$-2T_2 - \frac{1}{2} \vartheta \frac{\partial^2 T_2}{\partial \vartheta} + \frac{1}{2} \sigma^2 \frac{\partial^2 T_2}{\partial \vartheta^2} = Q(\vartheta, \sigma).$$

So the natural candidate for the solution to (41) is:

$$\widehat{I}_2 = T_2 - \frac{1}{2} \rho b \mu'.$$

References


Table 1  Model parameters

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<th>$\kappa_v$</th>
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$^1$ $\mu_J$ and $\sigma_J$ are mean and standard deviation of normally distributed variable log(1+$\Delta J$), where $\Delta J$ is the size of the jump.
Figure 1. Accuracy of the short term asymptotics of implied volatilities (pure diffusion models)
Parameters are taken from a) Bakshi et al. (1997); b) Duffie et al. (2000). The square of spot volatility is set equal to its long-run mean.
Table 2 Accuracy of the short term asymptotics based on parameters of Bakshi et al. (2000)

### a) implied volatilities

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Table 3 Accuracy of the short term asymptotics based on parameters of Duffie et al. (2000)

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Figure 2. Accuracy of the short term asymptotics of implied volatilities
Graphs depict implied volatilities of options and their approximations. Model parameters are taken from Bakshi et al. (1997); the square of spot volatility is set equal to its long-run mean.
Figure 3. **Accuracy of the short term asymptotics of implied volatilities**

Graphs depict implied volatilities of options and their approximations. Model parameters are taken from Pan (2002); the square of spot volatility is set equal to its long-run mean.

---

**a) one week options**

**b) one month options**
Figure 4. Accuracy of the short term asymptotics of implied volatilities
Graphs depict implied volatilities of options and their approximations. Model parameters are taken from Duffie et al. (2002); the square of spot volatility is set equal to its long-run mean.
Table 4 Accuracy of the short term asymptotics based on parameters of Bakshi et al. (2000)

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Table 5 Accuracy of the short term asymptotics based on parameters of Pan (2002)

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<td>8.255</td>
<td>0.225</td>
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<tr>
<td>½ year</td>
<td>4.003</td>
<td>17.94</td>
<td>0.468</td>
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<tr>
<td>3quarters</td>
<td>5.922</td>
<td>27.77</td>
<td>0.688</td>
<td>3.521</td>
<td>0.116</td>
<td>0.660</td>
<td>-0.026</td>
<td>-0.180</td>
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<tr>
<td>1 year</td>
<td>7.818</td>
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<td>0.150</td>
<td>0.854</td>
<td>-0.036</td>
<td>-0.248</td>
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b) option prices

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<td>RD</td>
<td>AD</td>
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<td>-0.025</td>
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<td>1.286</td>
<td>0.024</td>
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<td>-0.148</td>
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<td>-0.001</td>
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<td>½ year</td>
<td>0.613</td>
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<td>0.708</td>
<td>0.027</td>
<td>-0.180</td>
<td>-0.012</td>
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<td>-0.007</td>
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<td>1 year</td>
<td>0.660</td>
<td>1.940</td>
<td>0.244</td>
<td>0.963</td>
<td>0.042</td>
<td>-0.247</td>
<td>-0.018</td>
<td>-0.646</td>
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</tr>
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</table>

Note: AD and RD stand for the accuracy of the asymptotics based on AD and RD parameters, respectively.
Table 6 Accuracy of the short term asymptotics based on parameters of Duffie et al. (2000)

a) implied volatilities

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<td>AD</td>
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<tr>
<td>1 week</td>
<td>0.004</td>
<td>0.019</td>
<td>0.002</td>
<td>0.010</td>
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<td>0.000</td>
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<tr>
<td>1 month</td>
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<td>0.455</td>
<td>0.013</td>
<td>0.079</td>
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<td>0.021</td>
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<tr>
<td>2 month</td>
<td>0.252</td>
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<td>0.033</td>
<td>0.187</td>
<td>0.008</td>
<td>0.052</td>
<td>0.002</td>
<td>0.014</td>
<td>-0.001</td>
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<tr>
<td>1 quarter</td>
<td>0.393</td>
<td>1.823</td>
<td>0.055</td>
<td>0.307</td>
<td>0.013</td>
<td>0.085</td>
<td>0.003</td>
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<tr>
<td>½ year</td>
<td>0.785</td>
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<td>0.116</td>
<td>0.640</td>
<td>0.029</td>
<td>0.186</td>
<td>0.006</td>
<td>0.041</td>
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<tr>
<td>3 quarters</td>
<td>1.154</td>
<td>5.679</td>
<td>0.173</td>
<td>0.976</td>
<td>0.044</td>
<td>0.278</td>
<td>0.007</td>
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<tr>
<td>1 year</td>
<td>1.511</td>
<td>7.914</td>
<td>0.224</td>
<td>1.288</td>
<td>0.056</td>
<td>0.360</td>
<td>0.008</td>
<td>0.059</td>
<td>-0.018</td>
</tr>
</tbody>
</table>

b) option prices

<table>
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<tr>
<th>moneyness</th>
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<tr>
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<td>RD</td>
<td>AD</td>
<td>RD</td>
<td>AD</td>
<td>RD</td>
<td>AD</td>
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<tr>
<td>1 week</td>
<td>0.000</td>
<td>0.001</td>
<td>0.000</td>
<td>0.001</td>
<td>0.000</td>
<td>0.000</td>
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</tr>
<tr>
<td>1 month</td>
<td>0.003</td>
<td>0.024</td>
<td>0.001</td>
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<td>0.005</td>
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<td>0.003</td>
<td>0.000</td>
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<tr>
<td>2 month</td>
<td>0.016</td>
<td>0.096</td>
<td>0.002</td>
<td>0.019</td>
<td>0.001</td>
<td>0.012</td>
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<td>0.007</td>
<td>0.000</td>
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<tr>
<td>1 quarter</td>
<td>0.037</td>
<td>0.192</td>
<td>0.004</td>
<td>0.034</td>
<td>0.002</td>
<td>0.020</td>
<td>0.000</td>
<td>0.011</td>
<td>0.000</td>
</tr>
<tr>
<td>½ year</td>
<td>0.132</td>
<td>0.524</td>
<td>0.015</td>
<td>0.085</td>
<td>0.005</td>
<td>0.045</td>
<td>0.001</td>
<td>0.021</td>
<td>-0.002</td>
</tr>
<tr>
<td>3 quarters</td>
<td>0.246</td>
<td>0.851</td>
<td>0.031</td>
<td>0.146</td>
<td>0.010</td>
<td>0.069</td>
<td>0.002</td>
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<tr>
<td>1 year</td>
<td>0.359</td>
<td>1.126</td>
<td>0.049</td>
<td>0.208</td>
<td>0.014</td>
<td>0.092</td>
<td>0.003</td>
<td>0.030</td>
<td>-0.007</td>
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</table>
Figure 5. Calibration results and inferred relationships between four model parameters and spot volatility. The graph plots the results of the calibration based on the short term asymptotics, together with the non-parametric estimate and quadratic fit of the functional relationship between model parameters and volatility.
Table 7. Quadratic fits of the four relationships between model parameters and spot volatility inferred from the calibration

<table>
<thead>
<tr>
<th></th>
<th>constant</th>
<th>linear term</th>
<th>quadratic term</th>
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</thead>
<tbody>
<tr>
<td>$b_\rho$</td>
<td>-0.32*(-3.65)</td>
<td>7.63*(3.85)</td>
<td>-62.6*(-5.61)</td>
</tr>
<tr>
<td>$\mu$</td>
<td>-0.017(-1.28)</td>
<td>0.49(1.62)</td>
<td>-6.35*(-3.73)</td>
</tr>
<tr>
<td>$\lambda$</td>
<td>1.63(1.32)</td>
<td>-46.5(-1.66)</td>
<td>453.0*(2.62)</td>
</tr>
<tr>
<td>$E\Delta$</td>
<td>-0.035(-1.43)</td>
<td>0.05(0.09)</td>
<td>0.53(0.17)</td>
</tr>
</tbody>
</table>

Note: t-statistics are in parentheses. The * indicates that the parameter is significantly different from zero at the 5% confidence level.
Figure 6. Consistency check.
The graph offers a comparison of the vovol inferred from the calibration (non-parametric fit) and that estimated from the dynamics of spot volatility (non-parametric diffusion estimates), which is also inferred from the calibration.
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