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Two-particle scattering matrix of two interacting mesoscopic conductors

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We consider two quantum coherent conductors interacting weakly via long range Coulomb forces. We describe the interaction in terms of two-particle collisions described by a two-particle scattering matrix. As an example we determine the transmission probability and correlations in a two-particle scattering experiment and find that the results can be expressed in terms of the density-of-states matrices of the non-interacting scatterers.

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Recently there has been a growing interest in intriguing subjects such as quantum measurement, controlled dephasing and shot noise correlations which involve two or more separate mesoscopic conductors interacting via long range Coulomb forces. We highlight here only three recent experiments. The noise cross-correlation of two capacitively coupled quantum dots has been measured \cite{1} and the sign of this correlation was found to be tunable by the gate voltage. In another work a Mach-Zehnder interferometer \cite{2,3,4} coupled to a detector channel with shot noise was investigated both experimentally and theoretically \cite{5,6}. Ref. \cite{7} shows how the current through a quantum point contact, capacitively coupled to a quantum dot, can be used to determine the statistics of charge transferred through the quantum dot.

Several approaches have been developed to treat Coulomb coupled conductors. For systems in the tunneling limit, a particle number resolved master equation approach \cite{8,9} is often used. For systems well connected to contacts a self-consistent scattering approach has been developed and has been applied to dynamic conductance and charge fluctuations \cite{10,11,12} in good agreement with experiment \cite{13}. A momentum resolved treatment of long range Coulomb interaction is necessary to treat the Coulomb drag which one conductor exerts on another \cite{14,15}.

We are interested in a different formulation based on two-particle collision processes. We view two-particle interactions as the elementary process and derive a two-particle scattering matrix. In the weak interaction limit such an approach can be expected to have a wide range of applicability similar to Boltzmann equations with two-particle collision kernels. In contrast, in the strong interaction limit, it will be necessary to go beyond two-particle processes and permit for instance the interaction of a carrier in one conductor with a number of carriers in the other conductor \cite{16}.

Two-particle processes occur also if both particles are in the same conductor. Even in the absence of interactions, shot noise tests two-particle correlations and a predicted two-particle Aharonov-Bohm effect \cite{16} has recently been measured \cite{17}. With interaction a number of highly interesting effects have been discussed in systems with disorder \cite{18}, for conductance and pumping \cite{19} and noise \cite{20} in quantum dots. In distinction, we extend scattering theory and obtain results that are both very general yet have still an immediate physical appeal.

Interestingly the description of two-particle processes presented here involves, like the self-consistent scattering approach, a generalization of the Wigner-Smith delay time matrix \cite{21,22}. The Wigner-Smith matrix contains energy derivatives of the scattering matrix: its diagonal elements are proportional to the density of states generated by scattering states incident from a particular contact \cite{10}. Thus diagonal elements are useful to describe the piled up charges in a conductor in response to a voltage change at a contact. The off-diagonal elements of the matrix describe spontaneous charge fluctuations \cite{13,11}.

In Fig. 1 we show the type of system we are interested in. We consider two scatterers, both coupled to two non-interacting leads $i = L, R$. For simplicity we consider single channel leads. In the figure the scatterers are chaotic quantum dots (and we will refer to them as such), but our theory is valid for any scatterer. The Hamiltonians of the two (non-interacting) sub-systems read

$$H^{I} = H_{d}^{I} + H_{l}^{I} + V^{I}, \quad H^{II} = H_{d}^{II} + H_{l}^{II} + V^{II}. \quad (1)$$

The symbols I/II refer to the first/second conductor. The

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{fig1}
\caption{Two quantum dots coupled via the interaction $\lambda Q^{I}Q^{II}$. The operators $a^{\dagger I} (c^{\dagger I})$ create incoming electrons in the scattering states in the left (L) and right (R) leads of dot I (II), while $b^{\dagger I} (d^{\dagger I})$ are similar operators for outgoing electrons.}
\end{figure}
lead and dot Hamiltonians are given by

\[ H^I_1 = \sum_i \int dE \varepsilon_i |a_i(E)\rangle \langle a_i(E)|, \quad H^I_2 = \sum_i f_i^\dagger f_i \epsilon_i, \] (2)

\[ H^{II}_1 = \sum_i \int dE \varepsilon_i |c_i(E)\rangle \langle c_i(E)|, \quad H^{II}_2 = \sum_i g_i^\dagger g_i E_0, \] (3)

While \( a_i^\dagger(E) \) creates incoming carriers in lead \( i \) of the first dot, \( c_i^\dagger(E) \) is a similar operator for the second dot. Furthermore \( f_i \) (\( g_i \)) annihilates an electron with energy \( \epsilon_i \) (\( E_0 \)) in the first (second) dot. The coupling between leads and dots is described by

\[ V^I = \sum_i dE (a_i^\dagger(E) f_i W^L_i + H.c.), \] (4)

\[ V^{II} = \sum_i dE (c_i^\dagger(E) g_i W^R_i + H.c.). \] (5)

We will assume an interaction of the form

\[ H_c = \frac{\lambda}{e^2} \hat{Q} \hat{Q}^I, \] (6)

with \( \lambda \) a coupling energy and \( \hat{Q}^I = e \sum_i f_i^\dagger f_i \) (\( \hat{Q}^{II} = e \sum_i g_i^\dagger g_i \)) the charge operator on dots \( I \) (II).

Before proceeding to treat this interacting problem, we recall properties of a single non-interacting dot (here dot I). The single-particle scattering matrix \( S^I(E) \) relates operators which annihilate carriers in incoming and outgoing states, \( a_i = \sum_j S^I_{ij} a_j \). The charge fluctuations at frequency \( \omega = E - E' \) can be expressed in terms of a density of states matrix \( [10, 11] \)

\[ N^I(E, E') = S^I(E) S^I(E') - S^I(E) S^I(E') \] (7)

The diagonal elements of this matrix \( N^I_{LL} \) (and \( N^I_{RR} \)) are the injectance of the left (right) contact, i.e. the part of the total density of states associated with carriers incident in the left (right) lead. In the limit \( E \to E' \) the density of states matrix reduces to the famous Wigner-Smith delay time matrix discussed in the introduction.

We are interested in correlations generated by the Coulomb interaction between the two dots. Consider the particles leaving the two dots. The outgoing states in the outgoing states are created by operators \( b_i^\dagger(E) \) (\( d_i^\dagger(E) \)). The outgoing two-particle state \( b_i^\dagger(E_1) d_j^\dagger(E_2) \) depends on two-particle input states in all pairs of incident channels and depends on the energy of the incident carriers not only at energy \( E_1 \) and \( E_2 \) but also at \( E_1 - \omega \) and \( E_2 + \omega \) where \( \omega \) is the energy which can be exchanged in the collision process. As a consequence the relation between incoming and outgoing two-particle states is

\[ b_i(E_1) d_j(E_2) = \sum_{kl} \left[ S^I_{ik}(E_1) S^I_{jl}(E_2) a_k(E_1) c_l(E_2) + \int d\omega \delta S_{ik,jl}(E_1, E_2, E_1 - \omega, E_2 + \omega) a_k(E_1 - \omega) c_l(E_2 + \omega) \right]. \] (8)

Eq. \( 8 \) defines the two-particle scattering matrix. The first part of the relation is simply a product of the two single-particle scattering matrices, the second part contains the effect of the interaction. In this paper we will derive \( \delta S \) up to first order in the interaction energy \( \lambda \).

Let us first illustrate the derivation of the single dot scattering matrix (we choose dot I). We denote the \( n \)th eigenstate of the Hamiltonian \( H^I_1 + H^I_2 \) with energy \( E \) by \( \psi_{nE}^I \). The eigenstates are either localized in the dot or lead. Since the spectrum in the lead is continuous, matrix multiplication also involves an integral over the energy. We are interested in the eigenstate \( \psi_{nE}^I \) of \( H^I \) which approaches \( \psi_{nE}^I \) in the limit \( V^I \to 0 \). It is expressed by the Lippmann-Schwinger equation \( [23] \)

\[ |\psi_{nE}^I\rangle = |\psi_{nE}\rangle + G_{1E}^I v^I |\psi_{nE}\rangle, \] (9)

with \( G_{1E}^I = (E - H^I_1 - H^I_2 \pm i\eta)^{-1} \). Here \( \eta \) is a positive infinitesimal. It can be shown (see Ref. \( [23] \)) that \( |\psi_{nE}^I\rangle \) (\( |\psi_{nE}\rangle \)) satisfies the boundary condition for ingoing (outgoing) states. The scattering matrix relates the two and is therefore defined to be \( S^I_{nm}(E) \delta(E - E') = \langle \psi_{mE} | \psi_{nE}^I \rangle \). Using Eq. \( 9 \) one can rewrite it as

\[ S^I_{nm}(E) = \delta_{nm} - 2\pi i T^I_{nm}(E, E'), \] (10)

with transition matrix \( T^I_{nm}(E, E') = \langle \psi_{nE} | v^I | \psi_{mE}^I \rangle \). For later use we will not only be interested in the transition matrix between different lead states \( [24] \), but also between lead (I) and dot (II) states. Using the Lippmann-Schwinger equation we find

\[ T^{(II)}(E, E') = W^I [D^I(E')]^{-1} W^I, \] (11)

\[ T^{(I'd)}(E, E') = W^I [D^I(E')]^{-1} (E' - H^I_2), \] (12)

\[ T^{(I'd)}(E, E') = (E' - H^I_2) [D^I(E')]^{-1} W^I, \] (13)

with \( D^I(E) = E - H^I_1 + i\pi W^I W^I \). Combining Eqs. \( 10 \) and \( 11 \) gives the well-known scattering matrix relating incoming and outgoing lead states \( [23] \)

\[ S^I(E) = I^I - 2\pi i W^I [D^I(E)]^{-1} W^I. \] (14)

We will now turn to the case of two quantum dots. Without interaction (\( \lambda = 0 \)) the eigenstates are product states \( |\psi_{nE_1}^I \rangle \otimes |\psi_{mE_2}^I \rangle \equiv |\psi_{nE_1}^I \psi_{mE_2}^I \rangle \). Let \( |\zeta_{n,E_1,m,E_2}^\pm \rangle \) be the eigenstate of the Hamiltonian \( H^I \otimes \mathbb{1}^{II} \) \( \mathbb{1}^{II} \otimes H^{II} + H_c \) which approaches \( |\psi_{nE_1}^I \psi_{mE_2}^I \rangle \) as \( \lambda \to 0 \). It fulfills a modified Lippmann-Schwinger equation

\[ |\zeta_{n,E_1,m,E_2}^\pm \rangle = |\psi_{nE_1}^I \psi_{mE_2}^I \rangle + G_{E_1 + E_2 H_c}^I |\zeta_{n,E_1,m,E_2}^\pm \rangle, \] (15)

with \( G_{E_1}^I = (E - H^I_1 \otimes \mathbb{1}^{II} - \mathbb{1}^{II} \otimes H^{II} \otimes \mathbb{1}^{II} + i\eta)^{-1} \). The two-particle scattering matrix has matrix elements

\[ S_{nm,kl}(E_1, E_2, E_3, E_4) \delta(E_1 + E_2 - E_3 - E_4) = \langle \zeta_{n,E_1,k,E_2}^\pm | \zeta_{m,E_3,l,E_4}^\pm \rangle. \] (16)
With the help of the Lippmann-Schwinger equation, we find the interacting part
\[ \delta S_{nm,kl}(E_1, E_2, E_3, E_4) = \]
\[ -2\pi i \langle \psi_{kE_2}^{-1} \psi_{mE_3}^{-1} | H_c | \psi_{nE_1}^{1+} \psi_{vE_4}^{1+} \rangle + O(\lambda^2). \] (17)
For the last equality we have used Eq. (15) and expanded up to first order in the coupling energy \( \lambda \). Because the coupling term \( H_c \) is a direct product of operators working on dot I and dot II, we can write
\[ \langle \psi_{kE_2}^{-1} \psi_{mE_3}^{-1} | H_c | \psi_{nE_1}^{1+} \psi_{vE_4}^{1+} \rangle = \frac{\lambda}{c^2} \langle \psi_{nE_1}^{1+} | \hat{Q}^{1I} | \psi_{mE_3}^{1+} \rangle \times \]
\[ \langle \psi_{kE_2}^{-1} | \hat{Q}^{1I} | \psi_{vE_4}^{1+} \rangle. \] (18)

We have expressed everything in single dot quantities and we can use the single dot Lippmann-Schwinger equation to proceed. The operator \( \hat{Q} \) is the charge operator of the dot and therefore
\[ \langle \psi_{nE_1}^{1+} | \hat{Q}^{1I} | \psi_{mE_3}^{1+} \rangle = e \sum_k \langle \psi_{nE_1}^{1+} | \phi_{kE}^{(d)} \rangle \langle \phi_{kE}^{(d)} | \psi_{mE_3}^{1+} \rangle. \] (19)
We only sum over states in the dot as indicated by the superscript \( (d) \). Using Eq. (9) we find
\[ \langle \phi_{kE}^{(d)} | \psi_{mE_3}^{1+} \rangle = \delta_{km} + \frac{T_{1km}^1(E, E_3)}{E_3 - E + i\eta}. \] (20)
Combining Eqs. (17), (18), (19) and (20) we find that the two-particle scattering matrix between different lead states depends on the single-particle \( T \)-matrices between lead and dot. Using Eqs. (12) and (13) we calculate
\[ \delta S(E_1, E_2, E_3, E_4) = -2\pi i \frac{S_{11}(E_1) - S_{11}(E_3)}{2\pi i (E_2 - E_4)} \otimes \]
\[ \frac{S_{11}(E_2) - S_{11}(E_4)}{2\pi i (E_1 - E_3)}. \] (21)
Thus we have expressed the two-particle scattering matrix in terms of the scattering matrices of the uncoupled dots. Eq. (21) is the key result of this work. Its form reminds us of the density of states matrix Eq. (7).

The application of our two-particle scattering matrix to a transport experiment such as the experiment in Ref. 1 requires that we take into account Pauli blocking in the Fermi sea of the leads. This is not trivial and we will present our solution in a later work 21. Here we will illustrate the properties of the two-particle scattering matrix by considering a real two-particle scattering experiment. We suppose our scattering problems to be one-dimensional and denote the coordinates along the left lead of the first (second) dot by \( x \) (\( y \)). We will assume that at time \( t_0 \) a wave packet is created in each left lead, at positions \( x_0 \) and \( y_0 \). The dots are at positions \( x = 0 \) and \( y = 0 \). This means that we can describe the initial state in our system by
\[ |\Psi\rangle = \int dEdE' \alpha^I(E) \alpha^{II}(E') e^{i(k(E)x_0 + k(E')y_0)} \]
\[ e^{-i(E+E')t_0/h} a_{I}(E) c_{I}^{†}(E') |0\rangle. \] (22)
The functions \( \alpha^I(E) \) obey \( \int dE |\alpha^I(E)|^2 = 1 \), and \( k(E) > 0 \) is the wave vector corresponding to an energy \( E \).

Since we work with wave packets the timing is important: if both wave packets reach the dots at very different times they cannot interact. Under the assumption that the width of the wave packet \( 1/\delta k \) is much larger than \( v_F \tau_d \), the dot appears effectively point like and we can factorize the influence of the interaction into a contribution from the wave packet overlap and a contribution from the scattering matrices of the dots. Here \( v_F \) is the Fermi velocity and \( \tau_d \) the dwell time in the dot. In this limit it is the integral
\[ \mathcal{I} = \int dE dE' d\omega \alpha^I(E_1) \alpha^{II}(E_1 + \omega) \alpha^{II}(E_2) \alpha^{II}(E_2 - \omega) \]
\[ e^{i(k(E_1 + \omega) - k(E_1))x_0 + i(k(E_2 - \omega) - k(E_2))y_0}, \] (23)
that quantifies the overlap of the wave packets in the dot. For a Gaussian distribution \( \alpha^{II}(E) = e^{-(E - E_F)^2/(2\delta k^2)} / (2\pi\delta k^2)^{1/4} \) with \( \delta E \ll E_F \) (so that we can linearize the wave vectors around the Fermi wave vector \( k_F \)) we find \( \mathcal{I} = 2\sqrt{\pi}\delta \omega \exp \left( -(x_0 - y_0)^2/4\delta k^2 \right) \)
\[ \delta k \][1] the widths in \( k \)-space and \( \omega \)-space are related by \( \delta k = \delta E/(2\pi\delta \omega) \). With \( E_F \) the Fermi energy and \( m \) the mass. There is only an effect of the interaction if the wave packets are timed to reach the dots at about the same time, i.e. for \( |x_0 - y_0| \delta k \ll 1 \) (we assumed equal Fermi velocities). Furthermore \( \mathcal{I} \) vanishes linearly with \( \delta E \), because completely delocalized particles (plane waves) have a vanishing probability to be in the dot.

Let us first calculate the probability that the first particle leaves through the right lead, regardless of the behaviour of the second particle. We define the operator \( \hat{n}_{R}^{I} = \int dE b_{R}^{†}(E)b_{R}(E) \alpha^{II}(E) \) and we calculate
\[ \langle \hat{n}_{R}^{I} \rangle = \langle \Psi | \hat{n}_{R}^{I} | \Psi \rangle. \] (24)
We rewrite the state (22) in terms of output operators \( b^I \) and \( d^I \), using the inverse of Eq. (5). Eq. (24) now contains expectation values of the form
\[ \langle 0|d_{i_1}(E_1)b_{j_1}(E_2)b_{k_1}(E_3)d_{i_2}(E_4)|0\rangle = \delta_{n_1}(E_1 - E_3)\delta_{R_1}(E_2 - E_3) \delta_{R_2}(E_4 - E_3). \] In the limit \( \delta k \gg v_F \tau_d \) (discussed above Eq. (23)) the energy dependence of the scattering matrices can be neglected and we can evaluate all scattering matrices at the Fermi energy. We will therefore suppress the energy argument. We write
\[ \langle \hat{n}_{R}^{I} \rangle = T^{I} + \int dE \sum_p (S_{RL,p}^{1I} S_{RL,p}^{II} \delta S_{RL,pL} + S_{RL,p}^{1I} S_{RL,p}^{II} \delta S_{RL,pL}) \]
\[ = T^{I} - i\lambda \frac{\partial T^{I}}{\partial E} \hat{n}_{R}^{II}. \] (25)
Here $T^1 = |S^1_{LR}|^2$ is the transmission probability of the first scatterer. Thus due to the interaction the transmission probability of the first dot depends on the injectance of the left lead of the second dot defined in Eq. (7).

Secondly we calculate the cross-correlation between particles in the right leads

$$\langle \delta n^1_R \delta n^II_R \rangle = \frac{\lambda T}{2} \left[ \frac{\partial T^1}{\partial E} (S^II_{LR}N^II_{RL}S^1_{LL} + S^1_{LL}N^II_{LR}S^II_{RL}) + \frac{\partial T^II}{\partial E} (S^1_{LR}N^II_{RL}S^II_{LL} + S^II_{LL}N^II_{LR}S^1_{RL}) \right],$$

(26)

with $\delta n^1_{II_R} = n^1_{II_R} - \langle n^1_{II_R} \rangle$. The cross-correlation depends on the off-diagonal elements of the density of states matrix of Eq. (7). For two completely symmetric scatterers with $S_{LR} = S_{RL}$ and $S_{RR} = S_{LL}$ it disappears.

Let us work out Eq. (26) for two experimentally relevant systems. In a first example we assume that our second system is an edge state which is noisy because of the presence of a quantum point contact (QPC) left of it outside the interacting region, as shown in Fig. 2 (this is equivalent to assuming that the QPC has energy-independent transmission and reflection probabilities $T$ and $R = 1 - T$). We do not specify the first system. For an interacting region of length $L$ the density of states is $N^II = mL/2\pi\hbar^2k_F$. We find

$$\langle \delta n^1_R \delta n^II_R \rangle = -\lambda T \frac{\partial T^1}{\partial E} N^II RT.$$  

(27)

The factor $RT$ is a consequence of quantum partition of carriers at the QPC.

With the experiment of Ref. 1 in mind, we now assume that system II is a quantum dot with a Breit-Wigner resonance at energy $E_F = E^II$ (so $\partial T^II/\partial E = 0$ at $E_F$) and with rates $\gamma^II_L$ ($\gamma^II_R$) through the left (right) barriers. Conductor I is arbitrary. This gives

$$\langle \delta n^1_R \delta n^II_R \rangle = \lambda T \frac{\partial T^1}{\partial E} \frac{4\gamma^II_L\gamma^II_R}{\pi (\gamma^II_L + \gamma^II_R)^2}.$$ 

(28)

Similar to the experiment 1, the correlations Eqs. (26-28) can have different signs. The sign of the correlation depends on the asymmetry of the scattering matrix but also on the sign of the energy derivative of the transmission probability 22.

To conclude we have calculated the two-particle scattering matrix for two weakly coupled mesoscopic conductors and we have expressed it as a function of the non-interacting scattering matrices. We illustrated the properties of this matrix by calculating the transmission probability and cross-correlation for a two-particle scattering experiment. Our results can be expressed in terms of the density of states matrices of the uncoupled conductors. The approach developed here is very general and permits to treat a large class of systems.

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