Nonequilibrium, self-gravity and the fragmented interstellar medium

HUBER, Daniel

Abstract

In this thesis we study the formation of nonequilibrium structures in open self-gravitating systems, such as the fragmented interstellar medium (ISM). Our interest is particularly focused on the role of gravity. Thus, in a first project numerical simulations of self-gravitating galactic disks are carried out. We find that the competing gravitational and dissipative processes produce persistent patterns formed by transient, filamentary structures. In a second project, we numerically simulate spherical N-body systems and compare them with theoretical models. We find inconsistencies in the interval of negative heat capacity, substantiating the importance of microscopic physics and the lack of consistent theoretical tools for the description of self-gravitating gas. Finally, we carry out observations of the ISM and analyze its structure. These studies together with the numerical simulations lead to a picture of the ISM that is much more dynamical and less uniform than previously thought.

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Nonequilibrium, Self-Gravity and the Fragmented Interstellar Medium

THÈSE

présentée à la Faculté des sciences de l’Université de Genève
pour obtenir le grade de Docteur ès sciences,
mention astronomie et astrophysique

par

Daniel HUBER

de

Madiswil (BE)

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Le Doyen, Jacques WEBER
Preface

I do not care what methods a philosopher (or anybody else) may use so long as he has an interesting problem, and so long as he is sincerely trying to solve it.

Popper 1980

The Topic of this Thesis

Life is the result of an evolution from simplicity towards complexity. Thus the evolution of life is a process that is inconsistent with classical thermodynamics that predicts, according to the second law, an evolution towards homogeneity and maximum disorder. However, classical thermodynamics accounts for the evolution of closed systems and is thus not appropriate to describe the evolution of biological systems that are in general open to energy-flows.

Such as biological systems, physical systems can often not be approximated by closed systems. Indeed, energy-flows typically carry the systems away from thermodynamic equilibrium. Far from equilibrium the systems become unstable and may develop complex nonequilibrium structures with a higher degree of order.

An astrophysical system featuring a higher degree of order is the interstellar medium (ISM). Indeed, observations reveal hierarchically organized, fragmented cloud-structures from galactic disc scales down to solar system scales.

Certainly, the ISM medium is an open system. A permanent energy-flow through the scales may be maintained by large-scale energy injection due to the differential rotation of the galactic disk and small-scale viscous dissipation. However, besides the energy-flow, there is an other important physical factor that is not included in classical thermodynamics and may dominantly contribute to the structure shaping process in the ISM, namely, gravity.

Indeed, most astrophysical structures, from cosmological structures down to planets, are the result of gravitational instabilities. Yet, the structure growth during the non-linear phases of gravitational instabilities is not clearly understood.

Perhaps one of the fundamental reasons why fragmentation and structure formation involving gravitational instabilities appears so difficult is that gravity cannot consistently be included in a thermostatistical description of complex systems that is able to account for correlated motions and granular phases, properties that are typically present in dissipative, self-gravitating systems, such as the ISM.

Indeed, ignored in almost all works about ISM physics is that gravity is a non-shielded, long-range force violating the assumption of classical thermodynamics that particles exclusively interact with neighboring particles.

In this thesis we are concerned with the formation of nonequilibrium structures in open, self-gravitating systems in general and in particular in the ISM. To better understand the
nonlinear structure growth in these systems, numerical simulations are carried out. These simulations are realized with sticky particle methods in order to avoid possible consistency problems. That is, the dissipative medium is represented by \( N \) particles, that are governed by Newton’s equation of motion. The particles dissipate their energy during inelastic collisions according to a certain, phenomenological rule.

Furthermore, the structure of the ISM, from galactic disk scales down to sub-parsec scales is analyzed with the robust \( \Delta \)-variance method that allows to determine the power-law power-spectrum and to constrain the scale-range over which the structure of the ISM is statistically self-similar.

**Structure of the Thesis**

The thesis is divided up into three parts. The first part is dedicated to an general introduction. Here the object of interest, the structure of the ISM, is discussed. Furthermore, some topics that are relevant for the structure formation in self-gravitating systems and motivate the general properties of the numerical models are discussed. In the second part the numerical simulations are presented. The results of the numerical simulations were published in two articles, that are included in the thesis. The third part is dedicated to the analysis of observed ISM-structures. That is, the structures of molecular and dust regions in the the Milky Way and other galaxies are analyzed and discussed. A publication of the results of these analysis is in preparation.

The thesis has the following structure:

**Part I: General Introduction**, p. 5

1. Astrophysical Context
2. Structure Formation Outside of Equilibrium
3. Nonextensivity
4. Which Tool to Simulate Self-Gravitating Gas?

**Part II: N-Body Simulations of Dissipative, Self-Gravitating Systems**, p. 47

1. A Local Model of Self-Gravitating Disks
   - *Article I*: Lumpy Structures in Self-Gravitating Disks
2. A Numerical Study of Self-Gravitating N-Body Spheres
   - *Article II*: Long-Range Correlations in Self-Gravitating Systems

**Part III: The Structure of the Interstellar Medium:**
Observations and \( \Delta \)-Variance Analysis, p. 115

1. Abstract
2. Introduction
3. Application of the \( \Delta \)-Variance Analysis-Method

**Summary of the Thesis** (English Version), p. 159

**Résumé de la thèse** (Version française), p. 171
Abstract of the Thesis

In this thesis we study the formation of nonequilibrium structures in open, self-gravitating systems, such as the fragmented structures of the interstellar medium (ISM). Our interest is particularly focused on the role of gravity. Thus, in a first project numerical simulations of self-gravitating galactic discs are carried out. In order to increase the performance with respect to previous simulations, a new method using a time-dependent affine coordinate system is employed. Among other things, we find that in the disks persistent patterns formed by transient structures appear, whose intensity and morphological characteristic depend on the relative strengths of the competing gravitational and dissipative processes. In a second project, we numerically simulate spherical N-body systems and compare them with theoretical models. We find, in the interval of negative heat capacity equilibrium properties differing from theoretical predictions, which substantiates the importance of microphysics and the lack of consistent theoretical tools for the description of self-gravitating gas. Also, in the interval of negative specific heat, yet outside of equilibrium, the self-gravitating systems may fragment and establish long-range correlations. Finally, we carry out observations of the ISM and analyze its structure over a scale range that extends from sub-parsec scales up to galactic disk scales. These studies together with the numerical simulations lead to a picture of the ISM that is much more dynamical and less uniform than previously thought.

Acknowledgments

I feel obliged to all the employees of the Geneva observatory. The creative working atmosphere and the excellent infrastructure, that have enabled the completion of this thesis, are the result of their work.

I would like to particularly thank Prof. Daniel Pfenniger, the physicist working in the field of astrophysics and whose office door is always open to everybody, for directing me to a captivating subject, for his guidance and support, as well as for introducing me to the art of scientific writing. I appreciate very much his ability and courage to critically question generally accepted, astrophysical concepts, to point out possible inconsistencies and to draw unconventional approaches.

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I am indebted to the scientists and technicians of the First Physical Institute of the University of Cologne, who manage the KOSMA telescope at the Gornergrat by Zermatt, for introducing me to sub-mm observations of molecular clouds. Particularly, I am grateful to Dr. Volker Ossenkop and once more to Dr. Carsten Kramer for answering many questions and for the interesting discussions about molecular cloud structures. Also, I owe many thanks to Dr. Frank Bensch for providing me the Δ-variance code, for the support during its application, and for his fruitful collaboration.

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During the recent years it was a pleasure for me to work in the Galactic Dynamics Group of the Geneva Observatory, successfully run until October 1999 by Prof. Louis Martinet and since then by Prof. Daniel Pfenniger. I am grateful to all former and present day members
of the group, with whom I had the pleasure to collaborate, by name: Didier Greusard, Yves Revaz, Dr. Roger Fux, Veruska Muccione, and Dr. Daniel Friedli. I thank them for their contribution to the relaxed and inspiring atmosphere. In particular, I thank Dr. Maria Gomez for the many interesting discussions about science, politics, and life in general as well as for her friendship.

This thesis on is the result of a long process of data remodeling and information processing that was, of course, not accomplished without the so often troubled, but most frequently silently appreciated, computers. I am grateful to the entire computer team making sure that powerful numerical tools were functioning daily. Among the computer team I would like to give special thanks to Gilles Simond, Olivier Genevay, Dr. Paul Bartholdi, Dr. Denis Mégevand, Cédric Briner, and once more, Prof. Daniel Pfenniger for managing the operation of the Gravitor Beowulf cluster.

However, the thesis is not only the result of information processing, but also of nutrition processing and so I acknowledge the women of the cafeteria for preparing the lunches during the recent years.

I would like to thank my companions Marc Freitag for his friendly nature and Francesco Kienzle who takes credit for his effort in organizing weekly football matches. Thanks to Claudio Melo and Nuno Santos as well. I enjoyed the common football nights with beer and chips. Furthermore, I would like to use the opportunity to acknowledge Kilgore Trout.

Many special thanks to my parents Marlies and Toni Huber who always supported me during the long years of education and gave me the freedom to choose my way. Also, I thank my brother Thomas Huber and Emil Wächli for their help and support.

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# Contents

Preface .................................................. v
   The Topic of this Thesis ........................................ v
   Structure of the Thesis ......................................... vi
   Abstract of the Thesis .......................................... vii
   Acknowledgments ................................................ vii

I General Introduction ...................................... 5

1 Astrophysical Context ....................................... 7
   1.1 The Structure of the Interstellar Medium and Beyond .......... 7
   1.2 How Do Interstellar Clouds Form? ........................... 9
      1.2.1 Collisonal Agglomeration ................................ 11
      1.2.2 Shock-Driven Fragmentation ............................ 11
      1.2.3 Fragmentation and Phase Segregation Induced by Instabilities 13
   1.3 Star-Formation ............................................. 14
      1.3.1 Structure Properties of the Stellar Component .......... 14
      1.3.2 Star-Non-Formation .................................... 15
      1.3.3 Explanations for Star-Non-Formation .................... 16
   1.4 Gravity as an Agent of Structure-Formation ................ 17
      1.4.1 The Gravitational Model of Star-Formation .............. 17
      1.4.2 Similarity to Cosmological Structures ................. 18

2 Structure Formation Outside of Equilibrium ................ 19
   2.1 Introduction ............................................... 19
   2.2 Dynamical Systems ......................................... 20
      2.2.1 Hamiltonian Systems .................................. 20
      2.2.2 Dissipative Systems .................................. 20
      2.2.3 Summary .............................................. 21
   2.3 Chaos and Complexity ........................................ 22
      2.3.1 Introduction .......................................... 22
      2.3.2 Equation of Evolution ................................ 22
      2.3.3 Logistic Map ......................................... 23
   2.4 Nonequilibrium and Complexity ............................. 26
      2.4.1 Introduction .......................................... 26

1
III  The Structure of the Interstellar Medium:
Observations and $\Delta$-Variance Analysis

Abstract

7  Introduction

8  Application of the $\Delta$-Variance Analysis-Method

6.2.3  Why Studying Systems Subject to an Energy-Flow? 89
6.3  Article II: Long-Range Correlations in Self-Gravitating Systems 91
6.4  Complementary Discussions 111
6.4.1  Granularity and Criticality 111
6.4.2  A Possible ISM-Shaping Scenario 111
6.4.3  Comments on the Analysis Technique 113

Abstract 117

7  Introduction

7.1  The Need of a Statistical Description 119
7.2  Power-Spectrum 119
7.3  Problems of a Statistical Description 121
7.4  The Aim of this Study 122
7.4.1  Why Using the $\Delta$-Variance? 123
7.5  The $\Delta$-Variance 123
7.5.1  Principle 123
7.5.2  Correction for White Noise and Beam Smoothing 124
7.5.3  Correction for Large-Scale Trends 125
7.5.4  Error Analysis 125
7.5.5  Summary 126
7.6  Relation Between $\beta$ and Other Structure Parameters 126
7.7  Outline 127

8  Application of the $\Delta$-Variance Analysis-Method

8.1  L1512 and L1524 129
8.1.1  Sources 129
8.1.2  Observations 130
8.1.3  Results 131
8.2  The Outer Galaxy 132
8.2.1  The Data-Set 133
8.2.2  Results 134
8.3  M51 141
8.3.1  The Data-Set 142
8.3.2  Results 142
8.3.3  Discussion 148
8.4  Completion with Other Surveys 150
8.4.1  Supplementing Data-Sets 150
8.4.2  Results 151
8.4.3  Discussion 154
8.5  Conclusions 156
Summary of the Thesis (English Version)

Project 1: Lumpy Structures in Self-Gravitating Disks ........................................ 159
Project 2: Long-Range Correlations in Self-Gravitating Systems .............................. 162
Project 3: Structure Analysis Using the Δ-Variance .............................................. 166
Concluding Thoughts .............................................................................................. 166
Perspectives ............................................................................................................ 169

Résumé de la thèse (Version française)

Projet 1: Structures grumeleuses dans des disques auto-gravitants ......................... 171
Projet 2: Corrélations à longue portée dans des systèmes auto-gravitants ............. 175
Projet 3: Analyse de structure en utilisant la Δ-variance ................................... 178
Raisonnement final ................................................................................................. 180
Perspectives ............................................................................................................ 182

A Thermodynamic Limit for Power-Law Interactions ........................................... 187

B Structure Analysis Using the Δ-variance ............................................................ 189

Publications ............................................................................................................ 197

Bibliography ........................................................................................................... 199
Part I

General Introduction
Chapter 1

Astrophysical Context

1.1 The Structure of the Interstellar Medium and Beyond

In this thesis we are concerned with astrophysical non-equilibrium structures. In particular, we want to study the non-equilibrium structures of the cold interstellar medium (ISM). Here some structure properties of the ISM and related systems are discussed.

Observations of the ISM reveal fragmented structures over a scale-range of 7 dex from the Galactic disc scale down to sub-parsec scales of the order of $\sim 10^{-4}$ pc (e.g. Combes 2000).

Despite their irregular appearance the structures of the ISM seem not to be completely random but are hierarchically organized (see Fig. 1.1).

To some extent the hierarchical organization is scale-free and the structure properties can be described by characteristic power-laws. Observed power-law scaling relations suggesting a scale-free order are for instance:

- The perimeter-area relation of iso-intensity contours (e.g. Falgarone et al. 1991; Scalo 1990),

$$P \propto A^{d_{PA}/2},$$

where $P$ and $A$ are the perimeter and the area, respectively. The index $d_{PA}$ can be interpreted as a fractal dimension measuring the degree of irregularity of the iso-contours, i.e., of the clump surfaces. The analysis of intensity images mapping the ISM yields $1.2 \lesssim d \lesssim 1.5$.

- The cloud mass spectrum (e.g. Casoli et al. 1984; Stutzki & Guesten 1990; Kramer et al. 1998a),

$$dN/dM \propto M^{-\alpha}$$

where $N$ is the number of clumps with a certain mass $M$. The index $\alpha$ derived from the decomposition of cloud surveys into fragments is $1.4 \lesssim \alpha \lesssim 1.8$.

- The mass-size relation (e.g. Combes 1998; Elmegreen & Falgarone 1996),

$$M \propto R^{D_{MS}}$$

where $M$ is the mass of an interstellar region with a size of $R$. As the index of the perimeter area relation $d_{PA}$ the index of the mass-size relation $D_{MS}$ can be interpreted
as a fractal dimension. Yet, $D_{MS}$ does not describe the irregularity of clump surfaces, but measures to what degree the structure fills the embedding space. The decomposition of observed molecular structures into clumps yields, $1.6 \lesssim D_{MS} \lesssim 2.5$.

- The velocity-dispersion-size relation also referred to as Larson’s law (e.g. Larson 1981; Scalo 1985; Falgarone & Perault 1987; Myers & Goodman 1988),

$$\Delta v \propto R^{\delta_L},$$

where $0.3 \lesssim \delta_L \lesssim 0.5$ and $\Delta v$ is the spectral line width of an interstellar gas region with a size of $R$.

As stated above, these scaling-relations are typically determined after a survey of the interstellar medium has been decomposed into discrete clumps. However, often the ISM
structures are readily represented by discrete clouds and a description in terms of clumps, sheets and filaments forming a complex continuous network may be more appropriate (Scalo 1990; Green 1993).

A statistical description providing indications of a scale-free order, that can be derived without a preceding clump decomposition, is the power-spectrum of the spatial mass distribution (e.g. Green 1993; Waller et al. 1998; Bensch et al. 2001),

\[
P(k) \equiv |I_k|^2 = k^{-\beta},
\]

where \( k = |k| \) is the wave number and \( |I_k| \) is the amplitude of the Fourier mode. The power spectral index \( \beta \) derived from molecular cloud and HI surveys is \( 2 \lesssim \beta \lesssim 3.5 \).

Actually, a scale-free structure has a power-law power-spectrum, but the inverse is not valid, i.e., a power-law power-spectrum does not prove the existence of a scale-free structure, since it provides no information about the correlation of the Fourier mode phases (see chapter 7).

Despite these shortcomings (clump decomposition, incomplete statistical description) the individual analysis techniques provide important information, since they allow to study and to compare the structure of different tracers and populations, in different environments and at different scales.

For instance, a comparison of star-forming and quiescent clouds suggests that the structure is independent of the star-formation activity and the analysis of the extra-galactic system reveals structure properties similar to those found in the Galactic ISM.

Indeed, Larson (1979) found hints that the velocity-dispersion-size relation also holds for the stellar component and that it extends up to the size of galaxy clusters. Vogelaar & Wakker (1994) found power-law perimeter-area relations in high-velocity clouds; power-law power-spectra of HI emission were found in the Small and the Large Magellanic Cloud by Stanimirovic (1999, 2001), and Elmegreen et al. (2001a), respectively; measurements of the HI distribution in galaxies of the M81 cluster reveal a power-law perimeter-area scaling on the galaxy disc scale (Westpfahl et al. 1999).

On cosmic scales, up to about 100 Mpc, matter is also hierarchically organized (see Fig. 1.2). A common feature of the ISM and the cosmic structure is that the matter distribution can be characterized by comparable power-law mass-size relations. The cosmic and the interstellar index of the mass-size relations are, \( D_{\text{MS, Galaxies}} \approx 2 \pm 0.2 \) (Sylos Labini et al. 1998; Joyce et al. 1999) and \( D_{\text{MS, ISM}} \approx 1.6 - 2.5 \) (Elmegreen & Falgarone 1996; Combes 1998), respectively. That is, the precise value of the index \( D_{\text{MS}} \) does not seem to be universal, but a range between 1.5 and 2.5 appears frequent.

### 1.2 How Do Interstellar Clouds Form?

Various processes have been invoked in order to account for the structure of the ISM. Let us discuss this in turn.

The ISM covers many decades of density and temperature. Thus a wide range of physical factors and processes can be relevant for the structure of the ISM and the question arises, which of these processes contribute mainly to the ISM shaping. Let us quote some relevant factors: Energy is injected and dissipated by several processes; energy injection may be violent, thus producing shocks; magnetic fields pervade the ISM; self-gravity has to be taken
in account; supersonic turbulent motions appear; the ISM is very clumpy and cloud clumps may collide, dissolve or coalesce.

As a consequence of the wide range of physical processes at work in the ISM, various mechanisms have been proposed in order to account for the complex ISM-structure. The various mechanisms result from a different estimation of the relevance of the different physical factors. They can be divided into three general categories (for an overview see Scalo 1985; Elmegreen 1990; Blitz & Williams 1999). The general categories and some of the mechanisms thought to be important for ISM shaping are listed below.

1. Collisional agglomeration of smaller clouds

   • Random Coalescence

2. Shock-driven fragmentation
• Stellar induced fragmentation
• Turbulent fragmentation
• Fragmentation induced by Galactic density waves

3. Fragmentation or phase segregation induced by instabilities

• Gravitational instability
• Thermal instability
• Kelvin-Helmholtz instability
• Rayleigh-Taylor instability
• Magnetic Rayleigh-Taylor instability

Subsequently some of these items are briefly addressed:

1.2.1 Collisional Agglomeration

The collisional agglomeration model assumes that giant molecular clouds can result from the collision of pre-existing small molecular clouds (Oort 1954; Casoli & Combes 1982; Struck-Marcell & Scalo 1984).

Yet, Blitz & Williams (1999) claim that there is not enough mass in small molecular clouds to form the GMCs (Giant Molecular Clouds) through collisional agglomeration. Thus, they conclude that the collisional agglomeration of pre-existing molecular clouds as the main formation mechanism for GMCs seems to be untenable in any form. Some authors conjecture though that the amount of the Galactic molecular mass is underestimated because larger amounts of very cold (3K) and thus invisible molecular gas may exist (Pfenniger & Combes 1994; Pfenniger et al. 1994; Pringle et al. 2001). This may then query the latter conclusion.

A further weak point of the coalescence model concerns the time-scale of cloud formation by random collisions which may be too large compared to the cloud lifetime. Indeed, cloud dispersion due to star-formation is so effective that the time may be too short to accumulate sufficient mass for the formation of a star-forming giant molecular clouds (Elmegreen 1990).

Nevertheless, the basic point of the model, that small clouds or clumps collide and contribute thus to the formation of larger complexes must be true.

1.2.2 Shock-Driven Fragmentation

Shocks due to Spiral Density Waves and Star-Formation

Typically shocks are invoked together with instabilities in order to account for the complex network observed in the ISM. Shocks followed by instabilities may occur on different scales. For instance, spiral density waves induce large scale shocks.

Although spiral shocks induced formation of GMCs and stars seems to be indicated in many galaxies, it does not account for the gas compression, that shapes molecular clouds and leads finally to star-formation, in flocculent galaxies with weak or absent spiral arms.

On smaller scales shocks occur as a result of stellar processes. These shocks are the consequence of a primary spontaneous cloud formation. Thus, they cannot account for the formation of the first stars and molecular cloud structures in regions without stars. Nevertheless, the stimulated process, such as pressurized shell formation, are certainly at work too, and may even dominate cloud shaping in some particular regions.
Turbulence

Typically spectral line observations of the ISM show line widths that are much broader than the thermal line width, which substantiates the presence of supersonic turbulence. Thus turbulence models have been proposed in order to account for the structure of the ISM.

A turbulence model that has been invoked in order to explain some properties of the ISM is the Kolmogorov-type turbulence in which energy is injected at large scales and dissipated at small scales.

In the Kolmogorov scenario, energy, injected at large scales, cascades through an incompressible and dissipationless fluid until it is finally dissipated at smallest scales, where the cascade breaks down and energy is dissipated by viscosity. Particularly, Larson’s law has been interpreted as a sign of the Kolmogorov cascade. Indeed, a constant kinetic energy transfer $\dot{E} \propto v^2/(r/v) = \text{const.}$ yields a velocity-size relation of the form, $v \propto r^{1/3}$ (Fleck 1981; Combes 1998), which is, consistent with observations, at scales of the order of $\sim 0.1-100$ pc (e.g. Scalo 1985; Falgarone & Perault 1987; Myers 1983). However, contrary to this idealized picture, energy dissipation must occur in the real ISM also on intermediate scales, for instance in dissipative shocks of the compressible medium.

Actually, the Kolmogorov turbulence oversimplifies the physics of the ISM, however, as stated by Vázquez-Semadeni (2002) there is also an analogy, namely, in the sense that the global flow statistics is persistent, even though individual local structures are highly dynamical and transient.

Since there is no theory for dissipative compressible turbulence, numerical simulations of hydrodynamic turbulence have been carried out. These simulations suggest a picture sometimes referred to as turbulent fragmentation. Turbulent fragmentation with or without magnetic fields is currently a popular model to describe molecular cloud properties and star-formation, and often quoted in the literature. The idea of this models is that a highly dynamical network of shock compressed density fluctuations is generated by supersonic turbulence. Then, if a density fluctuation exceeds a critical mass (Jeans mass $M_\text{J}$), for instance in the intersection of two shock fronts, gravitational contraction sets in, which may lead to star-formation (e.g., Klessen et al. 2000; Klessen 2001).

Since, energy dissipation in shocks is very efficient, so that supersonic turbulence decays quickly (Field 1978). Classically Magnetic fields have been invoked in order to maintain the supersonic motion. The idea was that dissipative shocks would be transformed to non-dissipative Alfvén waves. However, numerical simulations suggest that despite the presence of strong magnetic fields, the decay time of turbulence is probably always far less than the free-fall time in molecular clouds (e.g., Mac Low 1999; Padoan & Nordlund 1999; Ostriker et al. 1999).

Thus, in order to maintain the turbulent structures over several dynamical times supersonic motion must be continually and strongly driven. Several energy sources which may have the potential to permanently drive turbulence have been invoked (see Mac Low 1999, for a discussion of different driving mechanisms):

- Differential rotation of galactic discs
- Turbulent, chaotic motion driven by gravitational collapses
- Ionizing radiation, winds and supernovae of massive stars
- Jets and outflows of protostars
• Spiral density waves

1.2.3 Fragmentation and Phase Segregation Induced by Instabilities

Gravitational Instabilities

The most straightforward mechanism that explains the large HI complexes on the kpc scale is the gravitational instability. Indeed, the mass and sizes of giant spiral arm cloud complexes correspond to the Jeans mass \( \approx 10^7 M_\odot \) and Jeans length \( \approx 2 \text{ kpc} \), respectively (Elmegreen 1979, 1994; Elmegreen & Elmegreen 1983).

Scale-free gravity may also explain the scale-free nature and the small scale structure of the ISM. Indeed, the contraction of a marginally Jeans-unstable isothermal cloud would lead to a reduction of the critical mass for instability \( M_J \propto \rho^{-1/2} \) and hence to the contraction of smaller regions within the original cloud (Hoyle 1953). The process continues recursively within the smaller fragments, leading to a hierarchical, fragmented structure obeying power-law relations (e.g. di Fazio 1986).

The fragmentation stops when the clumps become so dense that they are opaque to the heat radiation generated by their contraction. The break-down of the isothermal conditions determines the mass of the smallest clumps resulting from gravitational fragmentation, \( M_{\text{min}} = 3kTR/G\mu m_p \) (Rees 1976). Here \( k \) is the Boltzmann constant, \( T \) is the temperature, \( R \) the radius, \( \mu \) is the mean molecular weight and \( m_p \) is the proton mass.

At \( T = 3 \) K the minimum mass of a gas clump consisting of hydrogen and helium is \( M_{\text{min}} \approx 0.8 - 2.7 \times 10^{-3} M_\odot \), which is of the order of a Jupiter mass (Pfenniger & Combes 1994).

However, gravitational fragmentation is chaotic and thus sensitive to perturbations. Consequently one should expect that the properties of the clumps that result from recursive fragmentation are described by broader spectra.

Thermal Instabilities and Phase Segregation

Numerical studies suggest that thermal instabilities leading to a discrete phase segregation are inhibited in the ISM and that it is a second order effect in self-gravitating systems (Vázquez-Semadeni et al. 2000). Furthermore, in numerical simulations, the structures of the ISM are highly dynamical and transient, so that discrete pressure-confined phases cannot survive.

Thus Vázquez-Semadeni (2002) concludes that a discrete multi-phase medium in pressure equilibrium, as proposed for instance by Field et al. (1969) and McKee & Ostriker (1977), does not appropriately describe the turbulent continuum flow of the ISM.

Magnetic Instabilities

Elmegreen (1982a) and Elmegreen (1982b) claimed that the magnetic Rayleigh-Taylor instability (i.e. Parker instability, Parker 1966) dominates gravitational instability at the galactic disc scales in regions with densities lower than the mean spiral arm densities. Yet, even in such large-scale, low-density regions the gravitational energy-reservoir dominates the magnetic energy-reservoir suggesting that magnetic instabilities are only second order effects (Pfenniger 1996).
1.2.4 Summary

Actually a multitude of physical processes and factors are at work in the ISM. Yet, observations and simulations suggest:

- Coalescence is actually present in the dynamical ISM, but cannot account alone for the formation of molecular clouds and is rather a consequence of a more general driving process than a primary shaping agent.

- Magnetic and thermal instabilities are second order effects.

- Magnetic fields cannot prolongate the lifetime of shock-type, dissipative turbulence and consequently not retard the collapse of a gravitationally unstable cloud.

- Star-formation conditions a preceding gas compression. Thus star-formation is not a primary shaping process. Furthermore, molecular cloud structures of star-forming and quiescent regions are similar, substantiating that star-formation is not the dominant shaping agent.

There remain two primary physical agents that may be dominantly responsible for the ISM shaping, namely, gravity and shock-type turbulence. We refer to the role of gravity as a shaping-agent in Sect. 1.4.

1.3 Star-Formation

Stars form in dense regions of molecular clouds. They are the result of a gas compression process that starts at galactic disk scales and shapes the ISM throughout the scales. That is, a primary cloud shaping process precedes star-formation. Thus, the study of the ISM shaping mechanism provides the key for the understanding of the star-formation rate and the initial mass function (IMF).

Subsequently, the close relationship between star-formation and the ISM is pointed out. Furthermore, the star-formation rate and the molecular cloud lifetime are discussed.

1.3.1 Structure Properties of the Stellar Component

Some structure properties of the stellar component resemble those of the ISM and substantiate the close relationship between the ISM-shaping process and star-formation. Some of these properties are listed below:

- A scale free organization of star-forming regions is found on the kpc-scale by Elmegreen et al. (2001b) and Elmegreen & Elmegreen (2001). The authors find a fractal dimension, deduced from the luminosity size relation, that is consistent with the fractal dimension (derived from the mass-size relation) of the interstellar medium. On the pc-scale self-similar clustering of pre-main-sequence stars in molecular clouds were found by Nakajima et al. (1998) and Larson (1995). Yet, the latter non-equilibrium structures are destroyed after some dynamical times due to chaotic mixing as soon as the dissipative gas has been evacuated from the cluster by the ongoing star-formation process (Kandrup 2001; Miller 1964).
• Both, the mass spectrum of globular clusters and interstellar molecular clouds can be represented by a power law, \( dN/dM = M^{-\alpha} \), where \( \alpha \approx -1.6, \ldots, -2 \) (Elmegreen & Falgarone 1996; Heithausen et al. 1998). On smaller scales the Salpeter slope of the stellar initial mass function (IMF) is a little steeper, \( \alpha \approx -2.35 \) (Salpeter 1955).

1.3.2 Star-Non-Formation

Here we show that star-formation in molecular clouds is inefficient. Thus an understanding of the cloud shaping process demands also an explanation of the inefficiency of star-formation, i.e., of star-non-formation. This problem is addressed subsequently.

Star-Formation Rate

Since molecular clouds are the sites of star-formation, the star-formation rate (SFR) can be derived from the amount of available molecular gas. A simple estimation shows that star-formation is inefficient.

Indeed, the reservoir of molecular mass and the star-formation rate SFR are related as follows:

\[
SFR = \frac{\eta_{SF} M_{H_2}}{\tau_{SF}},
\]

where \( \eta_{SF} \) is the star-formation efficiency, that is, the total mass that forms stars per total molecular cloud mass, and \( \tau_{SF} \) is the star-formation time-scale.

If one takes the total mass of \( H_2 \) observed in the Galaxy \( M(H_2) \approx 2 \times 10^9 \, M_\odot \), assumes a star-formation efficiency of \( \eta_{SF} \approx 1 \) and a star-formation time-scale of \( \tau_{SF} \approx \tau_{dyn} \approx 4 \times 10^6 \, \text{y} \), where \( \tau_{dyn} \) is the dynamical time, one obtains an estimated star-formation rate of \( \approx 500 \, M_\odot/\text{y} \), which exceeds the current observed star-formation rate by a factor of about 100 (Scalo 1986; Evans 1999; Pringle et al. 2001). The discrepancy increases even more if one assumes that there are large amounts of unseen dense molecular gas.

Therefore, the star-formation efficiency and/or the star-formation time-scale have to be adjusted in order to be consistent with the observed star-formation rate. That is, star-formation must be inefficient and/or prolonged.

Lifetime of Molecular Clouds

Estimates of the molecular cloud dispersal time and the internal clump collision time suggest a star-formation time-scale of \( \tau_{SF} \approx 3 \times 10^7 \, \text{y} \) (Bash et al. 1977; Blitz & Shu 1980; Blitz 1993). Assuming such a time-scale, the star-formation efficiency is \( \eta_{SF} \approx 0.1 \) and the lifetime of the star-forming cloud is at least a factor of \( \approx 10 \) larger than the free fall time \( \tau_{ff} \sim \tau_{dyn} \). Thus, magnetic fields and driving sources have classically been invoked in order to maintain turbulence and thus to support molecular clouds against gravitational collapse.

However, Elmegreen (2000) questions this classical scenario and suggests that star-formation appears to go from start to finish in only one or two dynamical times. These conclusions hold for a dynamical range of \( \approx 3 \, \text{dex} \) and are based on a verity of recent observations, which are here briefly summarized:

• The high fraction of dense clouds containing star-formation suggests a short time \( \tau_{onset} \) between the formation of the star-forming cloud and the onset of star-formation (Jessop & Ward-Thompson 2000).
The low fraction of young clusters which have not cleared away their surrounding gas suggests a short dispersal time-scale \( \tau_{\text{disp}} \) (Leisawitz et al. 1989; Fukui et al. 1999), which is an upper limit for the star-formation time-scale \( \tau_{\text{SF}} \).

The existence of hierarchical structures in IR clusters suggests a short star-formation time-scale \( \tau_{\text{SF}} \), otherwise the substructures would be smeared out due to chaotic mixing before the cluster is formed.

The relation between the crossing time of clouds and their size \( \tau_{\text{cross}} \propto S^{x_1} \) is the same as those between the age difference of the star clusters and their spatial separation \( \Delta \tau_{\text{cluster}} \propto \Delta R^{x_2} \), i.e., \( \alpha_1 = \alpha_2 \). Thus the time-scale of star-formation is in average about one crossing time \( \tau_{\text{cross}} \approx \tau_{\text{SF}} \approx \tau_{\text{dyn}} \) (Elmegreen 2000).

Such a scenario suggests, that star-forming clouds have short lifetimes and are thus transient (see also Larson 2001).

A short star-formation time-scale of \( \tau_{\text{SF}} \approx 1 \tau_{\text{dyn}} \), implies a very low star-formation efficiency of the order of \( \eta_{\text{SF}} \approx 0.01 \). Subsequently, two possible explanations for this inefficiency are given.

### 1.3.3 Explanations for Star-Non-Formation

In order to account for the low star-formation efficiency, i.e., for star-non-formation, different scenarios have been proposed. The scenarios can be divided up in two main categories, i.e., in models where turbulence is dominant and in models where gravity is dominant. Subsequently, two typical scenarios are discussed.

1. A comparison of the virial mass with the estimated cloud-mass, deduced from the visible amount of CO by means of a conversion factor, suggests that many clouds may be gravitationally unbound. Therefore, it is often thought that the ISM shaping process is not driven by gravity, but by turbulence that produces many unbound structures. In such a scenario large amounts of molecular gas reside in gravitationally unbound clouds, being unable to form stars (Elmegreen 1999).

2. An alternative scenario suggests that the main ISM shaping process is gravitational fragmentation, that does not produce stars, but cold, low-mass clumps. Since the clumps are not homogeneously distributed, but form long-range correlations maintained by time-dependent boundary conditions, an ensemble of clumps does not collapse and form stars (Pfenniger 1998; de Vega et al. 1996). The scenario was developed to account for dark matter in the outer disk. In such an environment the clumps would have a mass of the order of Jupiter and sizes of the order of \( 1 - 10 \) AU. They are thought to be in thermal equilibrium with the cosmological background radiation at 3 K and may have a solid frozen core. Due to their low temperature, these gas clumps would be almost invisible for usual detectors (Pfenniger & Combes 1994; Pfenniger et al. 1994). A similar scenario may then also produce invisible, low-mass structures at \( \sim 3K \) in inner-disk dark molecular clouds, so that their mass would be systematically underestimated. Then, the reason for star-non-formation would not be a large amount of unbound gas, but a fragmentation process that produces objects with masses of the order of Jupiter that are too cold to be detected via emission lines.

*If \( \alpha_1 < \alpha_2 \) then \( \tau_{\text{cross}} < \tau_{\text{SF}} \)
1.4 Gravity as an Agent of Structure-Formation

Apparently a role of gravity in the universe is to form structures. Indeed, most astrophysical structures, from large-scale cosmological structures down to planets, result from gravitational instabilities.

However, in order to account for the non-equilibrium structures in the ISM, besides gravity, turbulence is often invoked as the dominant shaping agent.

Yet, turbulent chaotic motions in the ISM may be the effect of gravitational instabilities, meaning that the ISM shaping process is driven by gravity.

Thus, we focus our interest in this thesis on the role of gravity as an agent of structure formation. That is, we study structure formation and evolution, in general, in dissipative, self-gravitating systems and, in particular, in the ISM.

Clues for the important role of gravity in the ISM shaping process come from the large scale morphology of the ISM, from the gravitational model of star-formation and from the similarity of the ISM-structures to the large scale cosmological structures. This is subsequently discussed in greater detail.

1.4.1 The Gravitational Model of Star-Formation

Star-formation is closely related to the ISM shaping process and, presently, one of the most successful models of star-formation is the gravitational model. Accordingly to this model star-formation is triggered at large scales when the gas surface density $\Sigma_{\text{gas}}$ of a galaxy exceeds a critical value $\Sigma_{c}$ implied by the Safronov-Toomre stability criterion (Toomre 1964; Safronov 1960),

$$ Q = \frac{\kappa \sigma_v}{\pi G \Sigma_{\text{gas}}} = \frac{\Sigma_c}{\Sigma_{\text{gas}}} , $$

where $\kappa$ is the epicycle frequency and $\sigma_v$ is the velocity dispersion (Binney & Tremaine 1994). In regions where the surface density exceeds the critical value for star-formation is $Q \approx 1.4-2.0$ (Kennicutt 1989; Hunter et al. 1998; Elmegreen 1999) the overall star-formation rate follows the prescription,

$$ \Sigma_{\text{sf}} \propto \Sigma_{\text{gas}}^N , $$

where $N \approx 1-2$ (Kennicutt 1997).

The gravitational instability model has more predictive power than alternative star-formation models such as models based on cloud collisions. Indeed, the gravitational model accounts for the large-scale morphology of the ISM and the star-formation sites, i.e., their concentration into giant spiral arm cloud complexes of Jeans mass as well as for the overall star-formation rate Elmegreen (1999).

Of course, the gravitational model only suggests that star-formation is triggered at large scales by gravitational instabilities, but does not account for the succeeding process of gas compression. Thus, one may suppose that once gravity has triggered the process at large scales, the structure formation of the ISM is driven by other shaping agents. Yet, at smaller scales star-forming molecular clouds are gravitationally bound, which substantiates the important role of gravity in the star-formation process on various scales and by continuity one may suspect that gravity also dominantly contributes to the shaping of the ISM at intermediate scales.
1.4.2 Similarity to Cosmological Structures

As mentioned above similar hierarchical structures have been found in the ISM, in extragalactic systems and in the stellar component from the pc-scale up to cosmological scales. Indeed, Scalo (1985) pointed out that the hierarchical order at cosmological scales continue at sub-galactic scales both in the ISM, where clouds are formed by an ensemble of sub-clouds and in the stellar component, where the hierarchy is expressed by the sequence star-cloud, OB association, subgroups, open star clusters, and multiple stars. Thus it has been speculated that the similarity of sub-galactic and extra-galactic structures may reflect a common origin.

Since the observed hierarchical structures are scale-free to some extent, scale-free processes have been invoked in order to account for the structure formation, such as turbulence or gravitational instabilities.

Yet, the assumption that the structures over all these scales mainly result from the same scale-free shaping process, suggests that gravity is the dominant shaping agent, since, probably only gravity is able to be dominant over all these scales.
Chapter 2

Structure Formation Outside of Equilibrium

2.1 Introduction

The interstellar medium (ISM) has pronounced nonequilibrium structures. The existence of such structures may not be evident, since conventional thermodynamics predicts for an (micro-, grand-) canonical ensemble an irreversible evolution towards homogeneous equilibrium structures with maximum disorder.

However, from laboratory hydrodynamics and chemistry we know that irreversible processes, which are always associated with instabilities can also result in the formation of nonequilibrium structures.

Indeed, while in Hamiltonian and in closed* systems instabilities and irreversibility lead to the formation of smoothed equilibrium structures, in open dissipative systems instabilities may result in an irreversible evolution towards complex nonequilibrium structures.

Below we discuss irreversibility and the structure evolution in Hamiltonian and dissipative systems. In doing so, we shall meet some ingredients, that are necessary to form nonequilibrium structures in laboratory systems, namely, instabilities, dissipative processes and driving schemes.

These ingredients are also present in the ISM. Thus findings from nonequilibrium dynamics are relevant for the shaping process of the ISM and motivated in part the boundary conditions that are applied in the numerical models presented in this thesis.

Irreversibility is inconsistent with the fundamental equations of dynamics that are time-reversal and deterministic. But, if the fundamental equations are invariant with respect to time-reversal, how can they account for the complex nonequilibrium structures around us, which result from irreversible processes? In Sect. 2.5 we discuss the paradox and present some ideas that have been proposed to solve this problem, i.e., to reconcile thermal statistics and micro-dynamics.

*Here we refer to isolated systems and systems that can exchange energy and particles with a heat bath through its boundaries as closed.


2.2 Dynamical Systems

2.2.1 Hamiltonian Systems

We are principally concerned with dissipative systems. In order to realize the particularities of these systems let us briefly point out some characteristics of non-dissipative, i.e., Hamiltonian systems.

A characteristic of Hamiltonian systems is that the accessible volume in phase-space is conserved during their evolution. Among these systems, one can find particular systems which are stable and time-reversible\(^1\). An example for such a system is the motion of two self-gravitating bodies in an isolated system.

However, in general Hamiltonian systems are unstable and extremely sensitive to initial conditions. Actually, for these systems the volume in the 6D phase-space is conserved, but its shape is highly deformed as a result of instability. Indeed, any particular shape of the initial phase-space volume is washed out in the course of time. It is this deformation that gives the appearance of an evolution towards equilibrium. An example for such a system is given by the highly unstable Baker transformation (e.g. Nicolis & Prigogine 1989).

The evolution towards equilibrium represents an irreversible process that is “driven” by instabilities. As stated above, in dissipative systems the same processes may give rise to complexity and inhomogeneity. This is discussed subsequently.

2.2.2 Dissipative Systems

Contrary to Hamiltonian systems for which the volume of the phase-space is conserved, dissipative systems have a phase-space volume that shrinks with increasing time. The motion is no longer governed by canonical equations, but is generally described by a set of first order differential equations of the form,

\[ \frac{\partial X_i}{\partial t} = F_i(X_1, \ldots, X_n, r, t, \ldots), \quad (2.1) \]

where \( F_i \) is a function of the \( X_1, \ldots, X_n \) macroscopic system variables and their space derivatives, that depends explicitly on the position \( \vec{r} \) and the time \( t \).

Dissipative systems have in common that they are not invariant with respect to time-reversal, suggesting an irreversible approach towards a final state. Thus at least one equation of the above set, that describes the evolution of a dissipative system, expresses time-variance\(^2\).

Examples of dissipative processes are friction, diffusion and heat conduction. Let us consider the heat conduction in greater detail and show that it represents indeed a irreversible process.

If \( T \) is the temperature, \( t \) the time and \( \kappa \) the heat diffusivity coefficient, then heat conduction is described quantitatively by,

\[ \frac{\partial T}{\partial t} = \kappa \nabla^2 T, \quad \kappa > 0. \quad (2.2) \]

\(^1\)Here stability and reversibility refers to the trajectory of the system in the \( 6N \)-dimensional phase-space, where \( N \) is the particle number.

\(^2\)Time-variance means here that the system is not invariant to time-reversal.
The equation expresses that temperature variations are damped and disappear in the course of time. If time is reversed, one obtains the completely different law,

\[
\frac{\partial T}{\partial t} = -\kappa \nabla^2 T, \quad \kappa > 0,
\]

(2.3)
describing a situation in which initial temperature disturbances would be amplified rather than damped.

In closed systems dissipative processes tend to damp fluctuations and thus to eliminate complexity. This trend is expressed by the second law of thermodynamics. Accordingly to this law the entropy \( S \), representing the degree of disorder, increases monotonically until a maximum is reached,

\[
\frac{dS}{dt} \geq 0.
\]

(2.4)

However, real physical systems are open and this may radically change the role of dissipative processes. Indeed, if external constraints are imposed that maintain the system out of equilibrium, dissipative processes cause an asymptotic evolution towards a nonequilibrium stationary state, referred to as an attractor.

An attractor is reached when the volume of phase-space of the system is contracted to zero. We mentioned above that the volume of phase-space of a dissipative system continuously shrinks with increasing time, which leads to a contraction onto a surface of lower dimensionality than the original phase-space. Loosely speaking, this surface is called an attractor.

There are only three types of attractors for dissipative systems: fixed points, limit cycles and strange attractors. Whereas, fixed points and limit cycles are geometrically simple attractors on which the motion is regular, strange attractors have a fractal structure and are associated with chaotic motion (Lichtenberg & Lieberman 1983).

Typically a transition to chaos and strange attractors appears after a control parameter has passed a critical value. That is, when an increase of a control parameter has moved the system far away from equilibrium. Beyond such a critical value the equation of motion yields new solutions, which are typically associated with complexer structures.

### 2.2.3 Summary

Before we present in the following sections examples of dissipative systems that develop nonequilibrium correlations, let us briefly summerize and put in order the properties of the above discussed systems.

- **Hamiltonian Systems**

  - Stable Hamiltonian systems are invariant with respect to time-reversal. Thus their is no distinction between past, present and future.
  
  - Unstable Hamiltonian systems are characterized by orbital randomness and are variant with respect to time-reversal. There is no attracting set describing the asymptotical motion of the system in the \( 6N \)-dimensional phase-space. However, these systems are characterized by an irreversible evolution towards a dynamical equilibrium due to instabilities and chaotic mixing.
• Dissipative Systems

- Closed dissipative systems are variant to time-reversal and irreversible processes lead to uniformity and equilibrium as well.
- Open dissipative systems are characterized by asymptotic stability due to attracting sets. Far away from equilibrium these attracting sets give rise to complexity and can be characterized as being fractal. Dissipative systems are variant to time-reversal.

2.3 Chaos and Complexity

2.3.1 Introduction

Systems whose dissipated energy is not replenished in any form may actually develop and maintain complex nonequilibrium structures (Goldhirsch & Zanetti 1993; Jaeger et al. 1996; Luding & McNamara 1998), but the structures become static in the long-term limit.

Thus, in order to maintain dynamical nonequilibrium structures the systems have to be permanently driven by external forces.

An illustrative example showing the route towards chaos and complexity is the discrete Logistic map. The Logistic map represents the time evolution of a driven, damped rotator, i.e., of an open, dissipative system, that is permanently maintained from equilibrium by external constraints.

In order to illustrate the basic principles that lead to complex order in chaotic systems we discuss as an example the driven, damped rotator. That is, we discuss:

1. The equations of motion of the rotator.
2. The graphical representation of the system evolution with the Logistic map.
3. The appearance of bifurcations and complexity as a consequence of an increase of a control parameter, that represents external constraints and moves the system away from equilibrium.

Our treatment follows here in part a discussion of the problem by Gouyet (1996).

2.3.2 Equation of Evolution

The struck and damped rotator is a rather artificial system. However, it has some pleasant properties that can be easily mapped.

The systems consists of a rotator parameterized by the position angle $\phi$ and damped by friction $-\gamma \dot{\phi}$. The rotator is struck periodically by a force whose strength depends on the position, $Kf(\phi)$.

The equation of motion for such a system is,

$$\ddot{\phi} + \gamma \dot{\phi} = Kf(\phi) \sum_{n=0}^{\infty} \delta(t - nT),$$

where $n$ is an integer and $T$ is the period between two successive strikes.
2.3. CHAOS AND COMPLEXITY

Figure 2.1: Iteration of the logistic map. For $1.0 \leq r \leq 3.0$ the attractor is a fixed point. The iteration is shown for five different initial positions in the range $0 < x < 1$, that converge to the same fixed point. The four lines represent thus a contraction of the phase-space volume. The value of $r$ used in this example is indicated above the panel.

With $x = \phi$, $y = \dot{\phi}$ and $z = t$ the above equation can be reduced to a set of first order equations,

\[ \dot{x} = y \]
\[ \dot{y} = \gamma y + K f(\phi) \sum_{n=0}^{\infty} \delta(t - nT) \]  \( (2.6) \)
\[ \dot{z} = 1 . \]  \( (2.7) \)

Assuming that $x_n$ and $y_n$ is the angular position and the angular velocity, respectively, just before a strike occurs, one can find the discrete mapping (Schuster 1984),

\[ x_{n+1} = x_n + \frac{1 - \exp(-\gamma T)}{\gamma} [y_n + K f(x_n)] \]
\[ y_{n+1} = \exp(-\gamma T)[y_n + K f(x_n)] . \]  \( (2.8) \)

This map represents a discrete evolution of the struck rotator in phase-space. It consists of two coupled equations. However, the discrete evolution of the position $x_n$ can be decoupled from those of $y_n$ if particular conditions are assumed. Indeed, in the limit $\gamma \to \infty$ (strong damping), $K \to \infty$ with $\gamma/K = 1$ and by assuming $f(x) = (r - 1)x - rx^2$, the evolution of $x_n$ is decoupled and can be described by the so called logistic map,

\[ x_{n+1} = rx_n(1 - x_n) , \]  \( (2.9) \)

where $r$ is a control parameter in the range $0 \leq r \leq 4$.

2.3.3 Logistic Map

In Fig. 2.5 the bifurcation diagram of the logistic map is shown. In this diagram all attractors of the logistic map are represented.

Here we explore the behavior of the logistic map (see Eq. 2.9) in dependence of the control parameter $r$, representing external constraints that move the system away from equilibrium.

As mentioned above the logistic map is a particular representation of an open, dissipative system. Thus we expect an evolution towards an attracting set in the long term limit. That
Figure 2.2: Iteration of the logistic map. For $3.4 \approx r_3 \leq r \leq r_\infty \approx 3.6$ the attractor is a limit cycle. Here a cycle of four points is shown. All initial points in the range $0 < x < 1$ would lead to the same attractor. For clarity the iteration starting from only one initial position is here shown.

Figure 2.3: Iteration of the logistic map when the attractor is aperiodic, $r_\infty < r < 4.0$. This is a chaotic situation. However there is also some complex order. That is, the attracting set is a strange attractor, i.e., a structure with fractal properties. The particular sequence (i.e. the individual system trajectory, not the attractor) depends sensitively on the initial value $x_0$.

Figure 2.4: The attractor of the logistic map with $r = 3.57$ is shown at two different scales. Upper panel: The whole attractor. Lower panel: A magnification of the attractor in the range $0.8 < r < 0.9$. The pattern at the two different scales is almost identical, indicating a fractal ordering.
2.3. CHAOS AND COMPLEXITY

Figure 2.5: The bifurcation diagram shows the attractors $x_n$ as a function of $r$, where $r$ represent the strength of external constraints that move the system away from equilibrium. The complexity of the attractors increases with $r$. Finally, far from equilibrium strange attractors appear.

is, after a large enough number of strikes the evolution of the position angle can be described by an attractor.

Typically, the geometrical structure of an attracting set changes when a control parameter exceeds a critical value. This holds also for the logistic map. Below some typical attracting sets and the corresponding parameter ranges are listed. An overview is given in Fig. 2.5, where the so called bifurcation diagram of the logistic map is shown.

The evolution of the logistic map in dependence of the external control parameter $r$ can be summarized as follows:

1. $1.0 \leq r \leq 3.0$: In this range the position angle tends to a value $x^* > 0$ in the limit $t \rightarrow \infty$. Let us discuss what this means for the rotator. Above we said that $x_n$ represents the position just before a strike occurs. Then this angle stabilizes at $x_n = x^*$ after a large enough number of strikes has occurred. This is shown in Fig. 2.1.

2. $3.0 < r \leq r_3 \approx 3.4$: At $r = 3$ a first bifurcation appears and $x_n$ tends rapidly to a situation, where it oscillates between two values (see Fig. 2.2). Such an attracting set is called a limit cycle or more precisely a two point cycle. In fact, $x^*$ remains an accessible system state, but it is unstable.

3. $r_3 < r < r_\infty \approx 3.6$: At $r = r_3$ a new bifurcation appears and $x_n$ tends towards a four point cycle. A further increase of the parameter $r$ leads to further period doubling bifurcations that appear after shorter and shorter intervals.

4. $r_\infty \leq r \leq 4.0$: When $r_\infty$ is reached, the attracting set is no longer a limit cycle but a strange attractor characterized by a fractal dimension (see Fig. 2.3).

That the attracting sets beyond $r_\infty$ are really strange attractors with fractal properties is shown in Fig. 2.4. In this figure the attractor of the logistic map with $r = 3.57$ is shown on two different scales. Yet, despite the different scales the pattern is nearly self-similar. A property that is characteristic for fractals. The fractal dimension of this attractor is $D \approx 2.5$.

The attractors are obtained when a projection of the iteration diagram ($x_n$ versus $n$) on the $x_n$ axis is carried out.
2.4 Nonequilibrium and Complexity

2.4.1 Introduction

In the previous section we discussed the appearance of complexity in the context of nonlinear mechanics. An example of an open dissipative system showing a possible route towards chaos and strange attractors was presented.

Here we discuss if strange attractors can account for complex spatial (resp. phase-space) correlations, such as those observed in the ISM.

That is, we first introduce the term “spatio-temporal” structures. Then, we discuss the appearance of nonequilibrium structures and strange attractors in the context of thermodynamics. Finally, some relevant aspects of the Rayleigh-Bénard convection are discussed.

2.4.2 Spatio-Temporal Structures

Above we have shown that a dissipative system may evolve towards a strange attractor. A strange attractor describes the chaotic run of the system-trajectory in the 6N-dimensional phase-space, i.e., the temporal chaos of the system.

Yet, in this thesis we are mainly concerned with the hierarchical structure of the ISM and the question arises if the appearance of strange attractors can account for complex spatial correlations, since a chaotic temporal evolution may not necessarily stand for complex spatial patterns.

However, there are many examples, showing that strange attractors are related to complex spatial correlations and that the route to chaos may also be a route to structures with a complex spatial order. Such structures are referred to as spatio-temporal structures, meaning that the temporal chaos involves spatial pattern dynamics (Kapitaniak & Bishop 1999).

2.4.3 Nonequilibrium Thermodynamics

In this section we discuss how thermodynamics accounts for the appearance of nonequilibrium structures. The principle ideas may be summarized as follows (Prigogine & Kondepudi 1998; Glansdorff & Prigogine 1971; Nicolis & Prigogine 1989):

Real physical systems are often maintained outside of equilibrium by energy or matter flows. These flows are related to entropy flows. If the entropy flow leaving the system is larger than that entering, then the system evacuates its internal, by irreversible processes produced entropy to the outer world. Consequently order is created inside the system, which diverges from a classical thermodynamic equilibrium in a closed system.

Contrary to situations far from equilibrium, near equilibrium in the linear regime, the degree of order is not very pronounced. This is, because near equilibrium the system obeys a new extremum principle, that does not admit complex nonequilibrium structures. Namely, the principle of minimum entropy production replacing the maximum entropy principle at equilibrium and ensuring stability.

At equilibrium the entropy is a maximum and the entropy production is zero, \( dS/dt = 0 \). Outside of equilibrium the entropy production \( dS/dt > 0 \) determines the amount of entropy evacuated by the open system and thus the degree of order. Near equilibrium the entropy production is a minimum thus the deviation from equilibrium is minimal as well.

The situation changes far from equilibrium. Indeed, in the nonlinear regime the stability is no longer ensured by an extremum principle. Fluctuations can grow and the entropy
production may increase. If the equations of motion are nonlinear, bifurcations occur in general which lead to new and more stable spatio-temporal structures often endowed with a higher degree of order.

2.4.4 Rayleigh-Bénard convection

Here the above discussed ideas are illustrated with an example. That is, we discuss a thermodynamic system that far from equilibrium evolves towards a strange attractor and reveals complex pattern dynamics, referred to as Rayleigh-Bénard convection.

Consider a horizontal layer of fluid, between two parallel plates in a constant gravitational field. By heating the lower plate we can introduce a constraint that moves the system away from equilibrium.

If the temperature difference between the two plates is small ($\Delta T \ll T$), the system remains near equilibrium and the structure is determined by the minimum entropy production. That is, the system adopts a uniform temperature gradient and thus a density and pressure gradient that merely represent a modest degree of order.

By injecting more energy from below the system is moved further from equilibrium. If $\Delta T$ exceeds a critical value $\Delta T_c$, the state of rest (so far ensured by the minimum entropy production) becomes unstable and convection starts (Rayleigh-Bénard convection). Eddies appear in which the motion of myriads of molecules is correlated.

The eddies are referred to as Rayleigh-Bénard cells and are formed by ascending and descending currents. The cells have fixed spatial positions and two equiprobable rotation states. There is thus a first bifurcation between these two possible rotation states after $\Delta T$ has reached its critical value $\Delta T_c$. A schematic diagram of the system after the first bifurcation is shown in Fig. 2.6.

The patterns formed by the Rayleigh-Bénard cells can be visualized by an optical technique called shadowgraphy. The result of such a visualization is shown in Fig. 2.7.

As the temperature of the lower plate is increased, further bifurcations occur, followed finally by a transition to turbulence (see Fig. 2.8).

The turbulent motion of Rayleigh-Bénard convection is an example showing that fractal spatial correlations may appear together with strange attractors. Indeed, the evolution of
Figure 2.7: Patterns formed by Rayleigh-Bénard cells after the first bifurcation. The regular patterns, seen from above, result from warm ascending and cold descending flows. While the warm ascending fluid defocuses the light and appears dark, the cold falling fluid focuses the light and appears bright.

Figure 2.8: Turbulent spatio-temporal patterns appear if the temperature difference between the two plates is large enough. This means that a high degree of complexity is reached if the external constraints move the system far from equilibrium. The patterns are seen from above.
the turbulent system in the limit \( t \to \infty \) is described by a strange attractor and the spatial correlation of turbulence may be fractal as well. Let us discuss this in turn.

Approximate models of the Rayleigh-Bénard convection suggest the existence of strange attractors for \( t \to \infty \). The Lorenz system is such an approximate model. It consists of three first order equations with second degree cross terms. The equations result from an expansion of the equations of evolution (Navier-stokes, heat convection) in Fourier space followed by a truncation, where only three Fourier modes are retained.

If this system is maintained far from equilibrium by a control parameter the evolution can be described in the limit \( t \to \infty \) by a strange attractor referred to as Lorenz attractor. The Lorenz attractor is shown in Fig. 2.9.

The three mode Lorenz system is a very approximate model of the Rayleigh-Bénard convection. However, the existence of strange attractors is confirmed by expansions of the Lorenz equations up to fourteen modes (for a more extended discussion of the Rayleigh-Bénard convection and the Lorenz system see e.g., Lichtenberg & Lieberman 1983; Gouyet 1996).

Thus, the turbulent Rayleigh-Bénard convection shows that strange attractors are related to the appearance of dynamical, spatial patterns, that may even have fractal properties.

Indeed, the Kolmogorov model suggests that the eddies appearing in turbulent flows belong to a structure that is hierarchically organized and self-similar according to size (Kolmogorov 1941; Landau & Lifshiz 1959). Furthermore the model predicts a scale-free velocity-size relation, a property that is also present in the interstellar medium.
2.5 Irreversible Processes and the Arrow of Time

2.5.1 Introduction

The classical laws of dynamics agree with the properties of stable systems. They are deterministic and time-reversible. However, in reality stable systems seem to be the exception and unstable systems the rule.

Contrary to stable systems, the evolution of unstable systems is characterized through probabilities and irreversibility, i.e., through properties that are inconsistent with the classical laws of dynamics.

In the past this inconsistency have often been attributed to the macroscopic description of unstable systems that introduces approximations in the laws of dynamics. However, the study of chaotic systems that develop complex nonequilibrium structures suggests that probabilities and irreversibility are not only the result of a coarse grained, approximative description, but are fundamental.

Thus, in order to reconcile classical dynamics with the time-directed evolution of unstable chaotic systems, modifications of the classical laws of motion have been proposed. For the interested reader, this is subsequently discussed in greater detail.

2.5.2 Modified Newtonian Mechanics

Newton’s equation of motions are time-reversible and deterministic. If one knows the initial state of a dynamical system one can determine its state at any time in the future and in the past.

There is no distinction between past and future, since also the reverse process is allowed by the laws of motion. These properties are also valid in quantum mechanics where the time-evolution is described by Schrödinger’s deterministic and time-reversible equation.

However, the second law of thermodynamics describes an irreversible evolution towards maximum entropy and is thus directed in time. The braking of time-symmetry in statistical physics was often thought to be due to its approximate description, i.e., due to a lack of knowledge. That is, the appearance of the arrow of time was regarded as a phenomenological problem due to a limited human observer and an incomplete, coarse descriptions, that introduces approximations in the fundamental laws of dynamics.

To illustrate this let us consider an example. Two boxes are connected with a tube. One box contains many particles, while the other contains few particles. We expect that the number of particles levels out more and more, which would be a sign of irreversibility. Yet, if irreversibility was not more than that, it might indeed be an illusion, because, if we wait long enough, it may happen, that the particles reconcentrate in the same box. Irreversibility may then be explained by our limited patience.

However, as discussed above irreversibility appears not only in closed thermodynamic systems where the evolution obeys the second law of thermodynamics, but also in open, dissipative systems, where instabilities often result in an irreversible evolution towards complexity. Indeed, if a control parameter (expressing external constraints) exceeds a critical value the system loses its stability and the equation of motion yield new more stable solutions that typically are associated with a higher degree of order. The system evolves then with a certain probability to one of the newly accessible states.

The nonequilibrium structures persist as long as the system is maintained outside of equilibrium by the external constraints. Thus irreversibility can not be explained through a
time limited observation.

The world around us shows complex structures that seem to be the result of a temporal evolution. Irreversibility and probability accounts in open, dissipative systems for the development of such structures. Thus irreversibility and probabilities seem to be fundamental for a description of the Universe and should consequently be included in the basic laws of dynamics.

In order to overcome the contradiction between the irreversible processes of thermodynamics and time-symmetry of micro-dynamics Prigogine & Stengers (1996) and Prigogine (1998) propose to abandon the classical description of individual situations in terms of trajectories (or wave-functions), in favor of a statistical description.

Indeed, in unstable, chaotic systems the concept of trajectories becomes meaningless since trajectories that describe the system evolution diverge in the course of time, whatever their initial distance is.

Since, one can not determine an initial state with infinite precision, it would be more appropriate to use instead of trajectories, probability distributions.

Thus, new concepts of dynamics are necessary, where probability distributions do not express a lack of information in situations where exact initial conditions are not available, but are fundamental and irreducible as they do not apply to trajectories (Prigogine 1980, 1997).

2.5.3 Modified Quantum Mechanics

In order to reconcile thermodynamics and micro-dynamics a description in terms of probability distributions has been proposed. Actually, quantum mechanics has introduced a probabilistic element into the description of the micro-world, but, an a quantum mechanical description is confronted with the duality problem.

In order to solve the duality problem of quantum mechanics, a quantum mechanical description directly in terms of probability distributions instead of wave-functions has been proposed. Let us discuss this in turn (see the references above).

In quantum mechanics a physical state is described by a wave-function $\Psi$. This wave-function can be expanded in terms of eigenfunctions $u_n$ of the Hamilton operator $H$,

$$
\Psi(x, t) = \sum_n c_n \exp(-iE_nt) u_n(x),
$$

where $c_n$ are coefficients, $E_n$ are eigenvalues and $t$ is the time. According to the interpretation of quantum mechanics, after an energy measurement the system adopts with a probability of $|c_n|^2$ one of the energy eigenstates $u_n$ that corresponds to an energy $E_n$ (Sakurai 1985). In other words,

$$
\Psi(x, t) \xrightarrow{E \text{ measurement}} u_n(x).
$$

The change of the wave-function due to a measurement is referred to as the collapse of the wave function and constitutes the duality problem of quantum mechanics. Indeed, on the one hand the Schrödinger equation, which is time-reversible, describes the time evolution of a wave-function, on the other hand a collapse of the wave-function occurs in conjunction with a measurement, that causes a symmetry break and leads thus to irreversibility.
Therefore a formulation of quantum mechanics directly in terms of probability distributions and not in terms of wave-functions associated with a probability interpretation is needed.

A property of such a description should be that it is irreducible to trajectories or wave-functions and accounts thus for irreversibility as an inherent property of nonlinear chaotic systems (for an extended discussion see also, Prigogine & Stengers 1996).
Chapter 3

Nonextensivity

3.1 Introduction

Astrophysical systems, such as the ISM, often differ in two points from classical thermodynamic systems: 1) They are open and 2) self-gravity is an important factor. Both have the effect that the systems do not evolve towards the accustomed homogeneous equilibrium states, known from classical thermodynamics.

The effect of the first property was discussed in the previous chapter, where we showed that open dissipative systems may develop spontaneously spatio-temporal structures. In this chapter we are particularly interested in the consequences of the second property for thermodynamic systems.

Conventional, homogeneous thermodynamics is derived under the early and frequent use of the thermodynamic limit and extensivity. Yet, gravity is a non-shielded long-range force, violating the the assumption of classical thermodynamics that particles exclusively interact with neighboring particles. As a consequence the usual thermodynamic limit does not exist, and extensivity is violated.

Thus self-gravitating systems often show anomalous thermodynamic behavior and the question arises, what is the appropriate theoretical tool in order to account for such anomalies.

Subsequently, some thermodynamic anomalies of self-gravitating systems are presented. Furthermore, some general assumptions of classical thermodynamics that are violated in systems with long-range forces are pointed out. Finally, thermostatistical tools, that have been used to study the behavior of self-gravitating systems, are discussed.

3.2 Thermodynamic Anomaly of Self-Gravitating Systems

Few realized long ago that gravitating systems have anomalous thermodynamic characteristics. A fact that is often not taken into account in thermodynamical studies of astrophysical systems. Examples of such an anomalous thermodynamical behavior are the appearance of negative specific heat and the so called gravo-thermal catastrophe. Subsequently, these examples are discussed in detail.
3.2.1 Negative Specific Heat

An anomaly with respect to classical thermodynamics concerns the specific heat of self-gravitating system. Indeed, according to a theorem of classical statistical mechanics, specific heat is positive, but adding energy to a self-gravitating system makes it expand and cool down, i.e., self-gravitating systems have negative specific heat. Subsequently the proofs of these predictions are presented:

**Proof 1:** The specific heat of a canonical ensemble is positive. To show this let us consider a canonical ensemble. The mean energy of a such an ensemble (i.e. an ensemble in a heat bath with temperature $T$) is,

$$
\langle E \rangle = \frac{\sum_{i=1}^{W} E_i \exp(-\beta E_i)}{\sum_{i=1}^{W} \exp(-\beta E_i)} .
$$

(3.1)

Here, $\beta = 1/(k_B T)$, where $k_B$ is the Boltzmann constant, and $E_i$ is the energy of the $i = 1, \ldots, W$ possible microscopic configurations. Then, the specific heat is (Schrödinger 1948),

$$
C_V = \frac{d\langle E \rangle}{dT} = \frac{d\langle E \rangle}{d\beta} \frac{d\beta}{dT} = k_B \beta \langle (E_i - \langle E \rangle)^2 \rangle > 0 .
$$

(3.2)

**Proof 2:** The specific heat of a self-gravitating system is negative. Indeed, consider a system in virial equilibrium, $2T + U = 0$, where $T$ and $U$ are the kinetic and potential energy, respectively. The total energy of the system is, $E = T + U$. With, $T = 3k_B T/2$, the specific heat reads,

$$
C_V = \frac{dE}{dT} = -\frac{3}{2}Nk < 0 .
$$

(3.3)

These results are only apparently contradictory and there is no paradox if one regards the scope of the above results. In order to understand this, let us consider some properties of negative $C_V$ systems (Lynden-Bell 1998):

- A negative $C_V$ system cannot achieve thermal equilibrium with a large heat bath. Indeed, if owing to a fluctuation the heat bath become temporary too hot then thermal equilibrium cannot be reestablished by an energy flow from the bath to the system, because such a flow causes an increase of the temperature difference due to the negative specific heat of the system.

- Two negative $C_V$ systems in thermal contact cannot attain thermal equilibrium. In fact, one gets hotter and hotter by loosing energy, whereas the other gets colder and colder by receiving energy. Therefore negative $C_V$ systems cannot be divided in two independent parts. Consequently negative $C_V$ systems are nonextensive.

In summary, negative specific heat systems cannot form a canonical ensemble in thermal equilibrium and are nonextensive. Thus the scope of Proof 1 is valid for extensive canonical systems, whereas Proof 2 holds for nonextensive systems.
3.2.2 Gravothermal Catastrophe

A further example for the thermodynamic anomalous behavior of self-gravitating systems is the gravo-thermal catastrophe, that cannot be accounted for by conventional homogeneous thermodynamics.

The gravo-thermal catastrophe occurs when the energy of systems with Newtonian interaction potential falls below a critical value \( E < E_c \). Then the central parts of the system collapse and nothing can stop this collapse. However, if the short-distance singularity of the gravitational potential is removed by applying a potential regularization, there are always equilibrium states. Yet, a phase transition, separating a high energy homogeneous phase from a low energy collapsing phase with core-halo structure takes place at \( E \sim E_c \).

Subsequently, some results of gravothermal statistics with Newtonian and regularized interaction potentials are summarized.

First, we list some properties of gravitating systems with point-like particles (Antonov 1962; Lynden-Bell & Wood 1968; Chavanis et al. 2001):

- There are no global or local entropy maxima for an unconfined (not confined to a box) self-gravitating system. The system can always increase its entropy by taking a core-halo structure.

- No global entropy maximum exists for a self-gravitating system confined to a box. The system can always increase entropy by developing a dense and hot core surrounded by a low density halo.

- A local entropy maximum exists for a confined self-gravitating system, if the total energy is larger than the critical energy, \( E > E_c = -0.335GM^2/R \). This corresponds to a density contrast of \( \rho_c/\rho_0 = 709 \), where \( \rho_c \) and \( \rho_0 \) are the densities at the center and the border of the system, respectively.

- There is no local maximum for a system with \( E < E_c \). Thus there is no equilibrium to go and nothing can stop the collapse of the central parts. This is the so called gravo-thermal catastrophe.

The situation alters if the short-distance singularity is removed. Subsequently, some results of gravo-thermal statistics with non point-like, confined particles are listed (Hertel & Thirring 1971; Aronson & Hansen 1972; Follana & Laliena 2000):

- There is always an equilibrium state to go if the regularization is mild enough*. A phase transition separating a high energy homogeneous phase from a low energy collapsing phase, occurs in an energy interval with negative microcanonical specific heat.

- The high energy phase of a system with a mild regularization has a global entropy maximum that is close to the local maximum of the entropy with the unregularized potential.

- Below the critical energy there exists a global maximum as well if the regularization is mild.

*Follana & Laliena (2000) studied the problem by means of a softened potential. They achieve a softening by truncating to \( N \) terms an expansion of the Newtonian potential in spherical Bessel functions. For such a potential mild means, \( N < 30 \).
• If the regularization of the gravitational potential is sharp there are high-entropy mass distributions consisting of a small amount of condensed matter embedded in a homogeneous halo. The amount of condensed matter decreases if the regularization approximates the Newtonian potential. Thus Follana & Laliena (2000) suggest that this may indicate that at small scales the system is not well described by a smooth density, and granularity is playing a major role. That is, small dense grains may develop in a homogeneous self-gravitating halo.

3.3 The Forth Law of Thermodynamics

Self-gravitating systems are not the only ones showing anomalous thermodynamic behavior. Indeed, deviations from predictions of conventional thermodynamics have also been observed in dissipative and granular systems, respectively. Typically, these systems are nonextensive and violate thus the forth law of thermodynamics. Let us discuss this in turn.

Standard statistical mechanics assumes that a large number of atoms, molecules or particles ($\geq 10^{23}$) move independently (i.e. uncorrelated) in a system and interact only with nearby particles.

Macroscopic thermodynamic quantities $f$ of such systems are either extensive (i.e., additive) or intensive$^1$:

$f$ is extensive, if $f(A + B) = f(A) + f(B)$
$f$ is intensive, if $f(A + B) = f(A) = f(B),$

where $A$ and $B$ are two independent systems. The circumstance that macroscopic quantities are intensive or extensive cannot be deduced from first principles (“laws” of thermodynamics). Nevertheless, it is a very important characteristic of classical thermodynamics and is thus sometimes referred to as the forth law of thermodynamics (Landsberg 1972, 1984).

However, nature does not always satisfy extensivity and thermodynamic quantities, which are in standard Boltzmann-Gibbs (BG) statistics extensive may become nonextensive, i.e., superadditive (superextensive) and subadditive (subextensive), respectively,

$f$ is superadditive, if $f(A + B) > f(A) + f(B)$
$f$ is subadditive, if $f(A + B) < f(A) + f(B).$

Systems violating extensivity tend to do so because they have (Tsallis 1999; Latora et al. 2001),

• Long-range interaction potentials (such as gravity): The total energy density of a $D$-dimensional system with interaction potential $\propto r^{-\alpha}$ converges if the interaction potential is short-ranged, $\alpha > D$, and diverges if the interaction potential is long-ranged, $0 \leq \alpha \leq D$ (see Appendix A). This means that the total energy, which is for short-range forces extensive, becomes for long-range forces nonextensive.

• Long-time microscopic memory: A long-time microscopic memory can invalidate the

$^1$Examples of extensive quantities are energy and entropy. Whereas temperature and pressure are examples of nonextensive quantities.
accessibility assumption and micro-reversibility, both closely connected to a BG equilibrium (Binney et al. 1992).

- *(Multi-)*Fractal-like spatio-temporal structures: Such structures have scaling behavior with dimensionality deviating from the embedding space. Thus, the accessibility of the phase-space may be reduced and the ergodic hypothesis, a fundamental requirement for the applicability of BG statistical mechanics (Krylov 1944, 1979), may be violated.

### 3.3.1 Thermodynamic Limit

For nonextensive systems, there is no thermodynamic limit. Consider for instance a homogeneous self-gravitating system. For such a system energy is superadditive since it grows as $\propto G\rho_0^2V^{9/3}$ (Pfenniger 1998). Thus the energy density $\epsilon = E/V \propto G\rho_0^2V^{2/3}$, that is in conventional thermodynamics an intensive quantity, diverges in the limit $V \to \infty$.

Consequently, the usual thermodynamic limit, where $N \to \infty$ and $V \to \infty$, while intensive variables, such as the energy density, are kept fixed, does not exist (Votyakov et al. 2002).

### 3.4 Thermodynamics of Self-Gravitating Systems

#### 3.4.1 Introduction

Above we have shown that self-gravitating systems are nonextensive and violate the thermodynamic limit assumption. However, these concepts are tightly interwoven with standard homogeneous thermostatics.

Therefore, classical thermodynamic relations (e.g. conventional equations of state) cannot be applied to self-gravitating systems without reservation.

Consequently, thermodynamic quantities and relations describing the properties of self-gravitating systems have to be derived newly in a modified framework or from fundamental thermodynamical principles that do not invoke extensity and the thermodynamic limit.

Some thermostatistical techniques that have been used to study nonextensive systems are discussed subsequently.

#### 3.4.2 Micro-Canonical Approach

In order to study the thermal properties of a system one has to assume a statistical ensemble.

Here, we show that only the micro-canonical ensemble, i.e., a closed system with fixed energy $E$, particle number $N$ and volume $V$, can consistently describe a self-gravitating system.

Indeed, self-gravitating systems have negative specific heat and the thermodynamic limit concept is not applicable. Due to this properties, a heat bath realization is not possible and fluctuations cannot be ignored. Thus long-range forces cannot be described by the (grand-) canonical ensemble. Subsequently we discuss this in greater detail.

- **Negative Specific heat:** As discussed above, a negative specific heat system cannot achieve thermal equilibrium with a large heat bath, since thermal fluctuations of the system are unstable.

- **Thermodynamic Limit:** As it is not possible to go to the thermodynamic limit for self-gravitating systems, boundary effects and fluctuations cannot be neglected with respect
to the bulk. Since the details of the boundary conditions are important and fluctuations may become significant, it is questionable if self-gravitating systems can be controlled by a given heat bath temperature, without any further specification of the boundary conditions, as it is the case in a canonical ensemble (Gross 2001).

**Boltzmann Entropy**

For a consistent description of a nonextensive system one has to use a micro-canonical ensemble. For such an ensemble Boltzmann’s entropy is given through,

\[ S = k \ln W, \tag{3.4} \]

where \( k \) is a constant and \( W(E, N, V) \) is the number of micro-states. Then, the equilibrium state of a system can be found by maximizing Boltzmann’s micro-canonical entropy.

Since all system information is contained in \( W \) such an approach is exact in the sense that small scale granularity and particle correlations are taken into account. That is, the method describes the correct micro state of the system without restoring to any continuum approximation.

However, in practice, such an approach is often an impossible difficult task and so called mean field approximations have been introduced, that reduce the problem to a tractable level.

**Mean Field Approximation**

Ignoring granularity and correlations, and assuming a smooth density distribution, Boltzmann’s micro-canonical entropy per particle is given through (e.g. Padmanabhan 1990),

\[ s = S/N = - \int d^3r \rho(\vec{r}) \left[ \ln V \rho(\vec{r}) - 1 \right] + 3 \ln (E - \Phi/2)/2, \tag{3.5} \]

where \( E \) is the energy, \( N \) is the particle number, the density is normalized \( \int d^3r \rho(\vec{r}) = 1 \) and the gravitational potential is,

\[ \Phi = -G \int \frac{\rho(\vec{r}) \rho(\vec{r}')}{|\vec{r} - \vec{r}'|} d^3r d^3r'. \tag{3.6} \]

The maximum of Eq. 3.5 under the constraint \( M = \int d^3r \rho(\vec{r}) = 1 \) yields then the equilibrium state.

Such an approach has for instance been used by Follana & Lalien (2000) to study the collapsing transition of self-gravitating systems. In the article presented on page 91 some of their results are discussed in greater detail and compared with numerical simulations.

Above the mean field approximation was directly derived from the number of microstates. An analog approach is possible by assuming a smooth distribution function and using the Boltzmann-Gibbs (BG) entropy.

Let \( f(\vec{r}, \vec{v}; t) \) be the smooth (coarse grained) phase-space distribution function of a self-gravitating system, so that \( f(\vec{r}, \vec{v}; t) d^3r d^3v \) is the mass in the phase space cell \( (\vec{r}, \vec{v}; \vec{r} + d^3\vec{r}, \vec{v} + d^3\vec{v}) \) at time \( t \). Then the density, the total mass and total energy are given by,

\[ \rho = \int f d^3\vec{v}, \tag{3.7} \]

\[ M = \int \rho d^3\vec{r}, \tag{3.8} \]
and
\[
E = T + U = \frac{1}{2} \int v^2 f d^3 v d^3 r + \frac{1}{2} \int \phi f d^3 v d^3 r ,
\]
respectively. Then, the equilibrium state of the system can be found by maximizing the Boltzmann-Gibbs entropy,
\[
S = -\int f \ln f d^3 v d^3 r
\]
while energy \( E \) and mass \( M \) are kept constant.

Such a technique have for instance been used by Lynden-Bell \\& Wood (1968) in order to study the gravothermal catastrophe.

Actually, a mean field approximations makes complex systems mathematical tractable, but the smoothed system description contains much less information than it is provided by the exact micro-canonical description. That is, granularity is smoothed out and information about particle correlations get lost. Thus, the mean field approximation provides a appropriate description only as long as correlations are not significant. This is not the case in situations where phase transitions lead to complex nonhomogeneous order (Padmanabhan 1990).

### 3.4.3 Tsallis Thermal Statistics

Micro-canonical statistics can be developed without using the concepts of extensivity and the thermodynamic limit. Thus it provides a natural extension of thermal statistics that is able to describe the properties of some nonextensive systems (Gross 2001).

However, since ergodicity is a foundation pillar of micro-canonical statistics, it cannot account for long-range correlations appearing in some dissipative systems. Indeed, dissipative systems may evolve towards strange attractors with non-accessible phase-space regions and thus violate the ergodicity hypothesis.

Thus more radical generalizations of thermal statistics have been proposed in order to account for thermodynamic anomalies in nonextensive systems.

One of the most successful generalizations, including nonextensity and recovering extensivity as a particular case, is the formalism of Tsallis, that is actually not restricted to the micro-canonical description.

Here we discuss this generalization. Furthermore, problems that arise when the classical Navier-Stokes equations are used to simulate nonextensive fluids, such as the ISM, are pointed out.

Inspired by multifractals\textsuperscript{1} Tsallis proposed a generalization of the BG formalism by postulating a nonextensive entropy (Tsallis 1988; Curado \\& Tsallis 1991; Tsallis et al. 1998),
\[
S_q = k \frac{1 - \sum_{i=1}^{W} p_i}{q-1} \left( \sum_{i=1}^{W} p_i = 1 ; q \in \mathbb{R} \right) ,
\]
where the index \( q \), characterizes the degree of nonextensivity, \( p_i \) are the probabilities of the \( i = 1, \ldots, W \) possible microscopic configurations and \( k \) is a positive constant whose value depends on the particular units to be used (for convenience, we adopt here \( k = 1 \)). Then, the entropy

\textsuperscript{1}In the multifractal framework the quantity that is normally scaled is \( p_i^q \), where \( p_i \) is a probability associated with an event and \( q \) is any real number. Tsallis (1988) has used this quantity to generalize the standard expression of the entropy \( S \).
of Tsallis satisfies the particular relation, \( S_q(A + B) = S_q(A) + S_q(B) + (1 - q)S_q(A)S_q(B) \). In the limit \( q \to 1 \), the entropy of Tsallis becomes additive and reduces to the usual BG formula

\[
S_{q \to 1} = - \sum_{i=1}^{W} p_i \ln p_i .
\] (3.12)

Yet, for \( q \neq 1 \) the entropy becomes superadditive \((q < 1)\) and subadditive \((q > 1)\), respectively.

However, despite this deviation from BG thermostatistics for \( q \neq 1 \) it has been shown that a remarkable set of thermostatistical and dynamical properties are \( q \)-invariant. Among them are for instance, the H-theorem, the Ehrenfest theorem and the Onsager reciprocity. On the other hand there are properties which depend on \( q \) such as, the specific heat, the magnetic susceptibility as well as the Langevin and Fokker-Planck equations (for a more extend list of \( q \)-invariant and \( q \) dependent thermodinamical and statistical properties see Tsallis 1999).

Navier-Stokes Equations of Nonextensive Systems

In the context of the interstellar gas, the question arises about what are the consequences of nonextensivity for the classical equations of fluid dynamics. Are they \( q \)-invariant or not? Boghosian (1999) showed that the Navier-Stokes equations expressing conservation of mass and momentum are \( q \)-invariant, but the energy conservation equation is not. This may not be a serious problem in open systems, because energy is not conserved anyway. For instance, in weakly dissipative dynamical systems, it is expected that the exact form of the dissipative forces is irrelevant, since the long time behavior is an attractor (Pfenniger & Norman 1990; Huber & Pfenniger 2001). However, models for which the exact form of the cooling function is relevant must take into account the effect of nonextensivity in the energy budget.

As long as the above mentioned equations of the velocity moments are not a closed set, although they are mathematically correct, they have no physical content (Shu 1992).

In fluid dynamics the set of conservation equations can be closed by adding an equation of state, valid for a local thermodynamic equilibrium. However, the generalized thermodynamic equilibrium depends on \( q \) (Boghosian 1999; Latora et al. 2001).

Thus standard fluid dynamics is not consistent with Tsallis formalism, suggesting that it may not be an appropriate tool to model the dynamics of nonextensive interstellar gas.

Discussion

In fact, Tsallis generalized thermostatistics is very promising. Indeed, there are various systems whose thermostatistics seems to be best described by Tsallis formalism with an index \( q \neq 1 \) (e.g. Plastino & Plastino 1993; Boghosian 1996). However, the study of nonextensive systems with thermostatistical tools is still in its infancy and the proposed generalizations are so far not developed to a consistent theory.

For instance, Tsallis generalization cannot be deduced from first principles (laws of thermodynamics) and modifications have been proposed due to inconsistencies (Sumiyoshi et al. 2001).

\(^\text{9}\text{Indeed, } \lim_{q \to 1} p_i^q = \lim_{q \to 1} p_i^{q-1} = \lim_{q \to 1} p_i \exp(\ln p_i) \approx p_i (1 + [1 - q] \ln p_i). \)

Thus, \( \lim_{q \to 1} S_q = \left[ 1 - \sum_{i=1}^{W} p_i + (q - 1) \sum_{i=1}^{W} p_i \ln p_i \right] / (q - 1) = - \sum_{i=1}^{W} p_i \ln p_i. \)
Since, thermodynamical generalizations, such as Tsallis formalism, do up to now not allow to formulate a fully consistent theory and the micro-canonical approach can only describe complex systems in the mean field approximation under the assumption of ergodicity, appropriate and reliable thermostatistical tools for the description of long-range correlations in complex, dissipative systems are still lacking.

3.5 Summary

In this chapter we discussed some properties of nonextensive, self-gravitating systems. The main points of this discussion can be summarized as follows:

1. Self-gravitating systems are nonextensive and show anomalous thermodynamic behavior, such as negative specific heat and the gravothermal catastrophe.

2. (Grand-) canonical thermostatistics is tightly interwoven with the thermodynamic limit and extensivity. However, a thermodynamic limit does not exist for nonextensive systems. Consequently, the appropriate ensemble to study nonextensive systems is the micro-canonical ensemble.

3. A micro-canonical description of complex systems is often an impossible difficult task. To reduce the problem to a tractable level, mean field approximations have been used. Yet, such approximations smooth out the granularity of the system and do not take into account particle correlations.

4. A micro-canonical description of nonextensive systems still requires ergodicity. However, many structures of physical interest appear in non-ergodic systems, such as long range correlations in dissipative systems. In order to account for long-range correlations, thermodynamic generalizations with modified entropies have been proposed. An example is the formalism of Tsallis that does a priori not exclude the treatment of long-range correlations in non-ergodic systems.

5. Since many thermodynamic results are based on nonextensivity, one cannot apply without reservation conventional thermodynamics to self-gravitating, dissipative systems with long-range correlations, such as galactic discs or star-forming molecular clouds.
Chapter 4

Which Tool to Simulate Self-Gravitating Gas?

4.1 Different Methods

The methods that have been applied to simulate the dynamics of the ISM can be divided up into three categories, namely, continuum methods, smoothed particle hydrodynamics (SPH) and sticky particle methods. The three classes attach different importance to the clumpy nature of the observed ISM. Subsequently, the three methods are briefly presented. Furthermore, we give reasons for the use of sticky particle methods for the simulation of self-gravitating, dissipative media.

1. **Continuum methods:** The ISM is regarded as a continuum, i.e., a continuous substance with locally well defined thermodynamic quantities. The dynamics of the ISM can then be simulated with traditional equations of fluid dynamics, such as the Navier-Stokes equations (Shu 1992).

2. **SPH:** In SPH the medium is represented by \( N \) particles. Each particle is smoothed in space according to a function, called kernel, that represents the density distribution in space (Benz 1989, 1991). The resolution is given by the particle size and the particle size is given by the width of the kernel \( h \). Typically, thermodynamic forces are only calculated at scales \( \lesssim 2h = L_{\text{th}} \) and the influence of long-ranged gravity on a particle can be determined by \( N \)-body techniques (Müller 1997; Pux 1997). However, in order to determine the thermodynamic properties below \( L_{\text{th}} \) an equation of state, based on the assumption of a local thermodynamic equilibrium (LTE) up to the resolution scale \( h \), is used.

3. **Sticky Particle Methods:** The medium is represented by \( N \) particles governed by Newton’s equation of motion. That is the system-dynamics is simulated with \( N \)-body techniques. Thermodynamic quantities are not used. Thus there is no need of LTE in order to close the set of hydrodynamic differential equations by an equation of state. The particles dissipate their energy during inelastic collisions according to a certain rule and are referred to as sticky or dissipative particles (see e.g., Brahic 1977; Wisdom & Tremaine 1988; Toomre & Kalnajs 1991; Huber & Pfenniger 2001). Typically, the dissipation strength depends on the relative particle distances and relative particle velocities.
4.2 Violation of the LTE-Assumption

Since self-gravitating systems are nonextensive and molecular clouds reveal nonequilibrium structures down to the resolution limit of the observing instrument, the use of traditional thermodynamics may be inappropriate for the modeling of the ISM. Thus, we use in this thesis sticky particle methods for the simulation of self-gravitating, dissipative systems.

Consistency problems that arise when nonextensive systems are described with thermostatistical tools were discussed in detail in the previous chapter. Subsequently, some findings that are relevant for the simulation of the ISM are briefly summarized. Moreover, observational findings suggesting the violation of the LTE assumption are discussed.

4.2.1 Theoretical Point of View

Traditional thermostatistics is tightly interwoven with extensivity and the thermodynamic limit. Yet, self-gravitating systems are nonextensive and the thermodynamic limit does not exist for such systems. Thus, not all relations or results of traditional thermodynamics can be used for the description of gravitating systems.

Indeed, typically nonextensive systems do not evolve to the homogeneous equilibrium structures predicted by traditional thermodynamics. Furthermore, generalized thermodynamic formalisms that are currently developed and do incorporate nonextensivity, suggest that the energy conservation equation of fluid dynamics changes in nonextensive systems (Boghosian 1999).

4.2.2 Observational Point of View

Continuum methods applied to systems with spatially non-homogeneous conditions base on the assumption of local thermodynamic equilibrium (LTE) and along with this it is generally assumed that the functional properties that hold for equilibrium systems also hold among local thermodynamic properties. However, these idealization is only valid when (Carey 1999),

- the mean free path of the system constituents is smaller than the scale of density or pressure variation,

- the time-scale over which the system is permitted to reach equilibrium is much longer than the mean time between collisions.

Observations of molecular clouds reveal granular and clumpy structures down to the smallest accessible scales. The cloud clumps may violate the above mentioned conditions, so that the application of the classical Navier-Stokes equation is questionable. Let us discuss this in greater detail.

Observations of optical thin lines in the Orion molecular cloud show granular structures down to the resolution limit of $0.002 \, \text{pc}$ (Pauls et al. 1983; Mignone et al. 1989). Observations of scattering events of quasar light (Fiedler et al. 1994; Fey et al. 1996) and galactic absorption lines ($\text{HI, H}_2\text{CO}$) seen against background quasars (Marscher et al. 1993; Davis et al. 1996) suggest even clumpy structures at scales of $\sim 10 \, \text{AU} (\sim 10^{-4} \, \text{pc})$.

The presence of these density fluctuations suggests that the LTE-assumption is violated down to smallest scales. Thus the simulation of an interstellar system having the size of a giant molecular cloud with classical hydrodynamics is not possible due to the limited resolution, i.e., limited computational power. Indeed, even a 2D hydrodynamical simulation of a flow
on the 50 pc scale would require $\sim 10^{10}$ cells (or particles if SPH methods are used) due to deviations from thermodynamic equilibrium down to solar system scales. This surmounts the capacity of today's most powerful computers by at least a factor ten\textsuperscript{*}.

However, one might argue that it is not necessary to resolve the smallest clumps present in the ISM, because an ensemble of clumps at scales larger than the smallest observed density fluctuations, may already form a thermodynamic equilibrium. Indeed, the “mean-free path” of molecular cloud clumps is much larger than their size, so that a hydrodynamical description with particles of the size of clouds clumps may be allowed (Pfenniger 1994). Yet, Simon et al. (2001) found that even the smallest resolvable structures (mapping via radio spectroscopy of molecular lines) in the galactic molecular ring with sizes of the order of $\sim 0.1$ pc are supersonic and Fuller & Myers (1992) found that non-thermal motions dominate thermal motion down to scales of the order of $\sim 0.01$ pc. As long as turbulent correlated motion dominates no pressure equilibrium is reached and thus no equation of state can be defined. Assuming that thermal motions already start to dominate at scales of 0.01 pc, a 2-dimensional simulation of a GMC would still need $10^8$ particles (or cells) resp. $10^{12}$ for a 3-dimensional simulation.

To sum up, observations conflict with the assumption of a LTE made in codes using statistical gas physics. Furthermore, such codes ignore that gravitating systems are nonextensive, so that classical thermodynamic relations may be modified. Thus we completely do without statistical gas physics and apply sticky particle methods to study the evolution of gravitationally unstable, dissipative media.

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\textsuperscript{*}The largest computer simulation in the field of astrophysics ever done used $10^9$ particles (http://star-www.dur.ac.uk:80/moore/).
Part II

N-Body Simulations of Dissipative, Self-Gravitating Systems
Chapter 5

A Local Model of Self-Gravitating Disks

5.1 Introduction

A local, N-body model, whose basic physical ingredients are,

- Self-gravity
- Energy dissipation
- Differential rotation,

has been used to study the formation of long-range correlations in self-gravitating discs.

The results of these study as well as a detailed description of the model were published in the article *Lumpy Structures in Self-Gravitating Disks*.

Subsequently we first motivate the physics of the model and the applied technique. The article is presented on page 57. Finally, the different types of patterns and instabilities that appear in the simulations are discussed.

5.2 Motivation

5.2.1 Why Studying Open Systems?

Real physical systems are open and consequently often maintained outside of equilibrium by energy or matter flows. These flows are related to an entropy flow. If the entropy flow leaving the system is larger than those entering, order is created inside the system. Nonequilibrium thermodynamics teaches that far from equilibrium the system evolution is no longer governed by a thermodynamic extremum principle, thus the system becomes unstable, fluctuations can grow which may lead to more stable states. That is, far from equilibrium instabilities can create new spatio-temporal structures often showing a pronounced degree of order and obeying scaling relations (see chapter 2).

Inspired by what has been learned in non-equilibrium statistical mechanics Pfenniger (1998) suggested that in many situations open, gravitationally unstable systems, can reach a hierarchical fragmented state in dynamical equilibrium. A scenario is invoked in which large scale ordered motions, driven by time-dependent boundary conditions, are degraded
to smaller scales by gravitational instabilities. Then, negentropy (= negative entropy, i.e., order) cascades recursively down the scales until it is absorbed finally by small scale chaos and dissipative terms.

In this scenario gravitational instabilities are thought to lead not to "collapses" (gravitational runaway), but to self-organized nonequilibrium structures, as long as the system is maintained off equilibrium by an energy-flow due to large-scale, time-dependent boundary conditions and small-scale dissipation.

From laboratory experiments it is known that the trajectory describing the evolution of a dissipative system in phase-space becomes far from equilibrium chaotic and evolves typically towards a strange attractor (see chapter 2), meaning that in the long term nonequilibrium structures with persistent or recurrent statistical properties may be formed.

Astrophysical situations or systems where gravitational unstable media may establish long-range correlations are for instance (Pfenniger 1998):

- The formation of the first structures in the Hubble flow just after recombination.
- Cooling flows in galaxies and galaxy clusters.
- Spiral galaxies, as long as the amount of gas allows efficient energy dissipation.

Here, we want to check if self-gravitation in combination with time-dependent boundary conditions and a slight dissipation can produce inhomogeneous, lumpy and eventually self-similar structures, resembling those observed in galactic disks and molecular clouds.

For these purposes local simulations of self-gravitating disks are carried out. In these systems time-dependent boundary conditions are given through the differential disk rotation (shear-flow) representing a reservoir of large-scale, directed kinetic energy (von Weizsäcker 1951; Goldreich & Lynden-Bell 1965). Gravitational instabilities convert this energy to turbulent motion and heat. At small scales the energy is dissipated away by molecular collisions and shocks.

### 5.2.2 Why Using Dissipative Particles?

To simulate slightly dissipative galactic disks we use a sticky particle method. The use of such a method was motivated in detail in chapter 4. Let us briefly repeat the principal reasons:

- Classical thermodynamics is derived under the early and frequent use of extensivity (Gross 2001). Yet, gravitating systems are nonextensive. Thus, classical thermodynamics cannot by applied without reservation to self-gravitating systems.

- The LTE assumption may be violated in gravitational unstable, turbulent media. Indeed, in such systems long range correlations and matter motions propagate on the dynamical time scale $\sim \sqrt{\rho}$, comparable to the sound crossing time, so that thermalization cannot be established.

Pfenniger (1998) suspected that the discrepancy between the available theoretical tools, predicting local homogeneity after a few sound crossing times, and the highly hierarchical nonequilibrium structures, observed in the ISM may be a sign of these fundamental problems.

Thus, the self-gravitating medium at galactic disk scales is here represented by $N$ semi-collisional particles, referred to as dissipative particles, governed by Newton's equation of
motion. Such techniques have been proven to be a powerful tool for the simulation of granular systems, violating the mixing hypothesis of Boltzmann-Gibbs statistics at all scales, both in granular physics (e.g. Du et al. 1995; Jaeger & Nagel 1992; Luding et al. 1996) and in astrophysics (e.g. Brahic 1977; Wisdom & Tremaine 1988; Toomre & Kalnajs 1991).

Since we are concerned with the fundamental physical factors that may lead to the formation of long-range correlations in open, self-gravitating systems, our approach is not to include a maximum of physical ingredients present in complex physical objects, such as molecular clouds, but just the ones thought to be the most relevant for the problem on hand. Thus, several simplifying assumptions are made.

For instance, the matter is assumed to have an undefined mass composition with a slight dissipation. Then, for a normal spiral each particle may be considered as a mixture of stellar mass and gas, with a mean weak energy dissipation.

As long as energy dissipation remains weak the exact dissipation force may be irrelevant since the long term behavior of a dissipative system is a strange attractor (see chapter 2 and Pfenniger & Norman 1990). Thus the particles dissipate their energy via a simple linear friction force.

The physically simple and general framework of the model permits to draw conclusions for systems in which similar basic factors are present, such as molecular clouds and HI-disks.

5.2.3 Why Using Local Models?

As mentioned above we primary want to study processes that may give rise to spatio-temporal structures in self-gravitating disks, so we are not concerned with the overall density profile nor with bar instabilities.

Thus local simulations are carried out, since such simulations allow, in contrast to global ones, to study complex disk inhomogeneities independent of the global disk thickness and the particular dynamics at the disk center. Furthermore, using the same particle number local models can resolve smaller scales than global ones.

5.2.4 Simulating Large Disks with Local Models

The local model applied to study self-gravitating disks is described in the article on page 59ff. Subsequently, we summarise the principle, and explain why the local model can be used to study idealized disks on the kpc scale.

In local models everything inside a box (local simulation box) of a given size is simulated, and more distant regions in the plane are represented by replicas of the local box, in such a way that a particle leaving, for instance, the box at the rotationally leading border is re-fed at the trailing one. This resembles the periodic boundary conditions used in cosmological simulations with the complication that in galactic dynamics the periodic continuations must share the galactic shear due to differential rotation. Consequently, in order to keep the dynamics continuous during the re-feeding process, particles leaving the simulation box radially are re-fed at the opposite side at positions varying with time in accordance with the shear of the disk.

The origin of the particle coordinates in the local box is a reference point that moves on a circular orbit at distance $R_0$ from the galactic center with the orbital frequency $\Omega_0 = \Omega(R_0)$. In rotating Cartesian coordinates, the horizontal particle positions are given by $x = R - R_0$, $y = R_0(\theta - \Omega_0 t)$ and the vertical location by $z$. Inserting these coordinates into Newton's
equations of motion, a linearization with respect to \( x, y \) and \( z \) yields the local equations of motion, describing the system evolution under the influence of a external force field. In the present context they read (Hill 1878; Julian & Toomre 1966; Nakazawa & Ida 1988):

\[
\begin{align*}
\ddot{x} - 2\Omega_0 \dot{y} &= 4\Omega_0 A_0 x + F_x \\
\ddot{y} + 2\Omega_0 \dot{x} &= -\nu^2 y + F_y \\
\ddot{z} &= -\nu^2 z + F_z.
\end{align*}
\]

(5.1)

where \( \nu \) is the vertical frequency of the epicycle approximation (Binney & Tremaine 1994) and \( A_0 = -\frac{1}{2} R_0 (d\Omega/dR)_R \) is the Oort constant of differential rotation. Here \( A_0 = 1/2\Omega_0 \) meaning that \( V(R) = R\Omega(R) = \text{const} \). The second terms on the left hand side denote the Coriolis force. The term \( 4\Omega_0 A_0 \) gives the tidal force due to the global galactic disk potential and \( F_x, F_y, F_z \) are local forces due to self-gravitating particles.

(Demleitner 1998) showed that the local equation of motion can be derived under the assumption that 1.) the overall radial structure of the disc does not significantly change inside the box and 2.) the motions due to particle interactions in the local box are small compared to the circular velocity. Note that no explicit assumption limiting the extend of the local box is necessary, so that the local model can be used to study the dynamics of idealized discs on the kpc scale.

5.2.5 Why Using a Time-Dependent Coordinate System?

Hitherto Cartesian coordinates have been used in local models. In such coordinates the forces of the self-gravitating particles must be determined by direct summation. Thus, in order to increase the performance, we apply a time-dependent Coordinate system, allowing the gravitational force computation via a particle-mesh FFT-method.

Subsequently, we discuss this in more detail and add some complementary figures that should help the reader to follow the description of the model in the article.

Conventional Method: Cartesian Coordinates

In a Cartesian coordinate system \((x, y, z)\) the positions of the rectangular boxes (local simulation box and replicas) change with time due to the differential rotation \( d\Omega_0/dx \). The differential rotation causes a shear-flow, which is for a flat rotation curve, \( \dot{y} = -\Omega_0 x \). Thus the \( y \)-coordinate of the replicas is time dependent, so that a particle at \((x, y, z)\) has images at \((x + nL_x, y - nL_x \Omega_0 t + mL_y, z)\), where \( n \) and \( m \) are integers (Wisdom & Tremaine 1988). \( L_x, L_y \) and \( L_z \) are the sides of the local box. Thus, an initially \((t = 0)\) periodic arrangement of the boxes relative to a fixed Cartesian coordinate system can not be maintained (Wisdom & Tremaine 1988; Toomre 1990; Griv 1997; Daisaka 1999). As a consequence Fourier (i.e. FFT) methods cannot be applied and the forces of the self-gravitating particles have been determined in previous experiments by direct summation, meaning that the computation time for \( N \) particles is proportional to \( N^2 \). Thus, in order to reduce the burden of the force calculation it has been assumed that local gravitational forces vanish for larger particle separations due to cancelation (Toomre & Kalnajs 1991; Salo 1995). Thus a spherical region with radius \( R_{\text{max}} < \min(L_x, L_y) \) is chosen around each particle and only forces due to neighboring particles within this region are summed up. This and the time-dependent position of the shearing boxes are illustrated in Fig. 5.1.
New Method: Time-Dependent Affine Coordinate System

In the course of time the initial periodic arrangement of the boxes loose its periodicity only relative to a fixed Cartesian system. But not with respect to an affine coordinate system whose pitch angle changes in time in correspondence with the shear flow. Indeed, because the shear is linear in $x$ there is for all times an affine coordinate system $(x', y', z')$ on which the system is periodic.

Thus, using such a time-dependent coordinate system, the forces can be computed in Fourier space with the convolution method and the FFT algorithm (Press et al. 1986). Thereby the force computation time is reduced to be proportional to $N_c \log(N_c)$, where $N_c$ is the number of cells, taken here as proportional to the number of particles.

Actually, the FFT approach requires a system spatially isolated and/or periodic at all times. Here the system, representing the dynamics of an idealized disk, is isolated in $z$-direction. In the $x-y-$ plane the system is periodic, relative to the affine coordinate system, whose pitch angle $\alpha$ is determined by the shear,

$$\tan \alpha = [(-\Omega_0 t) \mod (L_y/L_x)].$$

The term, $\mod (L_y/L_x)$, expresses a periodical evolution of the pitch angle. Indeed, after $t = L_y/(L_x \Omega_0)$ the system recovers periodicity relative to a Cartesian coordinate system. Thus the pitch angle jumps from $\alpha = L_y/L_x$ back to zero and the process starts anew (see Fig. 5.2).

This back-jumping introduces discontinuities in the evolution of the pitch angle, which may lead to discontinuities in the force field. Yet, the discontinuities can be removed when, instead of one, two time-dependent coordinate systems are applied. Thus we apply two such coordinate systems (see article I on 111ff).

As discussed above, in the direct summation scheme the local gravitational forces acting on a particle result only from neighbors with distances smaller than $R_{\text{max}} < \min(L_x, L_y)$. In the Fourier scheme, these forces stem from neighbors resided in a region with size, $1.5L_x \times 1.5L_y \times L_z$. Thus in the models gravity has a longer range, when Fourier methods are applied. This is shown in Fig. 5.2 (b).
Conventional method: Cartesian coordinates, direct summation calculation.

Figure 5.1: The large dotted frame represents a section of the disk, being infinite in two directions, seen from above. Shown is the evolution of the position of the local simulation box and its replicas when fixed Cartesian coordinates \((x, y, z)\) are used. a.) The initial \((t = 0)\) periodic arrangement of the boxes. The six neighboring replicas of the local box (grey), are highlighted by solid lines, so that one can follow their evolution. The other replicas are dashed. b.) 1.) The periodic arrangement relative to the fixed Cartesian coordinate system is destroyed due to the shear flow, \(\dot{y} = -\Omega_0 x\). Here the position of the boxes is shown at \(t = L_y/(2L_x\Omega_0)\). 2.) Local gravitational forces acting on a particle (e.g. black dot in the local box) result only from interactions with neighbors inside a sphere with radius \(R_{\text{max}}\) (dash-dotted circle). Due to the regular arrangement of the boxes the position of the particles within this sphere but outside of the local box can be deduced from their copies inside the local box. Here the region with the corresponding copies are depicted by solid circular arcs. c.) After \(t = L_y/(L_x\Omega_0)\) the arrangement of the boxes recover periodicity relative to the fixed Cartesian coordinate system and the process starts anew at (a).
5.2. MOTIVATION

New method: Time-dependent affine coordinate system, FFT algorithm calculation.

Figure 5.2: The large dotted frame represents a section of the disk, being infinite in two directions, seen from above. Shown is the evolution of the position of the local simulation box and its replicas when a time-dependent affine coordinate system \((x', y', z')\) is used. 

\textbf{a.)} At \(t = 0\) the pitch angle of the time-dependent coordinate system is, \(\alpha = 0\), so that \((x', y', z') = (x, y, z)\), where the latter are Cartesian coordinates. To compare with the previous figure the six neighboring replicas of the local box (grey) are highlighted by solid lines. 

\textbf{b.)} 1.) The system remains periodic for all times but only on an affine coordinate system whose pitch angle change in time. Here the inclination of the time-dependent coordinate system is shown for, \(t = L_y/(2L_x\Omega_0)\). 

\textbf{c.)} In the Fourier scheme, the local gravitational forces acting on particles in the lower left quarter of the local box (dark zone), result from interactions with particles resided in the dash-dotted box. This box is shifted by \(L_x/2\) and \(L_y/2\), respectively, when particles in the other three quarters of the local box are considered. 

\textbf{c.)} When the time-dependent coordinate system reaches this inclination after \(t = L_y/(2L_x\Omega_0)\), the arrangement of the boxes and thus the system recovers periodicity relative to a Cartesian coordinate system. Consequently, the inclination jumps back to the position shown in (a) and the process starts anew.
Lumpy structures in self-gravitating disks

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Abstract. Following Toomre & Kalnajs (1991), local models of slightly dissipative self-gravitating disks show how inhomogeneous structures can be maintained over several galaxy rotations. Their basic physical ingredients are self-gravity, dissipation and differential rotation. In order to explore the structures resulting from these processes on the kpc scale, local simulations of self-gravitating disks are performed in this paper in 2D as well as in 3D. The third dimension becomes a priori important as soon as matter clumping causes a tight coupling of the 3D equations of motion. The physically simple and general framework of the model permits to make conclusions beyond the here considered scales. A time dependent affine coordinate system is used, allowing to calculate the gravitational forces via a particle-mesh FFT-method, increasing the performance with respect to previous direct force calculations. Persistent patterns, formed by transient structures, whose intensity and morphological characteristic depend on the dissipation rate are obtained and described. Some of our simulations reveal first signs of mass-size and velocity dispersion-size power-law relations, but a clear scale invariant behavior will require more powerful computer techniques.

Key words. methods: numerical – galaxies: structure, ISM – ISM: structure

1. Introduction

Classical gravity is scale free, i.e., self-gravitating systems may form similar structures on different scales. Indeed, observations of the interstellar medium, spiral disks and cosmic structures, do reveal similar characteristics. Although the structures in these systems are lumpy and inhomogeneous, they do not seem yet completely random.

The observations of molecular clouds reveal hierarchical structures with power-law behavior over several orders of magnitude in scale (Larson 1981; Scalo 1985; Falgarone et al. 1992; Heithausen et al. 1999). Larson (1979) found first hints that the power-law relation between velocity dispersion and size is also valid for stellar populations and that it extends beyond the size of Giant Molecular Clouds. Several observations confirm that the hierarchical structure of kinematically cold media is not only present in Milky Way molecular clouds, but is also found in other systems and on larger scales. For example, Vogelaar & Wakker (1994) found perimeter-area correlations in high-velocity clouds; power-law power spectra of HI emission were found in the Small and the Large Magellanic Cloud by Stanimirovic et al. (1999, 2001), and Elmegreen et al. (2000), respectively; measurements of the HI distribution in galaxies of the M 81 cluster reveal fractal structures on the galaxy disc scale (Westpfahl et al. 1999).

On cosmic scales, up to about 100 Mpc, matter is also hierarchically organized. A common feature of the ISM and the cosmic structure is that the matter distribution can be characterized by a comparable fractal dimension. The cosmic and the interstellar fractal dimensions are, $D_{\text{Galaxies}} \approx 2 \pm 0.2$ (Sylos Labini et al. 1998; Joyce et al. 1999) and $D_{\text{ISM}} \approx 1.6-2.3$ (Elmegreen & Falgarone 1996; Combes 1998), respectively. Thus the precise value of the fractal dimension does not seem to be universal, but a range between 1 and 2 appears frequent.

All this may suggest that a general scale free factor is mainly responsible for the matter distribution and the dynamics of cosmic structures, galactic disks and molecular clouds. Only one factor appears to be dominant over all these scales, namely gravity.

Gravo-thermal experiments on isolated systems show that typically two possible states are reached asymptotically, a high energy homogeneous state and a low energy collapsed state, with a halo-core structure (Lynden-Bell & Wood 1968; Hertel & Thirring 1971; Aronson & Hansen 1972).

Thus to produce more inhomogeneous structures self-gravitating systems must be open, such as be subjected to time dependent boundary conditions. On cosmic scales the Hubble flow represents a time dependent boundary condition, and develops lumpy structures. On galactic scales down to molecular cloud scales an energy flow, maintaining the system out of equilibrium, may be sustained by
the shear-flow and small scale dissipation. Indeed, gravita-
tional instabilities convert directed kinetic energy (shear-
flow) into thermal and turbulent motion (von Weizsäcker
1951; Goldreich & Lynden-Bell 1965). Turbulent motion
may then transport the energy through the scales until it
is dissipated away by radiation in molecular collisions and
shocks.

The lumpy distribution of matter reported by Toomre
(1990) and Toomre & Kalnajs (1991, hereafter TK) in
local shearing-sheet experiments of disks reminds us of
the ubiquitous inhomogeneous state of the ISM as well
as the flocculent structures of many spirals. The rele-
ance of these experiments for galaxies is supported by
the recurring spirals found in slightly dissipative complete
self-gravitating disk simulations by Sellwood & Carlberg
(1985) and many others (e.g., Miller et al. 1970). The
TK models confirm that purely self-gravitating systems
with time-dependent boundary conditions can produce
very chaotic inhomogeneous structures.

Here, in order to investigate in more detail the mat-
ter distribution produced by self-gravitation, shear and
dissipation we perform further such local shearing-sheet
experiments. To check if the resulting structures reveal
power-law relations we calculate the power-law indices
of the mass-size and the velocity dispersion-size relation. To
compare with earlier models, in particular with those of
TK, we start with 2D simulations and extend then the
model to 3 dimensions. This extension is important be-
cause as soon as dense clumps develop in a disk with hor-
izontal sizes comparable to or smaller than the supposed
thickness of the disk, motion transverse to the plane must
be strongly coupled to the motion in the plane, and the
2D approximation is no longer valid.

To obtain instructive models it is important not to in-
clude too many ingredients. We are primarily concerned
not with complex physical objects such as molecular
clouds, but with processes. So our approach is not to in-
clude a maximum of physical ingredients, but just the ones
that appear as the most relevant. We want to check if self-
gravitation in combination with time-dependent boundary
conditions and a slight dissipation can produce and main-
tain an inhomogeneous, lumpy and eventually self-similar
structures, resembling those observed in galactic disks and
molecular clouds.

The considered scales are of the order of $O(1-10)$ kpc.
Thus the transition regime between the molecular cloud
scale ($\approx 0.05$ kpc) and the galactic disk scale ($\approx 10$ kpc)
can be investigated. However, since the model is scale-
free, we can draw conclusions beyond the here considered
scales and thus eventually contribute to illustrate the scal-
ing laws observed in sub- or extra-galactic structures.

Preliminary results of our numerical experiments were
presented in Huber & Pfenniger (1999, 2001). Since then
we continued to improve our model and to collect more
experience, which led to new insights with respect to the
clustering simulation and scaling laws. In this paper we
discuss in detail the model and the results. Similar studies
have been presented by Semelin (Semelin 1999; Semelin &
Combes 2000).

In the next section, we justify the use of dissipative
particles in order to model the dynamics of self-gravitating
gas. The numerical model in presented in Sect. 3 and a
pseudo-code is given in Appendix A. In Sect. 4 we discuss
the methods used to analyze the structures resulting from
our shearing box simulations. The results of the 2D and
3D simulations are presented in Sect. 5. Finally, Sect. 6 is
dedicated to discussing limitations of the models.

2. Physical gas model

2.1. Hierarchical systems

For a hierarchical self-similar structure, one can define a
fragmentation efficiency (Scalo 1985),

$$f = \eta m_{L-1}/m_L,$$

where $m_{L-1}$ and $m_L$ are the mean masses of a fragment
at level $L-1$ and $L$, respectively. The factor $\eta$ is the
number of fragments formed at each level. The fragmenta-
tion efficiency $f$ indicates how much mass in a clump is
concentrated in subclumps. If $f$ is not very high ($<95\%$),
the smallest fragment masses become negligible after a
few levels and a hierarchical description is less relevant.
However, if $f$ is very high, several iteration steps can be
conducted and the bulk mass is still concentrated in the
smallest subclumps. As long as the bulk mass is concen-
trated in subclumps the interclump mass can be neglected.
For convenience we call the smallest clumps for which the
interclump medium can be neglected basic-clumps. If the
level of the basic clumps is zero, a clump at level $L$ is
formed by $\eta^L$ basic clumps.

Observations of the interstellar medium reveal a highly
inhomogeneous and clumpy structure (see, e.g., Dame et al.
2000; Tauber et al. 1991; Störzer et al. 2000). Moreover the structure is for a certain scale range hier-
archical. Assuming that the size of particles is larger or
equal to the size of the basic gas clumps, the structure of
the interstellar medium can be described correctly down
to the scale of the particles by the distribution of these
particles. A particle represents then the lowest resolvable
level, while clumps at higher levels, i.e., at larger scales
are represented by an ensemble of particles.

2.2. Dissipative particles

At the here considered scales, larger than tens of pc, the
description of the dissipative processes taking place in the
ISM is very complex and far from respecting the hypothe-
ses allowing the full application of Navier-Stokes equa-
tions. Thus the use of a traditional hydrodynamic code is
in no way “better” than the simpler approach adopted by
TK, where a simple small drag parameter is all what is
introduced as dissipation.
Indeed, we recall the following considerations:

1) Being long range, gravity breaks the fundamental assumption made in classical thermodynamics that interactions are short ranged. In turn, when gravity is sufficiently strong (i.e., the Jeans’ instability threshold is reached), supersonic chaotic motion is expected, as also systematically reported for the interstellar medium. This means that no local pressure equilibrium is reached at the scales over which turbulence exists. Down to the smallest scale at which supersonic turbulence exists no local thermodynamical equilibrium can be established, and thus no equation of state can be defined. A basic assumption allowing to derive the usual Navier-Stokes equations of fluid dynamics is missing. Besides, numerous observational evidences indicate non-thermal cloud clumps. For instance, Beuther et al. (2000) carried out multi-wavelength observations and compared the line ratios with radiation transport models. They found that models based on the assumption of a local thermodynamical equilibrium (LTE) cannot reproduce the observed data set. Due to the lack of a LTE down to smallest scales, thermal physics appears as an inappropriate tool to represent the statistical state of interstellar gas;

2) Fundamentally the Navier-Stokes fluid equations describe a) the local conservation of mass and momentum, with b) additional constraints such as local smoothness of the quantities subject to differentiation, c) an equation of state for closing the moment equations derived from Boltzmann’s equation, and d) phenomenological laws describing viscous forces. While the mass and momentum conservation laws are likely to be adapted even for such a clumpy medium as the ISM, the other constraints do not. In this context the energy equation is little relevant (is not a constraint) if no control can be performed on radiative processes, which operate on very short time-scale in the cold ISM. Since large but clumpy entities such as molecular clouds have a mean-free path much larger than their size, the dynamics of such systems may as well, in the present state of understanding, be described by semi-collisional, dissipative particles (Brahic 1977; Pfenniger 1998). Casoli & Combes (1982) studied the formation of giant molecular clouds through cloud collisions and coalescence in the molecular ring. They found that the ensemble of clouds never reaches a steady state. Thus they concluded that clouds are better described by particles than by a fluid. Another hint that the usual fluid equations are not better adapted to describe the ISM than sticky particles is that the rings in barred galaxies are never reproduced by the former but easily by the latter (e.g., Schwarz 1984);

3) In the ISM not only the cooling and heating processes are rapid with respect to global dynamics, but also the energy reservoir of global dynamics is much larger than the other available energy reservoirs represented by gas pressure, stellar radiation, cosmic rays, or magnetic fields (Pfenniger 1996). The virial theorem, expressing a balance of negative and positive energy reservoirs, is a useful tool to order the importance of respective physical factors according to their quantitative values. Since the energy budget at the galactic scale is dominated by dynamics, to first order the system is well described by conservative dynamics, and dissipative effects are of second order. In weakly dissipative systems the stable periodic orbits and fixed points of the conservative case are transformed into attractors, and chaotic orbits typically converge toward strange attractors with similar chaotic properties. Therefore one can naturally infer that the exact dissipative force is irrelevant as long as it remains weak, since the long term behavior is an attractor. The dissipative perturbation is weak when during the time-scale of interest the energy dissipated is small with respect to the total energy of the system. Therefore in this regime it is not necessary to know precisely how energy is dissipated, any weak factor leads to the same attractors (see Pfenniger & Norman 1990 for an extended discussion on the topic).

These considerations show that weakly dissipative particles are a permissible method to study the dynamics of interstellar gas at sufficiently large scales. The mass and momentum conservation is granted by the equations of motion, and the weakly dissipative regime by a simple linear friction law.

3. Numerical model

In previous studies of shearing sheet disks, the forces of the self-gravitating particles were computed by direct summation. Instead, we show that by using a time-dependent affine coordinate system we can represent the shear-flow in periodic coordinates. Consequently we can increase the computation performance by calculating the self-gravitational forces with the popular FFT-convolution.

3.1. Principle

Here we explain the principle of the local model for the 3D case. Ignoring all expressions with a z, yields the 2D case.

In a local model of a disk, everything inside a box of a given size is simulated, and more distant regions in the plane are represented by replicas of the local box (Toomre & Kalnajs 1991; Wisdom & Tremaine 1988; Salo 1995). The global galactic disk attraction made by components such as stars or dark halo not included in the local box is 1) cancelled to zeroth order by adopting a rotating frame, and 2) corrected to first order by the linear terms in the epicycle \( \kappa \) and vertical frequency \( \nu \) (Binney & Tremaine 1994).

In the same spirit as TK, the matter in the box has an undefined mass composition with a slight dissipation. For
a normal spiral each particle may be considered as a mixture of stellar mass and gas, with mean weak dissipation.

The origin of the particle coordinates in the local box is a reference point that moves on a circular orbit at distance $\mathcal{R}_0$ from the galactic center with the orbital frequency $\Omega_0 = \Omega(\mathcal{R}_0)$. In their model, TK used a rotating Cartesian coordinate system. The horizontal particle positions are then given by $x = \mathcal{R} - \mathcal{R}_0$, $y = \mathcal{R}_0(\theta - \Omega_0 t)$ and the vertical location by $z$. If $x, y, z \ll \mathcal{R}_0$, the orbital motion of the particles is determined by Hill’s approximation of Newton’s equations of motion (Hill 1878). In the present context, they read:

$$\begin{align*}
\ddot{x} - 2\Omega_0 \dot{y} &= 4\Omega_0 A_0 x + F_x , \\
\ddot{y} + 2\Omega_0 \dot{x} &= F_y , \\
\ddot{z} &= -\nu z^2 + F_z ,
\end{align*}$$

(2)

where $A_0 = -\frac{1}{2} \mathcal{R}_0 (d\Omega/d\mathcal{R}) \mathcal{R}_0$ is the Oort constant of differential rotation. $F_x, F_y$ and $F_z$ are local forces due to the self-gravitating particles, that should be small with respect to the global force field. Like TK and Griv et al. (1999) we use these equations also for simulation zones, where $x, y, z \ll \mathcal{R}_0$ is not valid for the most part, i.e., for galactic disc scales, meaning that non-linear higher order effects are not taken into account. However, since much of the gravitational force in any wavy disturbance stems from the nearest particles (TK, Julian & Toomre 1965; Toomre 1964), the conclusions of the model should be relevant for galactic disks, despite the violation of the linearity hypothesis. Indeed, the swing amplification theory, whose applicability to spirals has been well established, is based on the same assumptions.

In a Cartesian coordinate system $(x, y, z)$ the positions of the rectangular boxes (local simulation box and replicas) change with time due to the differential rotation $d\Omega/dx$. The differential rotation causes a shear-flow, which reads for a flat rotation curve, $\dot{y} = \Omega_0 x$. A particle at $(x, y, z)$ has images at $(x + n L_x, y - n L_y \Omega_0 t + m L_y, z)$, where $n$ and $m$ are integers (Wisdom & Tremaine 1988). $L_x, L_y$ and $L_z$ are the sides of the local box. An initially $(t = 0)$ periodic arrangement of the boxes relative to a fixed Cartesian coordinate system can not be maintained (see Salo 1995). As a consequence the forces of the self-gravitating particles have been determined in previous simulations by direct summation with upper and lower cut-offs, meaning that the computation time for $N$ particles is proportional to $N^2$.

However we can improve the performance by computing the forces in the Fourier space with the convolution method and the FFT algorithm (Press et al. 1986). Thereby the potential computation time is reduced to be proportional to $N_c \log(N_c)$, where $N_c$ is the number of cells, taken here as proportional to the number of particles. The FFT approach requires a system spatially isolated and/or periodic at all times. Here the system, representing the local dynamics of a disk, is isolated in $z$-direction. In the $x - y$-plane the system is periodic, but only on affine coordinate systems whose pitch angles change periodically.

The dark box in Fig. 1a represents the local box in a Cartesian coordinate system (solid mesh). In the initial state a certain local particle (star in the dark box) and its replicas (stars outside the box) are periodic relative to the Cartesian coordinate system. But then the particles are shifted by the shear and the periodicity relative to a rectangular coordinate system is lost. However because the shear is linear in $x$ there is for all times an affine coordinate system $(x', y', z')$ on which the system is periodic. Thus we modify our initially rectangular coordinate system with a time dependent pitch angle. The solid mesh in Figs. 1a–c represents an affine coordinate system in which the periodicity of the system is maintained. Its pitch angle is, $\alpha_b \leq 0$, for all times. Thus we call this coordinate system for convenience the backward mesh. The inclination of the backward mesh $\alpha_b$ is determined by the shear,$$
\alpha_b = \tan \alpha_b = [(-\Omega_0 t) \mod (L_y/L_z)].
\quad (3)
$$

Figure 1c shows the system at $t = L_y/(L_z \Omega_0)$. We can see that the periodic arrangement of the particle images corresponds to those in Fig. 1a. Consequently the system is again periodic on a rectangular coordinate system and we can replace the backward mesh in Fig. 1c with the one in Fig. 1a. Thus the inclination of the affine coordinate system jumps at $t = L_y/(L_z \Omega_0)$ from $\alpha_b = L_y/L_z$ to $\alpha_b = 0$. This accounts for the modulo function in Eq. (3). Thus, if $L_z/(L_y \Omega_0)$ is a multiple of the time-step, only a finite number of affine coordinate systems is necessary. As a consequence the corresponding kernels must be computed only once at the beginning of the simulation and stored for subsequent use.

For the computation of the forces of the self-gravitating particles one coordinate system in which the matter distribution is periodic at all times would in principle be enough (e.g., the backward mesh). However, in order to avoid discontinuities in the force field when the inclination of the coordinate system $\alpha_t$ jumps back to zero, we compute the forces additionally in a second affine coordinate system, in which the system is periodic as well. The dashed mesh in Figs. 1a–c represents this second coordinate system. Because its pitch angle is always, $\alpha_t > 0$, we call it the forward mesh. The inclination of the forward mesh $\alpha_t$ can be deduced from those of the backward mesh by:

$$\alpha_t = \tan \alpha_t = \alpha_b + (L_y/L_z).$$

(4)

The light box in Fig. 1 is the local computation box of the forward mesh. After the computation of the forces $F'$ in both coordinate systems, we add them with weighting factors in order to soften the effects of the abrupt transition at $t = L_y/(L_z \Omega_0)$ on the force field. The forces computed in the forward and the backward mesh are $F_b'$ and $F_f'$, respectively. Before adding the forces with the corresponding weighting factors, we transform them to Cartesian coordinates, $F_b' \rightarrow F_b, F_f' \rightarrow F_f$. The single components of the forces are transformed as follows:

$$\begin{align*}
F_{i,x} &= F_{i,x}' , \\
F_{i,y} &= F_{i,y}' + a F_{i,x}' ,
\end{align*}$$

(5)
D. Huber and D. Pfenniger: Lumpy structures in self-gravitating disks

Fig. 1. The dotted frame represents a section of the disk, being infinite in two directions, seen from above. The two meshes are affine coordinate systems on which the mass distribution is periodic. The dark and the light box represent the local computation box in affine coordinates, i.e., in the forward and the backward mesh respectively. a) The initial state of the two meshes \( t = 0 \). The inclinations of the meshes are \( a_b = 0 \) and \( a_f = (L_y/L_x) \). Thus the corresponding weighting factors are \( b = 1 \) and \( f = 0 \), respectively. Consequently, the forces in the Cartesian coordinates are for this situation, \( F = F_b \). Below the two meshes, the pitch angles of the affine coordinate systems are indicated, \( \alpha_f \) and \( \alpha_b \), respectively. The angle between the two meshes remains the same at all times \( (\alpha_f + \alpha_b = \arctan(L_y/L_x)) \). b) The meshes and the weighting factors at \( t = L_y/(2L_x \Omega_0) \). It is valid, \( a_b = -L_y/(2L_x) = -\alpha_f \) and thus \( F = F_b/2 + F_f/2 \). c) When the meshes reach these inclinations \( (t = L_y/(L_x \Omega_0)) \), they jump back to the positions shown in (a) and the process starts again without introducing discontinuities in the dynamics.

\[
F_{i,z} = F'_{i,z},
\]

where \( i = b, f \) for the backward, respectively for the forward mesh. Then the forces are weighted and added, \( F = bF_b + fF_f \). The weighting factors \( b \) and \( f \) are normalized \( (b + f = 1) \) and proportional to the mesh inclination, \( b = -a_b L_x/L_y \). Because forces are additive such a weighted force summation is permissible. The forces \( F \) correspond now to those in Eq. (2). That is, the inclined coordinate system are only used to compute the forces of the self-gravitating particles with the convolution method;
then they are transformed to a Cartesian coordinate system. The evolution of the system in the Cartesian coordinate system is given by Eq. (2).

The weighting described above, not only softens the effect of the abrupt change in time of the pitch angles, but also minimizes asymmetry effects due to the mesh inclinations. Asymmetry effects disappear for example completely when the inclination of one of the meshes is zero or when both meshes have the same inclination. In the first case (Fig. 1a) the weighting factor of the uninclined mesh is one and thus the forces are computed exclusively in the rectangular coordinate system. In the second case (Fig. 1b) the asymmetry effects in both inclined coordinate systems cancel each other out.

In an inclined coordinate system the gradient \( \nabla \) depends on the inclination. This must be taken into account by the calculation of \( F'_b \) and \( F'_f \). The Euler-Lagrange equations yield then the forces in an affine coordinate systems,

\[
\begin{align*}
F'_{i,x} &= a_i \frac{\partial \Phi}{\partial y^i} - a_x \frac{\partial \Phi}{\partial x^i} \\
F'_{i,y} &= -(1 + a_x^2) \frac{\partial \Phi}{\partial y^i} + a_y \frac{\partial \Phi}{\partial x^i} \\
F'_{i,z} &= \frac{\partial \Phi}{\partial z^i},
\end{align*}
\]

where \( i = \{b,f\} \) for the backward, respectively for the forward mesh.

### 3.2. Canonical equations

Pfenniger & Friedli (1993) shown that the use of a leapfrog finite difference approximation of Newton’s equations in a rotating reference frame with non-canonical variables lead to instability (“complex instability”) in the sense of von Neumann. This is not the case when canonical variables are used, then the stability or instability character is conserved between the leap-frog algorithm and the orbits. Therefore our model uses these equations. The canonical equations of motion with the momenta \( \{p_x, p_y, p_z\} \) are,

\[
\begin{align*}
\dot{x} &= p_x + \Omega_0 y \\
\dot{y} &= p_y - \Omega_0 x \\
\dot{z} &= p_z,
\end{align*}
\]

and

\[
\begin{align*}
p_x &= (4\Omega_0 A_0 - \Omega_y^2) x + F_x + \Omega_0 p_y \\
p_y &= -\Omega_y^2 y + F_y - \Omega_0 p_x \\
p_z &= -\nu^2 z + F_z.
\end{align*}
\]

These equations are invariant under the linear transformation

\[
\begin{align*}
x' &= x + k_x \\
y' &= y - 2A_0 t k_x + k_y \\
p_x' &= p_x + 2A_0 \Omega_0 t k_x - \Omega_0 k_y \\
p_y' &= p_y + (\Omega_0 - 2A_0) k_x,
\end{align*}
\]

where \( k_x \) and \( k_y \) are arbitrary numbers. Thus, whenever a particle leaves the local box \( L_x \times L_y \times L_z \) in the \( x \) or \( y \) direction and its image enters somewhat on the opposite side (in the affine meshes the image enters exactly at the opposite face), we also have to transform the canonical momenta and their time derivatives correspondingly to the rules given above.

### 3.3. Kernel

For the 2D simulations we use an isotropic interaction potential. However, in order to resolve the flat disk vertically an anisotropic kernel is necessary due to computational limits. Thus most of the 3D simulations are carried out with an anisotropic kernel having the form of a parallelepiped.

#### 3.3.1. Isotropic kernel

In affine coordinates the softened isotropic interaction potential has the form,

\[
\Phi = \begin{cases}
\frac{1}{r} & : r \leq \varepsilon \\
\frac{1}{r^4} & : r > \varepsilon,
\end{cases}
\]

where \( a \) is the mesh inclination and \( \varepsilon \) is the softening length. The advantage over a Plummer potential, used by TK and many others, is that this potential become a correct \( 1/r \) gravitational potential beyond the softening length. Thus there is no sum up of small errors of the gravitational force due to the many distant particles as in the case of a Plummer potential (Dehnen 2000).

#### 3.3.2. Anisotropic kernel

The simulation box, representing local dynamics of a disk galaxy on the kpc scale, is rather flat (\( L_z \ll L_x, L_y \)). Thus our 3D-model needs an anisotropic force resolution and consequently an anisotropic kernel. This will be explained more exactly in the following. To calculate the forces of the self-gravitating particles we use a particle-mesh method. This method consists of three steps. First, the particle masses are assigned to the nodes of a mesh, which we call simulation mesh. We do this in accordance with the cloud-in-cell (CIC) scheme (see, e.g., Hockney & Eastwood 1981). The masses at the nodes of the simulation mesh can be considered as new particles representing the mass distribution of the original particles. Second, the forces for the new particles are calculated on the simulation mesh nodes via the convolution method (Hockney & Eastwood 1981):

\[
\Phi_{ijk} = K_{ijk} \rho_{ijk} \\
\Phi = \Phi_{ijk},
\]

where \( \sim \) and \( \sim^{-1} \) are the Fourier and the inverse Fourier transform, respectively. \( K_{ijk} \) is the kernel and \( \rho_{ijk} \) is the mass density at a simulation mesh node \( r'_{ijk} \). If the mesh
tial. Pfenniger & Friedli (1993) optimized the softening by adapting the particle extent as well as possible to the cell triaxial ellipsoid. Here we adopt the particle extent to the cell shape as well. At a given time the cell shapes are all identical, typically, because of the shear, a non-rectangular parallelepiped. In the orthogonal case the corresponding analytical form of the potential is known (McMillan 1958). The analytical expression of such a potential is however quite cumbersome. Moreover we need also to describe the non-rectangular case. Thus we use a discrete realization of the particle mass. To this end we distribute the mass of the new particles over a refined discrete mesh having the same size as a cell of the simulation mesh. We call the refined mesh, new particle mesh. Simulation mesh and new particle mesh are shown in Fig. 2. To calculate the kernel $K_{ijk}$ at position $r'_{ijk}$ one has to sum up over all mass points of the discrete particle realization,

$$K_{ijk} = \sum_{u=1}^{N_u} \sum_{v=1}^{N_v} \sum_{w=1}^{N_w} \frac{1}{|r'_{ijk} - r'_{uvw}|},$$

(13)

where $N_u \times N_v \times N_w$ is the number of mass points representing a particle and $r'_{uvw}$ are the cell-centers of the new particle mesh (see Fig. 2). The positions are given in affine coordinates $r' = (x', y', z')$. In order to calculate the kernel the following coordinate transformation is thus carried out,

$$x' = x, \quad y' = y - ax, \quad z' = z,$$

(14)

where $a$ is the the inclination of the affine coordinates. The inclination is fixed by the pitch angle, $a = \tan \alpha$. Consequently the denominator in Eq. (13) has the form

$$|r'_{ijk} - r'_{uvw}| = (1 + a^2)(x'_i - x'_u)^2 + (y'_j - y'_v)^2 + (z'_k - z'_w)^2 + 2a(x'_i - x'_u)(y'_j - y'_v))^{1/2}.$$

(15)

Since the kernel needs to be evaluated only once for every possible inclination the cost of this procedure remains negligible.

Because the cell size of the simulation mesh determines the particle extent, the softening is automatically fixed by the choice of the simulation mesh.

It is important that the origin of the kernel represents the center of the simulation box in order to avoid a non-zero temporal mean velocity of the center of mass, which is introduced by an asymmetric description of the centrifugal force. Thus the positions $r'_{ijk} = (x'_i, y'_j, z'_k)$ are fixed as follows:

$$x'_i = \left(i - \frac{N_x}{2}\right) l_x: \quad i = 1, \ldots, N_x$$

$$y'_j = \left(j - \frac{N_y}{2}\right) l_y: \quad j = 1, \ldots, N_y$$

$$z'_k = \left(k - \frac{N_z}{2}\right) l_z: \quad k = 1, \ldots, N_z,$$

(16)

where $l_x \times l_y \times l_z$ is the size of a simulation mesh cell.

The positions representing the discrete mass distribution of the new particles $r_{uvw} = (x_u, y_v, z_w)$ are:

$$x'_u = \left(u - \frac{N_u}{2}\right) l_u: \quad u = 1, \ldots, N_u$$

$$y'_v = \left(v - \frac{N_v}{2}\right) l_v: \quad v = 1, \ldots, N_v$$

$$z'_w = \left(w - \frac{N_w}{2}\right) l_w: \quad w = 1, \ldots, N_w,$$

(17)

where $l_u \times l_v \times l_w$ is the cell size of the new particle mesh.

The system is not periodic in the $z$-direction. To suppress the images introduced by the FFT we use the classical doubling-up procedure (Hockney & Eastwood 1981),
which by doubling the size of the mesh over which the FFT must be performed exactly cancels all the images. Thus only the lower half of the entire mesh is relevant, i.e., only particles inside $-L_z/2 \leq z \leq 0$ are active and particles leaving this zone are considered as escaped.

3.4. Friction

In the ISM the collisional rate must depend on the clumping state, which must depend on the dissipation rate. Consequently, we expect a complex dependence of drag coefficients and mass density. However, as explained in Sect. 2, at the kpc scale the physics is dominated by the conservative gravitational dynamics and its concrete behavior should be weakly dependent on the particular dissipative factors, since dissipation mainly acts to ensure the convergence of the system toward the attractors determined by the conservative part of the system. Thus, and following TK, as dissipation factors we adopt linear friction terms, which should be weak in order to remain quasi-Hamiltonian (Pfenniger & Norman 1990).

Yet the collisional properties of the interstellar medium can be expected to differ along or transverse to the plane. To minimize the number of free parameters, we retain only two friction coefficients. The linear friction terms $-C_x \dot{x}$ and $-C_z \dot{z}$ added to the radial respectively to the vertical forces ($F_x, F_z$) in Eq. (8). There is no azimuthal friction in order to be consistent with a global angular momentum conservation.

3.5. Scaling, units, parameters

In order to fix a scale, the origin of our local model is located at a distance of $R_0 = 8$ kpc from the galactic center and rotates with an orbital frequency of $\Omega_0 = \theta_0/R_0$, where $\theta_0 = 210$ km s$^{-1}$. We assume for the general case a flat rotation curve. Moreover, we assume that the active disk has a surface density of $\Sigma_0 = 100 M_\odot/$pc$^2$.

As usual in local shear models of galactic disks the linear measure is the critical wavelength, i.e., the longest unstable wavelength in a zero-pressure disk,

$$\lambda_{\text{crit}} = \frac{4\pi^2 G \Sigma_0}{\kappa_0^2}. \quad (18)$$

The critical wavelength defines the scale for which the theory of swing amplification predicts the strongest response (Toomre 1981; Julian & Toomre 1965). For a flat rotation curve the epicyclic frequency is, $\kappa = \sqrt{2} \Omega_0$ and consequently the critical wavelength scales, with the parameter values indicated above, to $1 \lambda_{\text{crit}} = 12.32$ kpc. The disk scale height $z_0$ is then 0.024 $\lambda_{\text{crit}}$. However, the equations of motion are scale free and the model can, with an appropriate choice of the parameters, be rescaled at will.

The friction coefficients $C_x$ and $C_z$ of the friction terms $-C_x \dot{x}$ and $-C_z \dot{z}$ are in this work indicated in units of $1/\tau_{\text{osc}}$, where $\tau_{\text{osc}}$ is the period of the unforced epicyclic motion. The cooling times of the radial and the vertical damping are thus $t_{\text{cool}x} = 1/C_x \tau_{\text{osc}}$ and $t_{\text{cool}z} = 1/C_z \tau_{\text{osc}}$. For all models presented here, $\tau_{\text{osc}} < t_{\text{cool}}$ applies.

The time-step has to meet the following conditions:

$$\Delta t \leq 0.1 \min\{l_i/\sigma_i\}, \quad i = \{x, y, z\} \quad (19)$$
$$\Delta t = \frac{1}{k \frac{L_x}{L_y} \Omega_0}, \quad (20)$$

where $\sigma_i$ is the initial velocity dispersion ellipsoid, $l_i$ is the cell size and $k$ is an integer. According to Eq. (3) the evolution of the inclination of the backward grid is periodic with period $T = L_z/(L_y \Omega_0)$. The second condition guarantees that this period is a multiple of the time-step. Thus the number of possible grid inclinations and consequently the number of kernels is finite. In order to satisfy the above conditions the time-step is computed in two steps:

$$k = \left\lfloor \frac{10 L_x}{L_y \Omega_0 \min\{l_i/\sigma_i\}} \right\rfloor \quad (21)$$

$$\Delta t = \frac{L_x}{L_y \Omega_0 k}, \quad (22)$$

where $\lfloor \cdot \rfloor$ means to round to the next higher integer.

TK calculated the forces of the self-gravitating particles with direct summation. Thus they had to introduce an upper cutoff in order to limit the computational expenditure, meaning that beyond a certain separation the particles lost their mutual gravitational interaction. Their separation cutoff was equal to four times the softening length. This limited the dynamical range of gravity to 0.6 dex. They argued that a cutoff at larger separations did not affect the resulting structures. Thanks to the higher performance of the convolution method we can extend the dynamical range without increasing the computation time. This may be important in view of a self-similar matter organization in self-gravitating systems.

The parameters characterizing the model of TK are indicated in Tables 1 and 2. They carried out numerical shearing-sheet experiments for different particle densities $n$. Their friction coefficient is a function of the particle density $C_x = (3.5 \times 10^{-3})/n \tau_{\text{osc}}^{-1}$. In order to extend this study and to explore the resulting structures in dependence of the different parameters, we realize different versions of the shearing box model. These model versions are characterized by different parameter sets which fix the size of the simulation zone, the resolution, the particle density etc. For convenience we call these model versions in the following models. That is, a model denotes in the following a version of the shearing box model which is determined by a specific parameter set. The parameters of the models are indicated in Tables 3 and 4.

To be able to do some statistics of structures produced on scales with strongest swing amplification response we perform, like TK, simulations with a quite large simulation zone. Models 1–11 have such a large simulation box resp. simulation sheet in the 2D case, $L_x \times L_y(x \times L_z) = 6 \times 6(\times 0.8) \lambda_{\text{crit}}^3$. However since the dynamical range is
Table 1. Parameters characterizing the model of TK. Indicated are the size of the simulation zone, the number density of particles (surface density), the dynamical range, the gravitational potential of the particles and the number of dimensions.

<table>
<thead>
<tr>
<th>Model</th>
<th>$L_x \times L_y$</th>
<th>$n$</th>
<th>Dynamical range</th>
<th>Potential/Softening</th>
<th># Dimensions</th>
</tr>
</thead>
<tbody>
<tr>
<td>TK</td>
<td>$6.0 \times 8.0$</td>
<td>100–1200</td>
<td>0.6</td>
<td>Plummer</td>
<td>2</td>
</tr>
</tbody>
</table>

Table 2. Parameters of the TK model. The friction coefficient $C_x$ is a function of the particle density $n$. $N$ is the particle number, $\varepsilon$ is the softening length of the Plummer potential. The epicycle frequency $\kappa$ and Oort’s constant $A_0$ are indicated in units of $\Omega_0$.

<table>
<thead>
<tr>
<th>Model</th>
<th>$C_x/10^{-3}$</th>
<th>$[1/\tau_{esc}]$</th>
<th>$N$</th>
<th>$\varepsilon/\Lambda_{crit}$</th>
<th>$\kappa/\Omega_0$</th>
<th>$A_0/\Omega_0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>TK</td>
<td>3.5/n</td>
<td>4800–57 000</td>
<td>0.20</td>
<td>1.4</td>
<td>0.5</td>
<td></td>
</tr>
</tbody>
</table>

Table 3. We use 18 models to explore the different parameters. Besides the parameters presented in Table 1 for the TK model, we indicate here the mesh resolution $l_x \times l_y \times l_z$ and the dynamical range vertical to the plane. Var indicates the parameter, altered from run to run. The resulting structure are then explored as a function of this parameter. The gravitational particle potential is either isotropic or it is deduced from the discrete particle representation described in Sect. 3.3.2.

<table>
<thead>
<tr>
<th>Model</th>
<th>$L_x \times L_y \times L_z$</th>
<th>$l_x \times l_y \times l_z$</th>
<th>$n$</th>
<th>Dynam. range</th>
<th>Potential/Softening</th>
<th># Dim.</th>
<th>Var</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$6.0 \times 6.0 \lambda_{crit}^3$</td>
<td>$0.023 \times 0.023 \lambda_{crit}^2$</td>
<td>1820</td>
<td>1.5</td>
<td>-</td>
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<td>2</td>
</tr>
<tr>
<td>2</td>
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<td>1.5–2.5</td>
<td>-</td>
<td>Isotropic</td>
<td>2</td>
</tr>
<tr>
<td>3</td>
<td>$6.0 \times 6.0 \lambda_{crit}^3$</td>
<td>$0.188 \times 0.188 \times 0.013$</td>
<td>910</td>
<td>1.3</td>
<td>0.5</td>
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<td>3</td>
</tr>
<tr>
<td>4</td>
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<td>3640</td>
<td></td>
<td>Var</td>
<td>Var</td>
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</tr>
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<td>$0.188 \times 0.188 \times 0.013$</td>
<td>910</td>
<td>1.5</td>
<td>1.8</td>
<td>Anisotropic</td>
<td>3</td>
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<tr>
<td>6</td>
<td>$6.0 \times 6.0 \lambda_{crit}^3$</td>
<td>$0.188 \times 0.188 \times 0.013$</td>
<td>910</td>
<td>1.5</td>
<td>1.8</td>
<td>Anisotropic</td>
<td>3</td>
</tr>
<tr>
<td>7</td>
<td>$6.0 \times 6.0 \lambda_{crit}^3$</td>
<td>$0.188 \times 0.188 \times 0.013$</td>
<td>910</td>
<td>1.5</td>
<td>1.8</td>
<td>Anisotropic</td>
<td>3</td>
</tr>
<tr>
<td>8</td>
<td>$6.0 \times 6.0 \lambda_{crit}^3$</td>
<td>$0.188 \times 0.188 \times 0.013$</td>
<td>910</td>
<td>1.5</td>
<td>1.8</td>
<td>Anisotropic</td>
<td>3</td>
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<td>$0.094 \times 0.094 \times 0.013$</td>
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<td>1.8</td>
<td>Anisotropic</td>
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<td>1.8</td>
<td>Anisotropic</td>
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<tr>
<td>11</td>
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<td>Var</td>
<td>1.8</td>
<td>1.8</td>
<td>Anisotropic</td>
<td>3</td>
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<tr>
<td>12</td>
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<td>10100</td>
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<td>3</td>
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<tr>
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<td>$0.056 \times 0.056 \times 0.013$</td>
<td>10100</td>
<td>1.5</td>
<td>1.8</td>
<td>Anisotropic</td>
<td>3</td>
</tr>
<tr>
<td>14</td>
<td>$1.8 \times 1.8 \times 0.8$</td>
<td>$0.056 \times 0.056 \times 0.013$</td>
<td>10100</td>
<td>1.5</td>
<td>1.8</td>
<td>Isotropic</td>
<td>3</td>
</tr>
<tr>
<td>15</td>
<td>$1.8 \times 1.8 \times 0.8$</td>
<td>$0.056 \times 0.056 \times 0.013$</td>
<td>10100</td>
<td>1.5</td>
<td>1.8</td>
<td>Anisotropic</td>
<td>3</td>
</tr>
<tr>
<td>16</td>
<td>$1.8 \times 1.8 \times 0.8$</td>
<td>$0.028 \times 0.028 \times 0.013$</td>
<td>40450</td>
<td>1.8</td>
<td>1.8</td>
<td>Anisotropic</td>
<td>3</td>
</tr>
<tr>
<td>17</td>
<td>$1.8 \times 1.8 \times 0.8$</td>
<td>$0.028 \times 0.028 \times 0.013$</td>
<td>40450</td>
<td>1.8</td>
<td>1.8</td>
<td>Anisotropic</td>
<td>3</td>
</tr>
<tr>
<td>18</td>
<td>$1.8 \times 1.8 \times 0.8$</td>
<td>$0.028 \times 0.028 \times 0.013$</td>
<td>40450</td>
<td>1.8</td>
<td>1.8</td>
<td>Anisotropic</td>
<td>3</td>
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<td>19</td>
<td>$1.8 \times 1.8 \times 0.8$</td>
<td>$0.028 \times 0.028 \times 0.013$</td>
<td>40450</td>
<td>1.8</td>
<td>1.8</td>
<td>Anisotropic</td>
<td>3</td>
</tr>
</tbody>
</table>

limited due to computational limits we perform also simulations for a smaller local box, in order to resolve smaller scales. Therefore the simulation box of models 12–19 are reduced to $1.8 \times 1.8 \times 0.8 \lambda_{crit}^3$.

The time-step depends on the mesh resolution. The mesh resolution is fixed by the number of particles and the size of the simulation zone. The computation time of a simulation depends thus on the particle density $n$. We increase $n$ with respect to previous models based on direct force calculation up to a factor 30 and are furthermore able to perform the simulations in 3D. The code has been written in Matlab for its ease of use, but clearly a compiled language program would greatly improve its speed and memory usage. A pseudo code is given in Appendix A.

3.6. Code testing

In order to check our code we carry out two-body simulations and compare the results with analytical solutions of the Kepler problem. We use a non-rotating inertial frame, thus $\Omega_0 = A_0 = \nu = 0$ in the equations of motion (Eqs. (7) and (8)). However we compute the forces on the time...
dependent affine coordinate systems. Thus $\Omega_0 = \theta_0 / R_0$ in Eq. (3), where the inclination angles are determined. That is, we calculate the forces for a non-rotating isolated system with the help of “shearing Fourier meshes”. Furthermore we use the anisotropic kernel described in Sect. 3.3.2.

The vertical resolution of all simulations presented in this work is $l_z = 0.013 \lambda_{\text{crit}}$, but the resolution in the plane $l = l_x = l_y$ depends on the particle number. Thus we test our code for different resolutions $l$. The particle extension is fixed by the anisotropic kernel and is equal to the resolution.

In the initial state the velocities of the two particles are chosen, in the way that they move on circular orbits. Accordingly to theory the following holds at each time:

\begin{equation}
\Delta r = r - r_0 = 0
\end{equation}

\begin{equation}
\vec{r}_{\text{cm}} = 0,
\end{equation}

where $r$ is the relative particle distance at $t > 0$ and $r_0$ is the initial particle distance at $t = 0$. $r_{\text{cm}}$ is the center of mass. The orbital period for a particle with mass $m$ is,

\begin{equation}
T = 2\pi \sqrt{\frac{r_0^3}{2m}}.
\end{equation}

Particle mass, distance and velocities chosen for these tests yield a period of $T \approx$ galactic rotations.

These theoretical results are compared with the experimental results, i.e., with those resulting from our simulations. The code errors, arising from this comparison are shown in Figs. 3 and 4 for different resolutions. The resolution is indicated in units of $\lambda_{\text{crit}}$. During an orbital period $\Delta r / r_0$ oscillates around zero. The maximal error of the particle trajectory computed with the numerical model are then equal to the amplitude of this oscillation. The amplitude $(\Delta r / r_0)_{\text{max}}$ is plotted in Fig. 3. In this figure the relative error of the orbital period $\Delta T / T$ is indicated as well. Figure 4 reveals the acceleration of the center of

<table>
<thead>
<tr>
<th>Model</th>
<th>$C_x/10^{-3}$</th>
<th>$C_z/10^{-2}$</th>
<th>$N$</th>
<th>$\varepsilon$</th>
<th>$\nu/\Omega_0$</th>
<th>$\kappa/\Omega_0$</th>
<th>$A_0/\Omega_0$</th>
<th>Var</th>
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<tbody>
<tr>
<td>1</td>
<td>Var</td>
<td>-</td>
<td>65,520</td>
<td>0.2</td>
<td>-</td>
<td>1.4</td>
<td>0.5</td>
<td>$C_x = 40 \times 10^{-3}$</td>
</tr>
<tr>
<td>2</td>
<td>100</td>
<td>-</td>
<td>65,520</td>
<td>Var</td>
<td>-</td>
<td>1.4</td>
<td>0.5</td>
<td>$\varepsilon = 0.02 - 0.3$</td>
</tr>
<tr>
<td>3</td>
<td>Var</td>
<td>0.7</td>
<td>32,760</td>
<td>0.3</td>
<td>0.5</td>
<td>0.5</td>
<td>0.1 - 0.4</td>
<td>$C_x = 40 \times 10^{-3}$</td>
</tr>
<tr>
<td>4</td>
<td>100</td>
<td>0.7</td>
<td>131,040</td>
<td>Var</td>
<td>0.3</td>
<td>1.4</td>
<td>0.5</td>
<td>$C_x = 40 \times 10^{-3}$</td>
</tr>
<tr>
<td>5</td>
<td>Var</td>
<td>0.7</td>
<td>32,760</td>
<td>-</td>
<td>0.5</td>
<td>1.4</td>
<td>0.4</td>
<td>$C_x = 40 \times 10^{-3}$</td>
</tr>
<tr>
<td>6</td>
<td>Var</td>
<td>0.7</td>
<td>32,760</td>
<td>-</td>
<td>3.0</td>
<td>1.4</td>
<td>0.5</td>
<td>$C_x = 70 \times 10^{-3}$</td>
</tr>
<tr>
<td>7</td>
<td>140</td>
<td>Var</td>
<td>32,760</td>
<td>-</td>
<td>3.0</td>
<td>1.4</td>
<td>0.5</td>
<td>$C_x = 0.04 \times 10^{-3}$</td>
</tr>
<tr>
<td>8</td>
<td>140</td>
<td>0.7</td>
<td>32,760</td>
<td>-</td>
<td>Var</td>
<td>1.4</td>
<td>0.5</td>
<td>$\nu = 0.0 - 6.4$</td>
</tr>
<tr>
<td>9</td>
<td>Var</td>
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<td>131,040</td>
<td>-</td>
<td>0.3</td>
<td>1.4</td>
<td>0.5</td>
<td>$C_x = 40 \times 10^{-3}$</td>
</tr>
<tr>
<td>10</td>
<td>Var</td>
<td>0.7</td>
<td>131,040</td>
<td>-</td>
<td>3.0</td>
<td>1.4</td>
<td>0.5</td>
<td>$C_x = 0 - 120 \times 10^{-3}$</td>
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<tr>
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<td>70</td>
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<td>-</td>
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</tr>
<tr>
<td>12</td>
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<td>32,760</td>
<td>-</td>
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<td>1.4</td>
<td>0.5</td>
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</tr>
<tr>
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<td>Var</td>
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<td>-</td>
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<td>1.4</td>
<td>0.5</td>
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<tr>
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<td>32,760</td>
<td>-</td>
<td>0.3</td>
<td>1.4</td>
<td>0.5</td>
<td>$C_x = 0.04 - 40 \times 10^{-3}$</td>
</tr>
<tr>
<td>15</td>
<td>50</td>
<td>0.7</td>
<td>32,760</td>
<td>-</td>
<td>Var</td>
<td>1.4</td>
<td>0.5</td>
<td>$\nu = 0.0 - 6.0$</td>
</tr>
<tr>
<td>16</td>
<td>Var</td>
<td>0.7</td>
<td>131,050</td>
<td>-</td>
<td>0.3</td>
<td>1.4</td>
<td>0.5</td>
<td>$C_x = 20 - 40 \times 10^{-3}$</td>
</tr>
<tr>
<td>17</td>
<td>Var</td>
<td>0.7</td>
<td>131,050</td>
<td>-</td>
<td>3.0</td>
<td>1.4</td>
<td>0.5</td>
<td>$C_x = 20 - 40 \times 10^{-3}$</td>
</tr>
<tr>
<td>18</td>
<td>30</td>
<td>Var</td>
<td>131,050</td>
<td>-</td>
<td>0.3</td>
<td>1.4</td>
<td>0.5</td>
<td>$C_x = 0.04 - 40 \times 10^{-3}$</td>
</tr>
<tr>
<td>19</td>
<td>30</td>
<td>0.7</td>
<td>131,050</td>
<td>-</td>
<td>Var</td>
<td>Var</td>
<td>$\kappa = 1.4; 1.7; A_0 = 0.5; 0.25$</td>
<td></td>
</tr>
</tbody>
</table>

Table 4. Parameters of models 1–19. Contrary to the TK model $C_x$ is a free parameter. Moreover we have, because of the extension to three dimensions, a vertical friction coefficient $C_z$ and a vertical frequency $\nu$. The parameter range for which we explore the models is indicated in the last column.
3.7. Initial conditions

Because we are interested in the secular time behavior of the galaxy disk, the simulations are performed for \( t = 10 \) galactic rotations. In the initial state at \( t = 0 \) the particles are distributed uniformly in the \( x-y \)-plane. In the \( z \)-direction the particle distribution follows an isothermal law

\[
\rho \propto \text{sech}^2(z/z_0),
\]

where \( \rho \) is the density and \( z_0 \) is the disk scale height. The velocities at \( t = 0 \) are determined by the shear

\[
\begin{align*}
x &= 0 \\
y &= -2A_0 x \\
z &= 0
\end{align*}
\]

and the Schwarzschild velocity ellipsoid

\[
\begin{align*}
\sigma_x &= \frac{3.36G\Sigma_0 Q}{k} \\
\sigma_y &= \frac{\sigma_x \kappa}{2\Omega_0} \\
\sigma_z &= \sqrt{\pi G\Sigma_0 z_0},
\end{align*}
\]

where the Safronov-Toomre stability criterion is \( Q \geq 1 \) (Toomre 1964; Safronov 1960). This velocity distribution is a permissible assumption, because we represent the gas by dissipative particles.

4. Structure analysis

In order to characterize the structures resulting from the shearing box experiments, we determine the mass-size relation and the velocity dispersion-size relation.

4.1. Mass-size relation

We choose randomly a set of particles with distances \( r \leq L/4 \) from the center of the simulation box. The positions of the particles in the set are restricted to \( r \leq L/4 \) in order to avoid boundary effects in the analysis of the mass-size relation. We will refer to that at the end of this subsection. For each particle in the set we count the number of neighboring particles \( N(R) \) inside a certain radius (all particles in the simulation zone are considered as possible neighbors). If we repeat this for other values of \( R \) we can find the structure dimension \( D(R) \) via

\[
D(R) = \frac{\text{dln}(N)}{\text{dln}(R)}(R),
\]

where \( R \) denotes the scale. The mass-size relation is then

\[
N(R) \propto M(R) \propto R^{D(R)}.
\]
If the structure dimension is independent of the scale, \( D = D_t \), i.e., if \( D \) is constant or oscillates around a mean value, then the mass-size relation is a power-law (Semelin 2000)

\[
M \propto R^{D_t}.
\]

If furthermore \( D_t \) is non-integer, the structure is fractal (Mandelbrot 1982). We will check if \( D = D_t \) holds for the structures resulting from the shearing box experiments. However one has to take into account that these structures can never correspond to an idealized mathematical set, generated by means of an infinite number of levels. Rather they are the result of a finite simulation, modeling a finite physical system. Thus the structures resulting from our experiments can never be fractal beyond a lower and upper cutoff. An upper scale limit due to the numerical model is given by the size of the simulation box in the \( x \)-\( y \)-plane. On this scale the system becomes periodic, meaning that it can not be fractal. To avoid boundary effects as much as possible, we consider in our analysis only particles inside the simulation box. Therefore the fractal dimensions are only calculated for scales \( R \leq L/4 \). Moreover we determine the number of neighboring particles only for particles with a distance \( r \leq L/4 \) from the center of the simulation box. Consequently even for particles at \( r = L/4 \) all neighboring particles are inside the simulation box. A lower scale limit is due to the finite resolution of the simulation mesh. If the mesh cells have the size \( l_x \times l_y \times l_z \) and \( l = l_x = l_y > l_z \) then we do not expect to model correctly fractal structures below \( 2l \).

If however the structure dimension depends on the scale \( D = D(R) \), the structure dimension may simply be regarded as a statistical measure describing the clumpyness on the corresponding scale.

4.2. Velocity dispersion-size relation

There is observational evidence for Larson’s law

\[
\sigma \propto R^{0.4} \tag{31}
\]

on scales \( \mathcal{O}(0.1) \) to \( \mathcal{O}(100) \) pc with \( 0.3 \lesssim \delta_L \lesssim 0.5 \) (e.g. Larson 1981; Scalo 1985;Falgarone & Perault 1987; Myers & Goodman 1983). This power-law relation seems to extend beyond the 100 pc scale (Larson 1979).

In order to check if our model can reproduce these or similar velocity-size correlations on the kpc scale, we determine the velocity dispersion-size relations for the resulting structures. We use the same approach as for the determination of the fractal dimension, but now we calculate the velocity dispersion \( \sigma \) of the particles inside a certain radius \( R \). Then

\[
\delta(R) = \frac{d \sigma}{d \ln R}(R). \tag{32}
\]

What we said about the structure dimension applies also for \( \delta \). If \( \delta \) is constant or oscillates around a mean value over a certain scale range it may be regarded as the power-law index of Larson’s law on the kpc scale, \( \delta = \delta_L \). If however \( \delta \) depends on the scale, \( \delta = \delta(R) \), it may be considered as a statistical measure determining the velocity correlation.

5. Results

5.1. 2D simulations

To compare with earlier simulations, we start with 2D shearing sheet simulations (models 1, 2 in Tables 3, 4). The structures resulting from these simulations depend mainly on the relative strength of the competing gravitational and dissipation processes. Even without dissipation filamentary structures are already developed after \( \approx 1/2 \) galactic rotation. Yet, gravitational instabilities lead to a conversion of bulk kinetic energy (shear-flow) into random thermal motion. In this way the disk is heated up. Thus, if there is no or a too weak dissipation the initially arised filamentary structures are not maintained and smear out. However, with an appropriate dissipation strength filamentary structures can be maintained in a statistical equilibrium. If the dissipation strength is increased beyond the “equilibrium value”, the filaments become denser and denser, and clumps in filaments may be formed. If finally the dissipation dominates completely the heating process, hot clumps, collecting almost all the matter of the simulation zone are formed out of the filaments. Figures 5–7 show the change of the structure morphology for three different radial friction coefficients, i.e., dissipation strength. The friction coefficients are, \( C_x = 7 \times 10^{-3} \tau_{osc}^{-1} \), \( C_x = 140 \times 10^{-3} \tau_{osc}^{-1} \) and \( C_x = 210 \times 10^{-3} \tau_{osc}^{-1} \). To express the relative strength we call the corresponding dissipations “weak”, “middle” and “strong”. All three simulations reveal a fast fragmentation and structure formation. After one rotation around the galaxy center the characteristic striations appear already. The “weak” dissipation leads to a statistical equilibrium of the structure. This statistical equilibrium establishes after about 5 rotations and is maintained for the rest of the simulation. It has a persistent pattern, formed by transient structures. Contrary to the “strong” dissipation, where the structure evolution is dominated by dissipation, i.e., the dissipated energy can not be compensated by the heating mechanism, where no statistical equilibrium arises.

In order to characterize more precisely the resulting structures we compute with Fourier transforms the 2D autocorrelation function of the matter distributions shown in Figs. 5–7. The result is shown in Figs. 8–9. The figures reveal clearly the different morphology resulting from the simulation with the “weak” and the “middle” dissipation. Whereas the autocorrelation function reveals striations with a characteristic inclination for the simulation with a “weak” dissipation, these striations disappear for the “middle” and the “strong” dissipation.

Because we deal with self-gravitating systems which have negative specific heat for a certain energy range, energy dissipation does not mean necessarily a system cooling. Indeed, the “weak”, “middle” and the “strong” dissipation cool the corresponding system only during the first rotation. Then the systems are heated up. After some rotations heating and energy dissipation are balanced out.
and the velocity dispersions $\sigma_x$ and $\sigma_y$ reach a more or less stable level (see Fig. 11).

It is interesting to note that $\sigma_x > \sigma_y$ always holds for the "weak" and the "middle" dissipation strength. The same holds also for the "strong" cooling during the first six rotations, then this ordering is obviously destroyed by the formation of hot clumps.

In order to characterize the structure of the simulation terminal phase we determine the structure dimension $D$ and the index $\delta$. The longer term evolution of the structures may be superimposed by fluctuations\(^\text{1}\) on time-scales of the order of $\sim 1/2 \tau_{\text{rot}}$, where $\tau_{\text{rot}}$ is the time for a

\(^1\)To obtain an idea about these fluctuations see the evolution of the Schwarzschild velocity ellipsoid in Figs. 11 and 24.
rotation around the galactic center. In order to eliminate these fluctuations we indicate in this paper mean values of the structure dimension $D$ and the index $\delta$, determined during the last two rotations. In figures showing $D$ or $\delta$ we indicate in addition $1\sigma$ error bars.

The dimension $D$ and the index $\delta$ resulting from the structures shown in Figs. 5–7 are plotted in Figs. 12 and 13, respectively, as a function of the scale $R$. The vertical lines at the left of the curves are the $1\sigma$ error bars. Contrary, to the structure dimension $D$ the error bars of the index $\delta$ can vary up to a factor 7. Thus we indicate in Fig. 13 in addition the position and the size of the largest and the smallest error bars.

The stronger the dissipation is, the more filamentary resp. clumpy are the resulting structures and consequently the lower is the structure dimension. A comparison of the
Fig. 9. The autocorrelation function of the structures shown in Fig. 6, resulting from the simulation with the “middle” dissipation.

Fig. 10. The autocorrelation function of the structures shown in Fig. 7, resulting from the simulation with the “strong” dissipation.

structure dimensions $D$ with the indices of the velocity dispersion-size relation $\delta$ shows that $\langle \delta(R) \rangle$ increases with decreasing $\langle D(R) \rangle$, where $l < R < L$ (resolution: $l = l_x = l_y$, box size in the plane: $L = L_x = L_y$).

For the strong dissipation the mass-size relation can be approximated by a power-law for a scale range of roughly 1 dex, but the error bars are relatively large and the scale range is too small to call the corresponding structure scale-free or fractal.
As long as the structures are not completely dominated by hot clumps (strong dissipation) the velocity dispersion-size relation may be approximated by a power-law. However, also here the mean error bars are quite large with respect to the value of $\delta$, especially for the weak dissipation, where the resulting $\delta$-value would also be compatible with an uncorrelated velocity dispersion-size relation.

For the middle dissipation there is a little more evidence for a power-law relation over roughly 1 dex. However, also if there is a power-law relation the value of $\delta$ would be far away from the index of Larson’s law measured in molecular clouds ($0.3 < \delta_L < 0.5$).

The softening length in model 1 is quite large and corresponds to those of the TK model. In model 2 (Tables 3 and 4) we reduce the softening length $\varepsilon$, but we pay attention that $\varepsilon > \ell_x = \ell_y$ is always valid, i.e., that the softening length is always larger than the cell size of the simulation mesh. As expected the general tendency is that a smaller softening yields a stronger clustering and thus a smaller structure dimension. The structures resulting from simulations with a small softening length ($\varepsilon < 0.1$) become relatively fast very clumpy (after 2–3 rotations). In contrast to simulations with a strong dissipation, where also clumps are formed, the number of clumps remains from a certain moment on nearly constant, i.e., it does not decrease due to mergers. This is shown in Fig. 14. The structure dimension for this simulation is shown in Fig. 15.
5.2. 3D simulations

5.2.1. Isotropic kernel

We extend the models to 3 dimensions and carry on using an isotropic particle potential (models 3 and 4).

In models using a particle-mesh method the number of particles is determined by the number of mesh-cells and vice versa. Due to computational limits and in order to do a reasonably sized parameter study on the available machines we have to limit the number of particles to $N \approx 130000$. Thus it is not possible to resolve the system vertically with a softening length $\varepsilon \approx l_x \approx l_y$. The models using an isotropic potential reproduce thus only 2D dynamics in a 3D space. As expected we find...
therefore the same parameter dependence as in models 1 and 2, respectively. Compared with previous models we carry out here also simulations for a slightly larger softening length. These simulations reveal for a limited scale range approximately a velocity dispersion-size power-law relation. Figure 16 shows the structures resulting from 2 simulations with $\varepsilon = 0.3 \lambda_{\text{crit}}$. The indices $\delta$ resulting from these 2 simulations are shown in Fig. 17.

### 5.2.2. Anisotropic kernel

We use now the anisotropic kernel described in Sect. 3.3.2. Thus the softening length $\varepsilon$ is here no longer a free parameter. The softening of the gravitational potential is given by the resolution of the simulation mesh. With the anisotropic kernel we can resolve the system vertically, so that there is a vertical dynamical-range of 1.8 dex for all models with anisotropic kernel. Since the third dimension is now resolved we explore the structure in dependence on the friction coefficient $C_z$ (model 7) and the vertical frequency $\nu$ (model 8).

**Vertical friction coefficient $C_z$.** We start with simulations, where $C_x = C_z$. However, these simulations show that with such a dissipation it is not possible to maintain simultaneously strong density fluctuations and the disk thickness. That is, either the density fluctuations are maintained and the disk scale height tends towards zero, or the disk scale height is maintained nearly constant and the density fluctuations smear out. Therefore we choose for all further simulations $C_x > C_z$, i.e., $t_{\text{cool},x} < t_{\text{cool},y}$. These results justify a posteriori the choice of two friction coefficients. The velocity dispersions in the plane $\sigma_{xy} = (\sigma_x^2 + \sigma_y^2)^{1/2}$ are mainly controlled via $C_x$, whereas $\sigma_z$ and with it the disc scale height $z_0$ is principally driven by $C_z$. However, as soon as structures are formed with sizes comparable or smaller than the disk thickness, the dynamics in the plane and vertically to it are no longer independent. Thus the effect of $C_z$ on the structure depends also on $C_z$. The general effect of $C_z$ on the self-gravitating disk is the following: as long as the structure remains filamentary an increase of $C_z$ diminishes $z_0$ and $\sigma_z$. The solid curve in Fig. 18, resulting from a simulation performed with model 7, represents such a behavior. The effect of $C_z$ is also studied in models 14 and 18. There the structure changes from filamentary to clumpy due to an increase of $C_z$. As a consequence these systems may be heated up by further dissipation (see Fig. 18).

**Vertical frequency $\nu$.** In models 8 and 15 we study the effects of the vertical frequency $\nu$ on structure and dynamics. The vertical frequency determines the strength of the backward force due to a displacement from the galaxy plane. The backward force stems from the external galactic potential. Thus an increase of $\nu$ binds the particle stronger to the disk and diminishes $z_0$ (see Fig. 19).
In all 3D models the mean particle-velocity vertical to the plane $\langle v_z \rangle$ is not exactly zero, but oscillates with an amplitude of $\pm 0.1 \text{ km s}^{-1}$ and a frequency equal to the vertical frequency $\nu$.

Concerning the effect of $\nu$ on the structure, there are frequencies producing a clumpy structure, whereas some higher or lower frequencies produce a more filamentary structure. To show this we determined the minimal structure dimension,

$$D_{\text{min}} = \{ \min[D(R)] : l < R < L \}, \tag{33}$$

where $L = L_x = L_y$ (box size in the plane) and $l = l_x = l_y$ (resolution). The minimal structure dimension determines how strong the structures differ from a homogeneous matter distribution, i.e., the lower $D_{\text{min}}$ the more filamentary resp. clumpy the structure. We calculate $D_{\text{min}}$ for the structures resulting from simulations with different $\nu$. The result is shown in Fig. 20. The two simulations with $D_{\text{min}} \approx 1.7$ have a more clumpy structure than the ones resulting from the other simulations, which are more filamentary.

**Particle number $N$.** With model 11 we determine the structure dimension and the disk scale height as a function of the particle number $N$. We only alter the particle number. The mesh resolution and the friction coefficients $C_x$ and $C_y$ are kept constant. The 2D simulations of TK shown that a higher particle number leads to a stronger dissipation. Thus we expect a decrease in the disk scale height and a smaller structure dimension for higher $N$. We find a clear decrease of the disk scale height for an increasing particle number, but there is no clear trend of the structure dimension. This is because the disk is heated up in the plane due to the decreasing disk thickness.

We find that the effect on the structure due to a change of the particle number can be compensated with an appropriate choice of the dissipation strength, i.e., if we use a weaker dissipation for an increased particle number, the statistical properties of the resulting structures remain unchanged.

**Radial friction coefficient $C_x$ / large simulation zone.** In Sect. 5.1 we explored for the 2D models how the structure formation depends on the radial friction coefficient $C_x$. Here we study the connection between structure and radial friction in 3D. Considered are structures resulting from models with large simulation zone and anisotropic kernel (models 5, 6, 9 and 10).

Concerning the structure in the plane we find qualitatively the same dependence as in the 2D simulations. That is, the initially formed structures smear out if we don’t dissipate energy. However we still find correlations in the matter distribution after ten galaxy rotations. For an increasing radial dissipation the structures become denser and denser until finally hot clumps are formed out of filaments. Figures 21 and 22 show, as an example, the structures resulting from a simulation performed with model 10. The structures reach in this simulation very fast (after 2 rotations) a statistical equilibrium. The autocorrelation functions revealing the underlying characteristics of these structures are shown in Fig. 23. Figures 24 and 25 show the evolution of the velocity dispersions. These figures confirm that a statistical equilibrium is attained after $\approx 2$ rotations around the galaxy center.

The filaments resulting from these simulations ($C_x = 64 \times 10^{-3} \tau_{\text{osc}}$) are very dense and they are first signs of clump formation. A slight increase of the dissipation would thus turn the structure from filamentary to clumpy. Indeed, the structures in Fig. 26 resulting from simulations performed with the same model but with slightly higher radial dissipations $C_x = 66, 68, 70 \times 10^{-3} \tau_{\text{osc}}$, are already clumpy. For convenience we call the relative strength of the radial dissipations used in model 10 “weak”,

![Fig. 19. The disk scale height $z_0$ and the vertical velocity dispersion $\sigma_z$ as a function of the vertical frequency $\nu$ deduced from models 8 and 15, respectively. $z_0$ and $\sigma_z$ are mean values calculated during the last two galaxy rotations.](image1)

![Fig. 20. The minimal structure dimension $D_{\text{min}}$ as a function of $\nu$. The structure dimension is a mean value determined during the last two galaxy rotations. The corresponding structures result from simulations performed with model 15.](image2)
**Fig. 21.** The evolution of the particle positions seen from above the galaxy plane. The structures result from a simulation of model 10. The friction coefficient is $C_x = 6 \times 10^{-3} \pi_{osc}$ ("weak" dissipation). The number of rotations of the shearing box around the galaxy center is indicated at the top of each panel. Shown is each fourth particle.

**Fig. 22.** The particle positions inside the slice, $-0.05\lambda_{crit} < y < 0.05\lambda_{crit}$, seen along the direction of orbital motion (model 10, "weak" dissipation).
Fig. 23. The autocorrelation function of the particle distribution, presented in Figs. 21 and 22 (model 10, “weak” dissipation). Upper panels: autocorrelation function in the $x$-$y$-plane. Lower panels: autocorrelation function in the $z$-$x$-plane.

Fig. 24. The velocity dispersion components $\sigma_x$, $\sigma_y$ and $\sigma_z$ as a function of time $t$ (indicated in galaxy rotation units). The dispersions result from a simulation performed with model 10 and “weak” dissipation.

“middle”, “strong”, and “very strong”. In Fig. 27 the structure dimension $D$ is shown for the four different dissipation strengths. The structure dimensions are not constant over the corresponding dynamical range and are thus not fractal. However, the structure dimension resulting from the simulation with the “strong” dissipation has a dimension $1.5 < D < 1.8$ over the whole dynamical range in the plane and remains smaller than 2 also on scales where the disk thickness becomes important. Furthermore, the structure dimension has at $0.06 \lambda_{\text{crit}}$ the same dimension as at $1.6 \lambda_{\text{crit}}$ with a minimum in between. This is qualitatively a different behavior than those of the initial state. The slight increase of the structure dimension at the left and at the right may be induced by the small scale (resolution) and large scale cutoff (periodicity), respectively. This behavior is quite general for our simulations and it may be that a larger dynamical range produce a more constant structure dimensions, $D(R) \approx \text{const.}$
We do not find a velocity dispersion-size relation similar to a power-law for the simulations performed with model 10. However, we do find some hints for such a relation in the structures resulting from model 6. Figure 28 shows the velocity dispersion-size relation for two different simulations of model 6. The simulation with the weak dissipation produces filamentary structures whereas the simulation with the strong dissipation produces filaments and clumps, thus $\delta$ varies stronger and the error bars are larger for these structures. However, both simulations produce a velocity dispersion-size relation which may be approximated to first order by a power-law. Of course, the error bars are too large and the scale range is too small to call the corresponding velocity dispersion-size relations scale free. The resolution in model 6 is larger than those in model 10. Why this softer gravitation seems to reproduce better a power-law velocity dispersion-size relation is at the moment unclear.

Analog to the 2D case we find also in 3D simulations a systematic ordering of the velocity dispersion components. The ordering, $\sigma_x > \sigma_y > \sigma_z$, holds for quite a large range of dissipation strength. This is shown by simulations performed with model 10. The velocity dispersion components resulting from the simulation with the “weak” dissipation strength is shown in Fig. 24 and those
with the “middle”, “strong” and “very strong” dissipation strengths are shown in Fig. 29. Even if we don’t dissipate energy this ordering is still observed after 10 galaxy rotations. Only strongest clump formation can destroy the systematic ordering. From the definition of $x$ and $y$ in Sect. 3.1 it follows $\sigma_x = \sigma_R$ and $\sigma_y = \sigma_\phi$. To sum up, one can therefore say, that as long as our model produces structures resembling the lumpy matter distribution in spirals, it also produces an ordering of the velocity dispersion components ($\sigma_R > \sigma_\phi > \sigma_z$) corresponding to those of the solar neighborhood, or of $N$-body simulations of spiral disks (Pfenniger & Friedli 1991).

Radial friction coefficient $C_x$ / small simulation zone. In order to increase the resolution in the plane and to approach the vertical resolution we decrease the size of the simulation box. The structures resulting from these simulations differ from those in the large simulation box. We still find filaments for a “weak” dissipation and clumps for a “strong” dissipation (see Fig. 30). However, filaments and clumps, respectively, appear less numerous. From this point of view the structures in the small simulation box are not a fractal continuation of those in the large simulation box. However, this does not exclude the possibility that a simulation with a dynamical range incorporating the scale of the large and the small simulation box would produce scaling laws over the whole dynamical range.

Shear. The inhomogeneous structures appearing in the shearing-box simulations can only be maintained when the dissipated energy is compensated by an energy injection. The source of this energy is the shear motion. If the shear is reduced, the dissipated energy can no longer be balanced by the energy injection and the system collapses. A rotation curve, increasing with the square root of the radius ($v \propto \sqrt{r}$) reduces the shear-flow, $\dot{y} = -2A_0 x$, by a factor two. With model 19 we perform simulations for different rotation curves. That is, for the usual flat curve with $A_0 = 0.5 \Omega_0$, $\kappa = \sqrt{2} \Omega_0$ and for a curve increasing with the square root of the radius, $A_0 = 0.25 \Omega_0$, $\kappa = \sqrt{3} \Omega_0$. Such a choice of the rotation curve doesn’t reflect a realistic case but serves to explore the influence of the shear on the structure formation. However, a change of the rotation curve alters also the epicyclic frequency and consequently the Schwarzschild velocity ellipsoid as well as the critical wavelength $\lambda_{\text{crit}}$. Because we only want to examine the effect of the shear, simulations with different rotation curves are carried out with the same initial velocity dispersion. Moreover, the distances are indicated in units of kpc. The effect of a decreased shear is revealed in Fig. 31. The same friction coefficients as in the simulations with the flat rotation curve lead for the slightly increasing ($v \propto \sqrt{r}$) curve to a collapsed system.

5.2.3. Minimal structure dimension and maximal index $\delta$

In this paper we study mainly the structure dimension in dependence of the radial friction coefficient $C_x$, because this parameter determines principally the degree of structure inhomogeneity. Indeed, the parameters $C_x$ and $\nu$ serve mainly to prevent the disk from vertical dissolution and a change of the differential rotation or the particle number can be balanced by an appropriate choice of the dissipation strength.

The structure characteristics in dependence of $C_x$ are studied for different simulation boxes and resolutions. In order to compare the structures and the dynamics as a function of $C_x$ resulting from the different models we calculate the minimal structure dimension and the maximal index $\delta$ of the velocity dispersion-size relation. The minimal structure dimension is defined in Eq. (33). Correspondingly, the maximal index is,

$$\delta_{\text{max}} = \{\max[\delta(R)]: l < R < L\}. \quad (34)$$

The results are summarized in Fig. 32. The general trend is that the structures become denser and more clumpy for higher radial dissipation, leading to a smaller structure dimension and to higher velocity differences on the different scales of the system, i.e., to higher $\delta_{\text{max}}$. 

![Fig. 29. The velocity dispersion components $\sigma_x$, $\sigma_y$ and $\sigma_z$ as a function of time. The dispersions result from the simulation with the “middle”, “strong” and “very strong” dissipation strength, respectively (model 10). The matter distribution of the solar neighborhood, or of the small simulation box. The structures resulting from these simulations are shown in Fig. 26.](image-url)
Fig. 30. The evolution of the particle positions seen from above the galaxy plane. Shown is each forth particle. The structures result from simulations performed with model 16. We show 3 simulations with different dissipation strength. Upper panels: “weak” dissipation. Middle panels: “middle” dissipation. Lower panels: “weak” dissipation. The number of rotations of the shearing box around the galaxy center is indicated at the top of each panel.

6. Discussion

With respect to the models with the large simulation zone (models 1–11), the models with the small simulation zone (models 12–19) produce less fragmentation. That is, only a few weak filaments appear in simulations of these models as long as the dissipation is weak and an increase of the dissipation strength can not produce a more fragmented or filamentary structure, because clumps appear very quickly. These clumps become rapidly unphysically hot and collect almost all the matter of the simulation zone. The clumps may thus hinder the formation of a more fragmented structure in these models.

The formation of non-transient collapsed clumps, which have the tendency to attract more and more matter were also found in other numerical studies using dissipative or sticky particles in order to model the ISM (Huber 2001; Semelin & Combes 2000). The appearance of these clumps may thus be a generic problem of gravitational clustering simulations with dissipative particles.

In the following we discuss some aspects related with the formation of these non-transient and in our opinion unphysical hot clumps, which may hinder the evolution towards a structure, being hierarchical over a more extended scale-range. We discuss numerical, dynamical, and physical aspects of the problem.

6.1. Numerical problems

The size of dissipative particles, given by the softening length, is much larger than the smallest clumps in the interstellar medium. Thus the nearly homogeneous mass
distribution (smear out) over the softening length is not justified and it is not a priori clear that the inhomogeneous structures below the resolution scale do not affect the structures on larger scales.

Observations of the ISM reveal filamentary structures down to the smallest resolvable scales, thus it is unclear down to what scales the highly inhomogeneous structures continues. Moreover, a large scale range has probably to be taken into account in order to reproduce the observed structures. Thus it is an open question whether in the following years a resolution will be reached such that basic clumps can indeed be represented by dissipative particles. Taking into account the ubiquitous trend of gravitationally unstable media to produce sheets and filaments not only in the ISM but also in cosmological simulations, it might be that the particle model of the basic mass unit, as a rigid body conserving mass, is not adequate.

6.2. Dynamical aspects

One could argue that when the “mean free path” of cloud clumps is larger than their size and a particle description seems to be justified, physical cloud clumps collide and dissolve because they contain internal degrees of freedom due to the smaller subclumps moving inside them. Thus dissipative particle simulations should incorporate collisions with mass exchange (Pfenniger 1994). This is particularly relevant if the clouds collide with supersonic speed, that is, with velocities larger than their internal velocities.

Inherent dynamical properties of gravitational unstable media may cause further problems. Let us discuss two of them:

1) Typically clumps are subject to an anisotropic gravitational contraction that alter its morphological characteristics, i.e., clumps may become pancakes which become filaments. This was shown by Zeldovich (1970) and numerically confirmed by Kuhlman et al. (1996, hereafter KU). In their numerical experiments KU replaced particles in dense regions by clouds, made up of $2^3$ resp. $2^5$ particles. They found that only a small fraction collapses along all three directions and forms dense clumps. A further subdivision of the particles would probably lead to the same result. If the transformation from clumps to pancakes and to filaments on small scales is important for the appearance of the global structure, it may be problematic to model gravitational unstable media with particles, because a higher particle number would not solve the inherent problem of anisotropic clustering;

Fig. 3. Evolution of the matter distribution, seen from above the galaxy plane. The structures result from simulations with different rotation curves performed with model 19. Upper panels: flat rotation curve ($v = \text{const.}$). Lower panels: increasing rotation curve ($v \propto \sqrt{r}$). The number of rotations around the galaxy center is indicated at the top of each panel. Shown is each forth particle.
where $D$ is the mass scaling exponent, which would be the fractal dimension in a self-similar hierarchical system. $D$ is restricted to stay in the interval $[0-3]$ since mass is positive and space filling can not exceed the third power of scale. Then the density scales as

$$\frac{\rho_L}{\rho_l} = \left( \frac{R_L}{R_l} \right)^{D-3},$$

and consequently the crossing time $\tau_{\text{dyn}} \propto (G\rho)^{-1/2}$ scales as

$$\frac{\tau_{\text{dyn},L}}{\tau_{\text{dyn},l}} = \left( \frac{R_L}{R_l} \right)^{(3-D)/2}.$$

So in a hierarchical model the dynamical time (or crossing, or free-fall time) always decreases at smaller scales since $0 < D < 3$ (Pfenniger & Combes 1994). Thus the low scale structures evolve faster.

Self-gravitating $N$-body systems are chaotic and neighboring trajectories diverge exponentially (Miller 1964). This means also that any error propagates exponentially:

$$\Delta x \propto \Delta x_0 e^{\lambda t},$$

where $\Delta x_0$ is a small initial error at time $t = 0$, and $\Delta x$ is the error at time $t$. Now, the degree of chaos and thus the error evolution is determined through the maximum Liapunov exponent $\lambda$, which for small $N$-body systems is approximately inversely proportional to the dynamical time, $\lambda \propto 1/\tau_{\text{dyn}}$ (Miller 1994).

With this estimate, let us see how errors are amplified through the two adjacent levels. If $\Delta x_l$ is in fact the initial error at the lower level $l$, we get after one crossing time at level $L$, $\tau_{\text{dyn},L}$:

$$\frac{\Delta x_L}{\Delta x_l} = \exp \left( \frac{\tau_{\text{dyn},L}}{\tau_{\text{dyn},l}} \right) = \exp \left( \left( \frac{R_L}{R_l} \right)^{(3-D)/2} \right).$$

The error amplification becomes rapidly huge as soon as $D < 3$. Say, if $\frac{R_L}{R_l} = 2$ and $D = 1.5$, then $\frac{\Delta x_L}{\Delta x_l} = 5.38$. Across $n$ levels the error ratio at the highest level goes as $5.38^n$ and becomes much larger than the scale ratio, $2^n$. (It is easy to show that for $D < 2.264$ the error amplification is always larger that the scale ratio, while for $\frac{R_L}{R_l} > 2.72$ and $D > 2.265$ one can find hierarchical systems for which this problem does not occur.)

Thus we find an inherent dynamical problem, which can not be solved by using a higher resolution, because the increase of resolution exponentially increases the errors through the scales. Possibly such hierarchical systems might be dominated by numerical errors in simulations, and by small scale physics in real systems.

Fig. 32. Upper panel: the minimal structure dimension $D_{\text{min}}$ as a function of the radial friction coefficient $C_x$. Lower panel: the maximal index $\delta_{\text{max}}$ of the velocity dispersion-size relation as a function of the radial dissipation strength. $D_{\text{min}}$ and $\delta_{\text{max}}$ are mean values calculated during the last two galaxy rotations.

2) A further problem when simulating gravitational clustering is related to the exponential propagation of two-body relaxation in hierarchical $N$-body systems. If the mass distribution of the considered system is self-similar over a range of scales, at each scale the effective bodies can be viewed as the corresponding clumps. The hierarchical structure acts as a strong two-body relaxation amplifier since at each scale the effective number of bodies is strongly reduced. If this effective number is $O(10)$ the relaxation time at each level is of the order of the crossing time, so two-body relaxation is a major driver of evolution throughout the scales;

3) Let us describe in more detail the related problem of error propagation in a hierarchical system. First we show that in a gravitationally unstable medium developing a hierarchical structure, matter on smallest scales evolves faster.

A hierarchical mass distribution satisfies the scaling relation between two levels $L$ and $l$:

$$\frac{M_L}{M_l} = \left( \frac{R_l}{R_L} \right)^D,$$  (35)
6.3 Additional physics

Small scale physics, supporting the dissolving process of dense clumps, is not taken into account in our simple model. However, stellar winds, supernovae, jets and outflows may be important in the overall mass transport across the scales. These processes may ensure a cyclic matter flow by giving matter back to larger scales, which would condense back via gas cooling. Such a matter-flow may be crucial for sustaining the transient nature of hierarchical clumps for extended time.

7. Conclusions

The structure resulting from the local simulations of self-gravitating disks can be homogeneous, filamentary or clumpy depending on the relative strengths of the competing gravitational and dissipation processes. As long as the structure is mainly filamentary self-gravitation and dissipation ensure a statistical equilibrium, where repeated transient patterns are formed. If the dissipative processes begin to dominate the evolution, the structures turn from filamentary to clumpy. During the subsequent evolution the clumps become hotter and more massive.

In general, clumpy structures do not evolve towards a statistical equilibrium. However model 2 does show that it is also possible to establish a persistent pattern of clumps. These clumpy structures may show signs of a power-law mass-size relation. Some of our 2D as well as 3D simulations suggest also a power-law velocity dispersion-size relation.

However the scale range of the simulations appears too small to draw final conclusions about a fractal structure and an extension of Larson’s law beyond the size of molecular clouds. We can suggest a few reasons causing the discrepancy:

1) The numerical resolution should extend over several more decades of scale before the fragmentation stops to be dominated by finite scale range boundary effects;
2) A fundamental law is associated with the rigid point particle representation of mass which, by forcing a particular lowest scale boundary condition, would prevent the bottom-up building of scaling relations to match the observed ones. Also the propagation of errors may be super-exponential in a hierarchical organized medium because the error evolution at largest scales are not determined by the dynamical time of these scales, but by the much smaller dynamical time of the smallest scales. Systems with dimension lower than about 2.2 are particularly concerned. This point is relevant for particle simulations of gravitationally unstable systems, such as cosmological and thin disk ($D < 2$) simulations;
3) A key physical ingredient, such as mass cycling through the scales due to star formation, stellar activity, and gas cooling could be essential to sustain a fractal state of the ISM.

Finally, it is interesting to note that the anisotropy of the velocity-dispersion ellipsoid, resulting from our simulations, has systematically the same ordering ($\sigma_R > \sigma_\phi > \sigma_z$) and relative amplitudes as observed in the Galaxy and in N-body simulations of spirals. Since the models are deliberately a simplified representation of reality, we learn from this that this ordering may be due to a very general property of galactic disks, to be substantially self-gravitating in $z$, and to rotate differentially with a similar shear rate set by a constant rotation curve.

Appendix A: Pseudo code of the shearing box program

- Initialization of the particle positions and velocities.
- Calculation of the canonical momenta.
- The inclination of the meshes is $a_\theta = \lfloor (-\Omega_0 t) \mod (L_y/L_z) \rfloor$ and $a_\phi = a_\theta + (L_y/L_z)$, respectively. Thus there is a finite number of possible inclinations, $n = T/\Delta t$, where $T = L_y/(L_z\Omega_0)$ and $\Delta t$ is the time-step.

\textbf{do} $i = 1, n$

- The derivations of the Kernel $\nabla K(a_\theta)$ for the different possible inclinations $a_\theta$ of the forward mesh are calculated. $K$ in an $n_x \times n_y \times n_z$-matrix, where $n_x n_y n_z$ is the number of cells in the local simulation box.
- The points: $(n_x/2, n_y/2, 1 : n_z)$ of $\nabla K(a_\theta)$ are calculated for the backward mesh. The other Kernel points can be deduced from $\nabla K(a_\theta)$ of the forward mesh, making use of their symmetry.

\textbf{enddo}

\textbf{do} $i = 1, m$ (time propagation loop)

\textbf{do} $g = 1, 2$ (Do for forward and backward mesh, respectively)

\textbf{if} ($g = 1$)

- $a = a_f = \lfloor (-\Omega_0 t) \mod (L_y/L_z) \rfloor + L_y/L_z$. The inclination $a$ corresponds to the forward mesh.
- The particle positions of the Cartesian coordinates are saved ($y_c = y$).
- Transformation of the particle position: Cartesian coordinates $\rightarrow$ Forward mesh ($y = y - ax$).

\textbf{else}

- $a = a_t = a_t - L_y/L_z$. The inclination $a$ corresponds to the backward mesh.
- $y = y_c$. The particle positions are again those in Cartesian coordinates.
- $y = y - ax$. Transformation of the particle position: Cartesian coordinates $\rightarrow$ Backward mesh.
• The missing points in the $\nabla K(a_i)-$matrix are deduced from $\nabla K(a_i)$, making use of their symmetry.

do (time propagation loop)

• Transformation of $\nabla K(a)$ into Fourier space. $(\nabla K(a) \rightarrow \tilde{\nabla K}(a))$.

• Force weighting

• Force interpolation in accordance with the CIC-method.

• Transformation of the forces. Affine mesh $\rightarrow$ Cartesian coordinate system.

do (g = 1, 2)

• The particle positions are again those in Cartesian coordinates ($y = y_c$)

• Force weighting

• Calculation of the new canonical momenta and particle positions by means of the implicit canonical finite difference approximation (Pfenniger & Friedli 1993).

enddo

if (z-coordinate of a particle lies outside the local box.)

• The particle is considered as escaped.

elseif (x- and/or y-coordinate lies outside the local box.)

• Reentrance at the opposite side with appropriate canonical momenta (see Eqs. (9)).

do (storage condition)

• Calculation of the particle velocities.

• Storing of the particle positions and velocities to the disk.

do (time propagation loop)

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5.4 Complementary Discussion

5.4.1 Patterns and Instabilities

Here the different types of patterns and instabilities that appear in the above presented simulations are summarized and discussed.

Structures resulting from local simulations of self-gravitating, dissipative disks can be homogeneous, wavy, filamentary or clumpy, depending on the relative strength of the competing gravitational and dissipative forces. The transitions between the different states are smooth.

Since the disks are heated up due to gravitational instabilities leading to a conversion of the bulk kinetic energy (shear-flow) into random thermal motion, disk inhomogeneities cannot grow and tend to smear out, if energy dissipation is too weak. However, with an appropriate choice of the dissipation strength, persistent wavy or filamentary patterns formed by transient structures appear.

The here studied self-gravitating disks are stable against axisymmetric perturbations (i.e. \( Q \gtrsim 0 \)) suggesting that the large scale structure growth is triggered by non-axisymmetric perturbations induced in the models by the \( \sqrt{N} \) gravitational Poisson noise. Evidence for this scenario comes from Toomre & Kalnajs (1991) who analyzed the correlations of the wave-patterns in the linear regime and found that they agree with predictions of the swing-amplification formalism (Goldreich & Lynden-Bell 1965; Julian & Toomre 1966; Toomre 1981).

If the dissipation strength is increased, fluctuations grow, the system becomes nonlinear and secondary instabilities enforce the structure growth, so that the wavy structures turn to filamentary structures and, in the case of strong dissipation, to clumpy structures.

Secondary instabilities appearing in the nonlinear regime of swing amplified large scale inhomogeneities have been studied by Kim & Ostriker (2001). They carried out magneto-hydrodynamic simulations of gaseous shearing disks. The long-range correlations appearing in these simulations are qualitatively similar to those resulting from our \( N \)-body model, suggesting that the fundamental agents of structure formation on the kpc scale are gravity, shear and dissipation, and that magnetic fields and gas pressure are second order effects.

As long as the structure is mainly filamentary, self-gravitation and dissipation ensure a statistical equilibrium, where repeated transient structures are formed. These structures, appearing on the kpc scale, are spiral arm like and thus on average non-isotropic.

When the dissipative structures begin to dominate the evolution process, clumps form inside the filaments. Typically, the clumps do not disperse, meaning that their growth is not limited in time, which leads to gravitational runaway. In general, the clumpy structures do not evolve to a statistical equilibrium.

To sum up, on the kpc scale statistical equilibrium states of filamentary structures have been found. Structures on smaller scales become rapidly clumpy and non-transient, in the sense that they do not disperse in the course of time. Gravitational runaways (collapses) occur and a statistical equilibrium cannot be maintained.
Chapter 6

A Numerical Study of Self-Gravitating $N$-Body Spheres

6.1 Introduction

$N$-body models have considerably extended our understanding of the evolution of gravitating systems. Yet, despite this success many fundamental problems remain.

Indeed, models accounting for nonlinear structure growth are not always consistent with observations and a better understanding appears necessary. For instance, cold dark matter (CDM) simulations conflict at galactic scales with observations (Moore et al. 1999; Bullock et al. 2001; Bolatto et al. 2002)* and no theory of the ISM is presently able to predict the conditions of star-formation.

Typically, $N$-body models are inconsistent with observations in situations where the growth of singularities, triggered by instabilities, is allowed. In these situations the outcome may be strongly dependent on the small-scale physics and non-gravitational perturbations. In other words, far from equilibrium, where the growth of fluctuations is allowed, gravitational systems are sensitive to perturbations due to their chaotic nature.

Thus we study here the evolution of perturbed $N$-body spheres. That is, the effect of the following perturbations is investigated:

- Small scale repulsive forces
- Dissipative factors
- External forcing

Actually the here studied models are scale-free, however, due to their initially, extended spherical geometry the results may be particularly relevant for systems at sub-galactic scales, such as molecular clouds.

The results of these studies and the details of the corresponding models were published in the article Long-Range Correlations in Self-Gravitating $N$-Body Systems.

Subsequently, we first motivate the study of perturbed self-gravitating spheres. The article is presented on page 91. Finally, a complementary discussion is added.

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*Problems that arise in the CDM simulations are for instance: 1) The central density profile of dark matter halos is too steep with respect to observations; 2) The number of satellite systems orbiting around a larger galaxy is too high with respect to observations.
6.2 Motivation

A spatially isolated, spherical $N$-body model is used in order to study three physical situations, namely, micro-canonical systems, the gravitational collapse of strongly dissipative systems and systems that are subject to an energy-flow. Below, the study of these situations is briefly motivated.

6.2.1 Why Studying Micro-Canonical Systems?

In order to study with the same experiment the equilibrium properties of systems in dependence of their energy, a weak dissipation scheme is applied.

In this way the gravitating systems are in quasi-equilibrium states, which allows us to compare numerical simulations and findings, resulting from theoretical studies of self-gravitating micro-canonical systems.

These studies predict a collapsing transitions, at a critical energy value $\varepsilon_c$ in the interval of negative specific heat, separating a high-energy homogeneous phase from a collapsed phase with a core-halo structure.

The theoretical predictions result from a micro-canonical approach that is based on a continuum approximation (mean field approximation, see chapter 3). This approximation washes out any information of the system granularity and particle correlations.

Yet, this information is present in the numerical simulations. Thus we can check the effect of granularity and particle correlations on the evolution of self-gravitating, isolated systems in the interval of negative specific heat. Furthermore, we want to check how the system evolution is modified when small-scale repulsive forces are introduced.

The study of the evolution in the interval of negative specific heat is important, since in this interval the system becomes unstable and nonequilibrium structures may spontaneously develop due to first order phase transitions (Lynden-Bell 1998, 1996; Lynden-Bell & Lynden-Bell 1977).

6.2.2 Why Studying the Gravitational Collapse of Strongly Dissipative Systems?

Elmegreen (2000) proposed a star-formation scenario with short star-formation time-scales. According to this scenario, the lifetimes of star-forming clouds (whose dynamical range extends at least over 3 dex) are a few free-fall times (see chapter 1). Thus, the collapse of star-forming clouds due to their own gravity must not be delayed or prolonged by a permanent energy injection or magnetic fields.

This may then suggest that the structures of star-forming clouds, but also of quiescent clouds may be formed during gravitational collapses at the free-fall time-scale. Indeed, observations of star-forming and quiescent clouds reveal similar structure properties, suggesting that the same principle shaping agents are at work.

In order to check this possibility, different effective energy-dissipation-schemes that provoke a gravitational contraction on the free-fall time-scale are applied in the models. This allows us then to study if such rapidly contracting systems can develop long-range phase-space correlations, such as those observed in the ISM and how they do depend on the different dissipation schemes.
6.2. MOTIVATION

Perturbation potential (external forcing)

![Graph showing perturbation potential](image)

Figure 6.1: The perturbation potential as a function of time is shown along the line $x = y = z = 0, \ldots, 1$. The time is indicated in units of dynamical times. The dimensionless potential is given in units of $GM/R$, where $M$ is the total mass and $R$ is the radius of the system. Amplitudes and frequencies are controlled by two parameters, $\gamma$ and $\delta$, respectively. **Left panel:** $\gamma = 0.1$, $\delta = \pi/\tau_{dyn}$. **Right panel:** $\gamma = 0.3$, $\delta = 1/\tau_{dyn}$.

6.2.3 Why Studying Systems Subject to an Energy-Flow?

The above described scenario suggests that molecular cloud structures are short-lived. However, there are alternative scenarios.

For instance, according to the classical Blitz & Shu (1980) picture the typical lifetime of star-forming clouds is of the order of $\sim 3 \times 10^7$ y. That is, the lifetime is more than an order of magnitude longer than a gravitational cloud collapse would take in the absence of any support.

Furthermore, it has been suggested that large amounts of the fragmented interstellar matter are in dynamical equilibrium, so that, even though local properties are transient, the global flow statistics can be maintained without the need of newly supplied matter (see chapter 1).

In order to check if such persistent flow patterns can be formed in the models, we revert to ideas that have been develop in nonequilibrium dynamics (see chapter 2). That is, the systems are maintained outside of equilibrium, by a continuous energy-flow. That astrophysical systems with such boundary condition may indeed form persistent long-range correlations has been shown by the simulations of self-gravitating shearing boxes (see chapter 5).

In shearing box simulations the energy-flow is maintained by differential rotation and dissipation. However, here we are concerned with molecular cloud scales at which differential rotation can be neglected. Thus we chose an other forcing scheme. That is, the dissipated energy is continuously replenished by large-scale potential perturbations. These perturbations are linear combinations of stationary waves that do not inject energy. A visual representation of the potential perturbations is shown in Fig. 6.1. The analytical form of the perturbation is given in the article on page 93.

If our system represents a molecular cloud, then perturbations of the above described
form may be due to star clusters, clouds or other massive objects passing irregularly in the vicinity.
Long-range correlations in self-gravitating \( N \)-body systems

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Abstract. Observed self-gravitating systems reveal often fragmented, non-equilibrium structures that feature characteristic long-range correlations. However, models accounting for non-linear structure growth are not always consistent with observations and a better understanding of self-gravitating \( N \)-body systems appears necessary. Because unstable gravitating systems are sensitive to non-gravitational perturbations, we study the effect of different dissipative factors as well as different small and large scale boundary conditions on idealized \( N \)-body systems. We find, in the interval of negative specific heat, equilibrium properties differing from theoretical predictions made for gravo-thermal systems, substantiating the importance of microscopic physics and the lack of consistent theoretical tools to describe self-gravitating gas. Also, in the interval of negative specific heat, yet outside of equilibrium, unforced systems fragment and establish transient long-range correlations. The strength of these correlations depends on the degree of granularity, which shows that the mass and force resolution should be coherent. Finally, persistent correlations appear in model systems subject to an energy flow.

Key words. gravitation – methods: \( N \)-body simulations – galaxies: ISM – ISM: structure

1. Introduction

Most astrophysical structures result from gravitational instabilities, from large-scale cosmological structures down to planets. Yet, among the least understood topics in astrophysics we find galaxy formation and star formation, which both involve fragmentation and the non-linear growth of structures occurring during the non-linear phases of gravitational instability.

Perhaps one of the fundamental reasons why fragmentation and structure formation via gravitational instability appears so difficult is that we lack consistent theoretical tools allowing to combine gravity with gas physics. Indeed, it is too often ignored that classical thermodynamics does not hold for gravitating systems, because these are non-extensive in the thermodynamical sense (Landsberg 1972, 1984; Tsallis 1999; Plastino & Plastino 1999). Actually, many other natural systems do not respect the requisites of thermodynamics. Such systems often feature interesting phenomena such as growing long-range correlations or phase transitions. Among the symptoms of a fundamental deep problem in gravitating systems is the appearance of negative specific heat (Lynden-Bell & Lynden-Bell 1977; Lynden-Bell 1998), which was seen for a long time as a paradox in statistical mechanics, since negative specific heat was thought to be impossible.

Presently, the only available approach to follow the nonlinear phases of gravitational instabilities is to carry out numerical simulations. Among all the existing methods, \( N \)-body techniques are thought to be the most effective to simulate the continuous case as well as the granular phases of self-gravitating systems.

Yet, despite the considerable success of these methods in reproducing many observed features, many fundamental problems remain. As mentioned above, the fragmentation and structure formation are not clearly understood. Related to this, CDM simulations conflict with observations at galactic scales (Moore 1999; Bullock et al. 2001; Bolatto et al. 2002), and no theory of the ISM is presently able to predict the conditions of star formation. Most of the time the star formation process relies on recipes with few physical constraints.

In situations where \( N \)-body simulations are successful (e.g. hot stellar systems), gravitational dynamics is sufficient to account for their main global properties and additional microscopic physics can be neglected. But when gravitational instability via fragmentation involves small-scale physics, the outcome may be strongly dependent on the properties of the small-scale physics. In other words, in situations where the growth on singularities triggered by gravity is allowed, the chaotic nature of gravitating systems make them sensitive to the perturbations induced by non-gravitational physics.

Therefore it is important to understand the properties of \( N \)-body systems subjected to various perturbations.
For these purposes, a numerical study of perturbed, self-gravitating $N$-body systems is carried out.

Among the relevant perturbations we expect that boundary conditions at small and large scales, as well as dissipative factors, can play a key role. In order to characterize the individual effects of perturbations, in the tradition of analytical models, one is advised to deliberately use simplified models.

A study of dissipative systems is important because such systems may develop long-range correlations. In the typical ISM, radiative cooling is very effective and induces a temporary energy flow leading the system far from equilibrium (Dyson & Williams 1997). From laboratory experiments it is well known that systems outside of equilibrium may spontaneously develop spatio-temporal structures (Glansdorff & Prigogine 1971; Nicolis & Prigogine 1977; Prigogine 1980; Melo 1994).

A permanent energy flow is induced when energy loss due to dissipation is replenished, that is, when the system is continuously driven, e.g., by time-dependent boundary conditions. Such systems may develop persistent long-range correlations. Astrophysical examples of this are the growth of structures in cosmological simulations or the long-term persistence of filamentary structures in shearing flows (Toomre & Kalnajs 1991; Huber & Pfenniger 2001a; Wisdom & Tremaine 1988; Salo 1995; Pfenniger 1998). Among other things, the effect of time-dependent potential perturbations on dissipative self-gravitating spheres is studied in this paper.

In the next section we briefly review some theoretical results of the thermodynamics of self-gravitating isothermal spheres. The model is presented in Sect. 3 and the applied methods to carry out the structure analysis for the simulated systems are explained in Sect. 4. The results are presented in Sects. 5–7. In Sect. 5 quasi-equilibrium states of $N$-body models are compared with analytical predictions. Section 6 is dedicated to a study of long-range correlations appearing in the interval of negative specific heat during the collapsing transition of gravitating systems. Finally, in Sect. 7 the evolution of systems subjected to an energy-flow is discussed.

2. Gravo-thermal statistics

By confining a gravitational system within a sphere, we can compare our numerical work with previous analytical studies using the tools of statistical mechanics and point out some discrepancies. Although the theory, sometimes referred to as gravo-thermal statistics, is not fully consistent, since the applied Gibbs-Boltzmann entropy is derived under the assumption of extensivity (this inconsistency was recalled, e.g., by Taruya & Sakagami 2001), it yields some important and instructive results.

Before we present the model and the results let us here briefly review some theoretical findings (see Padmanabhan 1990 for an extended discussion on the topic).

Antonov (1962) and Lynden-Bell & Wood (1968) studied theoretically the thermodynamics of self-gravitating isothermal spheres and found anomalous behavior when compared with classical thermodynamics of extensive systems. An example of such an anomalous behavior is the so-called gravo-thermal catastrophe that occurs when an isothermal gas of energy $E(<0)$ and mass $M$ is released within a spherical box of radius greater than Antonov’s radius, $r_A = 0.335GM^2/(-E)$. The analytical model predicts that for such systems there is no equilibrium to go and nothing can stop the collapse of the central parts.

Antonov’s and Lynden-Bell & Wood’s investigations have based on point-like particles. Since then, several studies have been carried out with modified (i.e., non point-like) particle potentials. Hertel & Thirring (1971) modified the gravitational interaction potential by introducing short-distance repulsive forces due to quantum degeneracy and Aronson & Hansen (1972) investigated the behavior of self-gravitating hard-spheres (see also Chavanis & Sommeria 1998). Finally, Follana & Laliena (2000) applied softened interaction potentials. They all found qualitatively the same result: Unlike the model of Lynden-Bell & Wood, there is always an equilibrium state for finite size particles. However, a phase transition, separating a high energy homogeneous phase from a low energy collapsing phase with core-halo structure, occurs in an energy interval with negative microcanonical specific heat,

$$c_V = -\beta^2 \left( \frac{\partial \epsilon}{\partial \beta} \right)_V < 0,$$

where $\epsilon$ is the total energy and $\beta = 1/T$ is the inverse temperature.

There are fewer theoretical works done with the grand canonical ensemble (e.g. de Vega et al. 1996) where mass would be allowed to be exchanged with the environment, therefore we limit the scope of this study to the canonical ensemble, that is, energy can be exchanged with the environment, but not the mass and angular momentum.

Note that an ensemble allowing also the exchange of angular momentum would be very relevant for astrophysical situations, but few theoretical works have been made on this important aspect (Laliena 1999; Fliegans & Gross 2001).

3. Model

A spatially isolated, spherical $N$-body system is studied. The $N$ particles of the system are accelerated by gravity, and forces induced by boundary conditions and perturbations, respectively. In all, a particle can be accelerated by five force types: 1) confinement 2) energy dissipation due to velocity dependent friction 3) external forcing 4) long-range attraction (self-gravity) and 5) short-range repulsion. The five force types are discussed in detail in Sects. 3.2–3.5.

3.1. Units

A steep potential well confines the particles to a sphere of radius $R \approx 1$ (see Sect. 3.2). The total mass $M$ and the
gravitational constant \( G \) equal one as well, \( G = M = 1 \). The dimensionless energy is the energy measured in units of \( GM^2/R \), that is in our model \( \approx 1 \). Thus energies and other quantities with energy dimension (like the temperature) can be considered as dimensionless.

### 3.2. Confinement

In order to keep the model simple and to enable a comparison with established theoretical results of canonical isothermal spheres we apply a confinement that prevents gravitationally unbound particles from escaping and keeps thus the particle number constant.

The confinement is realized through a steep potential well

\[
\Phi_{\text{conf}} \propto R^{16},
\]

where \( R \) is the distance from the center of our spatially isolated system. The walls of the potential well are steep, so that the particles are only significantly accelerated by the confinement when they approach \( R = 1 \). However, the potential walls must not be too steep, to avoid unphysical accelerations near the wall, due to a discrete time-step.

Unlike non-spherical reflecting enclosures, the applied potential well conserves angular momentum.

### 3.3. Energy dissipation

The effect of different dissipation schemes is studied: local dissipation, global dissipation and a scheme similar to dynamical friction. For convenience we call the latter hereafter “dynamical friction”.

#### 3.3.1. Local dissipation

The local dissipation scheme represents inelastic scattering of interstellar gas constituents. That is, friction forces are added that depend on the relative velocities and positions of the neighboring particles.

A particle is considered as a neighbor if its distance is \( r < \lambda \epsilon \), where \( \epsilon \) is the softening length and \( \lambda \) is a free parameter. The friction force has the form:

\[
F_i = \begin{cases} 
\frac{\Lambda \left( \frac{m^2 r}{(r^2 + \epsilon^2)^{\eta}} \right)^{\eta^2 - \eta} \left( \frac{V \cdot r}{r} \right)^{\zeta} \cos \beta}{r^2 - \eta} & : r < \lambda \epsilon \quad \text{and} \quad V \cdot r < 0 \\
0 & : \text{otherwise},
\end{cases}
\]

where \( V \) is the relative velocity of two neighboring particles, \( \zeta, \eta \geq 0 \) are free parameters and \( i = x, y, z \). The term \( V \cdot r/r = V \cos \beta < 0 \) ensures a convergent flow and that dissipation affects only the linear momentum. As a consequence, angular momentum is locally conserved.

For \( \epsilon \ll r < \lambda \epsilon \) the friction force is

\[
F_i \propto \frac{(V \cos \beta)^{\zeta}}{r^{\eta}} e_i,
\]

where \( e_i \) is the \( i \)-th component of the unity vector. Thus the friction force increases with the relative velocity and decreases with the distance, provided that \( \zeta, \eta > 0 \).

The condition \( r < \lambda \epsilon \) ensures the local nature of the energy dissipation.

#### 3.3.2. Global dissipation

The global dissipation depends not on the relative velocity \( V \), but on the absolute particle velocity \( v \). The global friction force is

\[
F_i = -\alpha v_i,
\]

where \( \alpha \) is a free parameter. The same friction force was already used in the shearing box experiments of Toomre & Kalnajs (1991) and Huber & Pfenniger (2001a, 2001b).

#### 3.3.3. Dynamical friction

Setting the velocity-dispersion \( \sigma = 1 \), which is the dispersion of a virialized self-gravitating system with \( M = R = 1 \), Chandrasekhar’s dynamical friction formula can be parameterized to a good approximation by,

\[
F_i = \Gamma \frac{v_i}{(4 + v^3)},
\]

where \( \Gamma \) is a free parameter (Chandrasekhar 1943; Binney & Tremaine 1994). In contrast to global dissipation this scheme has the feature that \( F_i \) does not increase asymptotically with \( v \). There is a maximum at \( v \approx 1.3 \). Thus high speed particles in collapsed regions no longer dissipate energy and gravitational runaway is impeded.

### 3.4. Forcing scheme

If the dissipated energy is replenished by a forcing scheme the system can be subjected to a continuous energy-flow. Here the energy injection is due to time-dependent boundary conditions, that is, due to a perturbation potential. This perturbation should on the one hand be non-periodic and quasi-stochastic, on the other hand it should provide an approximately regular large-scale energy injection in time-average. A simple perturbation potential, meeting these conditions, has the linear form,

\[
\Phi_{\text{pert}}(x, y, z, t) = \gamma [B_x(t)x + B_y(t)y + B_z(t)z],
\]

where \( x, y \) and \( z \) are Cartesian coordinates, \( \gamma \) is a free parameter determining the strength of the perturbation, and

\[
B_x(t) = \sum_{i=1}^{3} A_{x,i} \sin(\omega_{x,i} t + \phi_{x,i}),
\]

\[
B_y(t) = \sum_{i=1}^{3} A_{y,i} \sin(\omega_{y,i} t + \phi_{y,i}),
\]

\[
B_z(t) = \sum_{i=1}^{3} A_{z,i} \sin(\omega_{z,i} t + \phi_{z,i}),
\]

where \( t \) is the time. The amplitudes \( A_{j,i} \) and the phases \( \phi_{j,i} \) are arbitrary fixed constants and remain unchanged.
for all simulations \((j = x, y, z)\). The frequencies are given through \(\omega_{j,i} = \delta \Omega_{j,i}\), where \(\Omega_{j,i}\) are arbitrary, but not rationally dependent constants (to avoid resonances) as well. \(A_{j,i}, \phi_{j,i}\) and \(\Omega_{j,i}\) lay down the form of the potential. Then, the amplitude and the frequency of the perturbation are controlled by two parameters, namely \(\gamma\) and \(\delta\), respectively (Huber 2001).

The perturbations are a linear combination of stationary waves that do not inject momentum.

If our system represents a molecular cloud then a perturbation similar to those described above can be due to star clusters, clouds or other massive objects passing irregularly in the vicinity. Indeed such stochastic encounters must be quite frequent in galactic disks and we assume that the average time between two encounters is,

\[
\tau_{\text{pert}} = \frac{1}{f_{\text{pert}}} \gg \tau_{\text{dyn}},
\]

where \(f_{\text{pert}}\) is the mean frequency of the encounters and \(\tau_{\text{dyn}}\) is the dynamical time. Thus, these encounters can provide a continuous low frequency energy injection on large scales.

Such a low frequency forcing scheme induces at first order particle motions, which are transformed in the course of time to random thermal motion, due to gravitational particle interaction.

### 3.5. Self-gravity, repulsion and force computation

Interaction forces include self-gravity and repulsive forces whose strength can be adjusted by a parameter. Repulsive forces may be a result of sub-resolution processes such as star-formation or quantum degeneracy. The particle interaction potential reads,

\[
\Phi_p(r) = -\frac{Gm}{\sqrt{r^2 + \epsilon^2}} \left(1 - \frac{\epsilon}{r^2 + \epsilon^2}\right),
\]

where \(m\) is the particle mass and \(r\) is here the distance from the particle center. On larger scales \((\gg \epsilon)\) the potential is a Plummer potential \(\Phi_{Pl}\) (Plummer 1911; Binney & Tremaine 1994). On small scales \((\ll \epsilon)\) the deviation from a Plummer potential and the strength of the repulsive force is determined by the parameter \(\xi\). The interaction potential becomes for instance repulsive in the range \(|r| < \epsilon \sqrt{3(\xi - 1)}\) if \(\xi > 1/3\) (see Fig. 1). If \(\xi = 0\), \(\Phi_p = \Phi_{Pl}\) holds.

The interaction forces are computed on the Gravitor Beowulf Cluster\(^1\) at the Geneva Observatory with a parallel tree code. This code is based on the Fortran Barnes & Hut (1986, 1989) tree algorithm, and has been efficiently parallelized for a Beowulf cluster. It is available on request.

The time-integration is the leap-frog algorithm with uniform time-step, which ensures the conservation of the symplectic structure of the conservative dynamics. The time-step is,

\[
\Delta t \leq 0.1 \frac{\epsilon}{\sigma_v},
\]

where \(\sigma_v\) is the velocity-dispersion of the initial state.

The accuracy of the force computation is given through the tolerance parameter, which for the studies presented in this paper is \(\theta \leq 0.58\).

### 3.6. Code testing

In order to test the code, the evolution of the angular momentum is checked. We find that the angular momentum \(J\) of a system with 10 000 particles and local dissipation scheme, that is initially \(J = 3.4 \times 10^{-5}\) in units such that energy is dimensionless, remains small, \(J < 4 \times 10^{-5}\). This is illustrated by means of two examples. Two simulations with equal dissipation strength are carried out. One with short range repulsion, \(\xi = 2/3\), and the other without, \(\xi = 0\). After 10 crossing times the angular momentum is \(J = 3.3 \times 10^{-5}\) and \(J = 3.8 \times 10^{-5}\), respectively. The deviation from the initial value is larger for the simulation without short-range repulsion. This is because the dissipation leads in this case to a stronger mass concentration, i.e., to shorter particle distances. Indeed, the potential energy after 10 crossing times is \(U \approx -12\) and \(U \approx -17\) for the simulation with and without short-range repulsion, respectively.

### 3.7. Parameters

The different dissipation schemes, the forcing scheme and the interaction potential with the short-distance repulsive force, have several free parameters. In order to do a reasonably sized parameter study the particle number has to be limited to a maximum of \(N = 160 000\). However, the absolute particle number is not important, but the effect of changing it is. Thus, \(N\) also is varied.

\(^1\) http://obswww.unige.ch/~pfennige/gravitor/gravitor.html
Table 1. Model parameters, characterizing energy injection (perturbation potential), dissipation, interaction potential (self-gravity and repulsion) and mass resolution.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Range</th>
<th>Description</th>
<th>Effect</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\delta$</td>
<td>$1/\tau_{\text{dyn}} , \ldots , 2\pi/\tau_{\text{dyn}}$</td>
<td>Perturbation frequency</td>
<td>Perturbation potential</td>
</tr>
<tr>
<td>$\gamma$</td>
<td>$0, \ldots , 0.7$</td>
<td>Perturbation factor</td>
<td></td>
</tr>
<tr>
<td>$\Lambda$</td>
<td>$0, \ldots , 5000$</td>
<td>Local dissipation factor</td>
<td></td>
</tr>
<tr>
<td>$\lambda$</td>
<td>$2\epsilon, \ldots , 50\epsilon$</td>
<td>Dissipation volume</td>
<td></td>
</tr>
<tr>
<td>$\eta$</td>
<td>$0, 1, 2$</td>
<td>Relative distance power</td>
<td>Local dissipation</td>
</tr>
<tr>
<td>$\zeta$</td>
<td>$0, 1, 2$</td>
<td>Relative velocity power</td>
<td></td>
</tr>
<tr>
<td>$\alpha$</td>
<td>$0, \ldots , 50$</td>
<td>Global dissipation factor</td>
<td>Global dissipation</td>
</tr>
<tr>
<td>$\Gamma$</td>
<td>$0, \ldots , 30$</td>
<td>Dynamical friction factor</td>
<td>Dynamical friction</td>
</tr>
<tr>
<td>$\epsilon$</td>
<td>$0.0046, 0.050$</td>
<td>Softening length</td>
<td></td>
</tr>
<tr>
<td>$\xi$</td>
<td>$0, 1/3, 2/3, 1$</td>
<td>Repulsion strength</td>
<td>Interaction potential</td>
</tr>
<tr>
<td>$N$</td>
<td>$6400, \ldots , 160000$</td>
<td>Particle number</td>
<td>Mass resolution</td>
</tr>
</tbody>
</table>

The model parameters and their ranges are indicated in Table 1.

4. Correlation analysis

In order to check if the perturbations of the gravitational system induce characteristic correlations in phase-space, the indices $D$ and $\delta$ of the mass-size relation and the velocity-dispersion-size relation are determined.

The index of the mass-size relation can be found via,

$$D(r) = \frac{d \log M}{d \log r} (r),$$

where $M$ is the sum of the masses of all particles having a relative distance $r$. The mass size relation is then

$$M(r) \propto r^{D(r)},$$

where $r$ can be considered as the scale. For a homogeneous mass distribution and for a hierarchical fragmented, fractal structure the index is a constant. For the latter case the index is not necessarily an integer and lies in the interval, $D_L < D < D_S$, where $D_L$ and $D_S$ are the topological dimension and the dimension of the embedding space, respectively (Mandelbrot 1982).

The index of the velocity-dispersion-size relation is determined by,

$$\delta(r) = \frac{d \log \sigma}{d \log r} (r),$$

where $\sigma$ is the velocity-dispersion of all particles having a relative distance $r$. Observations of the interstellar medium suggest a constant index $\delta = \delta_L$ on scales $\mathcal{O}(0.1) - \mathcal{O}(100)$ pc with $0.3 \lesssim \delta_L \lesssim 0.5$. This is expressed by Larson’s law (e.g. Larson 1981; Scalo 1985; Falgarone & Perault 1987; Myers & Goodman 1988)

$$\sigma \propto r^{\delta_L}.$$  

In order to preclude the effect of boundary conditions on the scaling relations, upper and lower cutoffs have to be taken into account. The lower cutoff is given by the softening length. An upper cutoff arises from the final system size. Thus, the scope of application is for the velocity-dispersion-size and the mass-size relation, $\epsilon < r < 2R_{90}$, and $\epsilon < r < R_{90}/2$, respectively, where $R_{90}$ is the radius of the sphere centered at the origin which contains 90% of the mass (see Huber & Pfenniger 2001a).

5. Results: I. Properties of equilibrium states in numerical and analytical models

Here, some $N$-body gravo-thermal experiments of systems with weak dissipation, i.e., of systems in quasi-equilibrium are presented. The results are compared with theoretical findings.

In order to cover a range of energy with the same experiment, we introduce a weak global dissipation scheme allowing us to describe a range of quasi-equilibrium states. Here weak means that $\tau_{\text{dis}} > \tau_{\text{dyn}}$, where $\tau_{\text{dis}}$ is the dissipation time-scale. Then, the results can be compared with theoretical equilibrium states.

Follana & Laliena (2000) theoretically examined the thermodynamics of self-gravitating systems with softened potentials. They soften the Newtonian potential by keeping $n$ terms of an expansion in spherical Bessel functions (hereafter such a regularized potential is called a Follana potential). This regularization allows the calculation of the thermodynamical quantities of a self-gravitating system. The form of their potential is similar to a Plummer potential with a corresponding softening length. Figure 2 shows a softened Follana potential with $n = 10$, and a Plummer potential with $\epsilon = 0.05$.

In their theoretical work Follana & Laliena found for a mild enough regularization ($n < 30$) a phase transition below the critical energy $\epsilon_c \approx -0.335$ in a region with negative specific heat. The transition separates a high energy homogeneous phase from a low energy collapsed phase with a core-halo structure.

We want to reproduce these findings by applying a Plummer potential ($\xi = 0$). Furthermore, the effect of a small-scale repulsive force ($\xi > 1/3$) is studied.

For these purposes, simulations with a weak global dissipation strength, $\alpha = 0.025$, are carried out.
The dissipation time is then \( \tau_{\text{dis}} = 80 \tau_{\text{dyn}} = 56.6 \tau_{\text{ff}} \), where the free fall time is \( \tau_{\text{ff}} = \tau_{\text{dyn}}/\sqrt{2} \).

Before we discuss the results, let us briefly present some model properties. The initial state is a relaxed, unperturbed and confined \( N \)-body sphere with total energy \( \epsilon = 1 \).

Assigning a particle a volume of \( 4\pi \varepsilon^3/3 \), the volume filling factor is,

\[
V_{\text{ff}} = \frac{N\varepsilon^3}{R^3},
\]

where \( N \) is the particle number and \( R \) is the radius of the system. The volume filling factor is set as \( V_{\text{ff}} = 1 \), meaning that the force resolution is equal to the mass resolution. Then, the particle number is \( N = 8000 \).

The applied weak dissipation strength maintains the system approximately thermalized and virialized (see Fig. 3). Here a system of \( N \) particles is called virialized when the moment of inertia \( I \) is not accelerated,

\[
\frac{\dot{I}}{2} = 2T + \sum_{i=1}^{N} F_ir_i \approx 0,
\]

where \( T \) is the kinetic energy, \( r_i \) is the location of the \( i \)-th particle and \( F_i \) is the sum of all forces acting on the particle (gravitation, friction, confinement).

The evolution of the inverse temperature \( \beta = 1/T \) as a function of the total energy \( \epsilon \) for two simulations with \( \xi = 0 \) and \( \xi = 2/3 \), respectively, as well as the semi-analytical curve calculated by Follana & Laliena are shown in the left panel of Fig. 4.

For the simulation with the Plummer potential the interval of negative specific heat agrees with theoretical predictions. Also, in accordance with predictions, a phase transition takes place, separating a high energy homogeneous phase from a collapsed phase in a interval with negative specific heat. Yet, the simulated phase transitions occur at higher energies. To illustrate this the evolution of the Lagrangian radii \( \kappa \) are shown in the middle and the right panel of Fig. 4.

In the high energy homogeneous phase the system is insensitive to the short-distance form of the potential. Thus all systems enter the interval of negative specific heat at the same energy. However, the collapsed phase is sensitive to the short-distance form. That is, the system with repulsive short-distance force re-enters the interval of positive specific heat at a higher energy than those without such forces (see left panel of Fig. 4).

Furthermore, the collapsed phase resulting from a simulation with a Plummer potential is hotter and denser than those resulting from a simulation with repulsive small scale forces. This can be seen in Fig. 4 as well.

For a smaller softening length, theory expects a phase transition at higher energies. Yet, already in simulations with a mild softening \( \epsilon = 0.05 \) the collapsing transition occurs straight after the system has entered the interval of negative specific heat, which is shortly after the system has become self-gravitating. Thus, the collapsing transition cannot occur at significantly higher energies for a smaller softening length.

However, for systems with smaller softening length, the energy at which the system reenters the zone of positive specific heat after the collapse changes, i.e., it is shifted to smaller energies. This is shown in Fig. 5, where two simulations with \( N = 160000 \) and \( \epsilon = 0.05 \) resp. \( \epsilon = 0.01 \) are compared.

Due to the higher particle number the dissipation time is in these simulations reduced to \( \tau_{\text{dis}} = 4\tau_{\text{dyn}} \). Thus, dynamical equilibrium is not as perfect as previously. That is, the acceleration of the moment of inertia \( I \) attains a maximum value of, \( \dot{\bar{I}}/T = 0.14 \), where \( T \) is the kinetic energy. However, in general dynamical equilibrium is well approximated and we expect, due to the experience with simulations with varying particle number and dissipation.
Fig. 4. Comparison of theoretical predictions and simulated systems. The simulations are carried out with \( N = 8000 \) and the softening length is, \( \epsilon = 0.05 \). Consequently, the volume filling factor is, \( V_{\text{ff}} = 1 \). Left: Inverse temperature \( \beta \) versus energy \( \varepsilon \) for models with softened potentials. The solid line indicates the theoretical result with the Follana potential, \( n = 10 \). The other curves depict the evolution of the simulated systems. Crosses: repulsive potential \( (\xi = 2/3) \). Dots: Plummer potential \( (\xi = 0) \). The circle indicates the initial state of the simulations. The range of negative specific heat corresponds to the range where the slope of \( \beta(\varepsilon) \) is positive. Middle: The dotted lines describe the evolution of the Lagrangian radii \( \kappa \) for the simulation with the Plummer softening potential \( (\xi = 0) \). Each curve depicts the radius of a sphere containing a certain mass fraction. The different mass fractions are: \( \Delta M/M = \{5\%, 10\%, 20\%, \ldots, 80\%, 90\%, 95\%\} \). The solid line depicts the theoretical 95%-Lagrangian radius for the Follana potential. The dashed vertical line depict the moment when the system becomes fully self-gravitating. The dash-dotted and the dotted line mark the interval of negative specific heat. Right: Idem for the simulation with the short range repulsive force.

Fig. 5. Same as Fig. 4 for two simulations with \( N = 160000 \) and with a Plummer potential \( \xi = 0.0 \). Dots: \( \epsilon = 0.05 \), \( V_{\text{ff}} = 20.0 \). Crosses: \( \epsilon = 0.01 \), \( V_{\text{ff}} = 0.16 \).

time, that the effect of the temporal deviation from equilibrium does not affect qualitatively the results presented in Fig. 5.

The volume filling factor of the systems with \( N = 160000 \) and \( \epsilon = 0.05 \) is \( V_{\text{ff}} = 20 \), meaning that the mass resolution is a factor 2.7 larger than the force resolution. Thus the system is less granular than in the previous simulations with \( N = 8000 \) and \( V_{\text{ff}} = 1 \) (see Fig. 4). Yet, this does not affect the simulated quasi-equilibrium states and the deviation from theoretical predictions remains.

5.1. Discussion

Some predictions made by analytical models, such as the interval of negative specific heat, agree with findings resulting from \( N \)-body models. Yet, discrepancies also appear, such as the way the collapsing phase transition develops.

At first glance the deviating results are surprising. Yet the small scale physics is different in the two models, which may account for the discrepancies. Indeed, thermostatistics assumes a smooth density distribution and is thus not able to account for two-body relaxation effects in granular media. However, a certain degree of granularity is an inherent property of \( N \)-body systems. This may then lead to discrepancies, especially in the unstable interval of negative specific heat, where phase transitions may be sensitive to small scale physics.

While thermostatistics is too smooth to account for microscopic physics in granular self-gravitating media such as the interstellar gas, two-body relaxation is often too strong in \( N \)-body systems due to computational limitations. This is particularly so in high force resolution simulations where the force resolution is larger than the mass resolution. We refer to this in Sect. 6.1 where we discuss, among other things, the effect of the granularity on long-range correlations appearing in the interval of negative specific heat.

A further discrepancy between nature, analytical models and \( N \)-body models may be due to the entropy used...
in gravo-thermal statistics (Taruya & Sakagami 2001). Indeed, the entropy used in analytical models to find equilibrium states via the maximum entropy principle is the extensive Boltzmann-Gibbs entropy, that is in fact not applicable for non-extensive self-gravitating systems.

Generalized thermostatistics including non-extensivity are currently developed (Tsallis 1988; Sumiyoshi 2001; Latora et al. 2001; Leubner 2001). These formalisms suggest that non-extensivity changes not all but some of the classical thermodynamical results (Tsallis 1999; Boghosian 1999), which agrees with our findings.

Because currently it is a priori not known which thermostatical properties change in non-extensive systems and consistent theoretical tools are not available, analytical results must be considered with caution.

6. Results: II. Transient long-range correlations in gravitational collapses

6.1. Effect of dissipation

Properties of equilibrium states differing from theoretical predictions in the interval of negative specific heat were found (see above). Let us now study the nonlinear structure growth in this interval for an increased dissipation strength, i.e., outside of equilibrium, when \( \tau_{\text{dis}} \lesssim \tau_{\text{H}} \).

As previously, a relaxed, unperturbed, confined N-body sphere with \( \varepsilon = 1 \) serves as the initial state for the simulations.

In order to dissipate the energy different dissipation schemes are applied. Before we discuss the appearance of long-range correlations in unstable dissipative systems let us briefly discuss the effect of the different dissipation schemes on the global system structure.

The different applied dissipation schemes (see Sect. 3.3) lead to collapsed phases with different global structures. That is, they have different mass fractions contained in the core and the halo. A typical ordering is \( D_{\text{global}} > D_{\text{dyn}} > D_{\text{local}} \), where \( D_{\text{global}}, D_{\text{dyn}} \) and \( D_{\text{local}} \) are the density contrasts resulting from simulations with a global dissipation scheme, dynamical friction and a local dissipation scheme, respectively. The density contrast is, \( D \equiv \log(\rho_{\text{cm}}/\rho_0) \), where \( \rho_{\text{cm}} \) and \( \rho_0 \) are the center of mass density and the peripheric density, respectively. This means that for a global dissipation scheme almost all the mass is concentrated in a dense core, whereas a local dissipation scheme can form a persistent "massive" halo.

The evolution of the mass distribution resulting from simulations with the different dissipation schemes is shown in Fig. 6. The collapse of the inner shells takes in all systems about the same time. Yet, the uncollapsed matter distributed in the halo is for the global dissipation scheme less than 2\% whereas it is 10\% for the local dissipation scheme.

Next, temporary long-range correlations that develop in unforced gravitating systems are presented depending on the different system parameters.

A sufficiently strong global dissipation, i.e., \( \tau_{\text{dis}} \lesssim \tau_{\text{H}} \), of a gravitating system leads during the nonlinear phase of the collapsing transition to fragmentation and long-range phase-space correlations, so that the index, \( \delta(r) \), of the velocity-dispersion size relation, \( \sigma \propto r^{\delta(r)} \), becomes positive.

Figure 7 shows the evolution of the velocity-dispersion size relation, i.e., of \( \delta \) for different dissipation strengths. The relations result from 3 different simulations with global dissipation schemes and different dissipation strengths. The dissipation strength is given through the parameter \( \alpha \). Here \( \alpha = 1.0, 5.0, 9.0 \). This corresponds to \( \tau_{\text{dis}} = 2.0, 0.4, 0.2 \tau_{\text{H}} \).

While the velocities remain uncorrelated in the simulation with \( \alpha = 1.0 \), \( \delta \) becomes temporarily positive over the whole dynamical range in the simulations with the stronger dissipation.

The velocity correlations start to develop at largest scales after the system has become self-gravitating. After the system has entered the interval of negative specific heat the correlation growth is accelerated. It attains a maximum and finally disappears when the collapse ends. The dynamical range over which \( \delta > 0 \) during maximum correlation is \( \approx 2 \) dex.

Correlations at small scales straight above the softening length are stronger for a stronger energy dissipation.

The maximum correlation established in the negative specific heat interval persists for \( O(0.1) \tau_{\text{H}} \) and is characteristic for the applied dissipation strength and softening. For instance, the simulation with the strongest dissipation develops during 0.3 \( \tau_{\text{H}} \) a roughly constant \( \delta \) over a range.

![Figure 6. Evolution of the Lagrangian radii for a system with global dissipation scheme, dynamical friction and a local dissipation scheme, respectively. The Lagrangian radii depict here and in the following figures, spheres containing the following mass fractions: \( \Delta M/M = \{5\%, 10\%, 20\%, \ldots, 80\%, 90\%, 98\%\} \).](image-url)
Fig. 7. Evolution of the index, $\delta(r)$, of the velocity-dispersion-size relation, $\sigma \propto \delta(r)$, during the collapsing transition for three simulations with global dissipation scheme and different dissipation strength. The correlations result from simulations that were carried out with 160 000 particles. The solid, the dashed and the dotted vertical lines indicate the scope of application of the simulation with $\alpha = 1.0$, $\alpha = 5.0$ and $\alpha = 9.0$, respectively. The lower cutoff is given by the softening length, that is here $\epsilon = 0.01$, and the upper cutoff by $2R_{90}$. The time is indicated above each panel. The corresponding evolution of the Lagrangian radii and the specific heat are shown in Fig. 8.

Fig. 8. Evolution of the Lagrangian radii (top) and the specific heat (bottom) for three simulations with global dissipation and different dissipation strength. The dashed line marks the moment, when the system becomes fully self-gravitating. The dash-dotted line indicates the moment when the system enters the zone of negative specific heat. The dotted line marks the “end” of the collapse. The evolution of the corresponding phase-space correlations are shown in Fig. 7.
of 1.5 dex that resembles Larson’s relation. Yet, correlations at small scales decay rapidly and the index $\delta$ even becomes negative at intermediate scales.

The end of the collapse and with it the disappearance of the correlated velocity structure is marked by a diverging negative specific heat $c_V \rightarrow -\infty$. This is shown in Fig. 8.

Systems with dynamical friction and local dissipation also develop velocity correlations.

Figure 9 shows the correlations resulting from simulations with a local dissipation scheme. Contrary to the global dissipation scheme, a strong local dissipation does not extend the collapse of the whole system. Thus, local friction forces that are strong enough to develop correlations lead also to a fast collapse and correlations accordingly persist a short time compared to simulations with global dissipation.

The corresponding evolution of the Lagrangian radii and the interval of negative specific heat are shown in Fig. 10.

In the dynamical friction scheme, energy dissipation depends, as in the global dissipation scheme, on the absolute particle velocity. Thus a strong dynamical friction extends the gravitational collapse and the “lifetime” of the correlations. Yet, the observed phase-space correlations are weaker, compared to those appearing in simulations with global and local dissipation, respectively, meaning that the reduced dissipation strength for fast particles, accordingly to the dynamical friction scheme, destroys the correlations.

6.2. Effect of short distance regularization

So far the velocity correlations dependent on the different dissipative factors were studied. Next the effect of different short distance regularizations of the Newtonian potential,
removing its singularity, are checked. That is, simulations with and without short distance repulsive forces and with different dissipation strengths are carried out. Energy is dissipated via the global dissipation scheme.

In Fig. 11 the velocity correlations resulting from three simulations with three different regularizations are compared. The regularizations are characterized by two parameters, namely, the softening length $\epsilon$ and the parameter $\xi$ that determines the strength of the repulsive force. The softening length and $\xi$ of the three simulations, compared in Fig. 11, are, $(\epsilon = 0.01, \xi = 0.0), (\epsilon = 0.01, \xi = 2/3)$ and $(\epsilon = 0.05, \xi = 0.0)$, where $\xi = 0.0$ means that a Plummer potential is applied and $\xi > 1/3$ means that short distance repulsive forces are at work (see Sect. 3.5). The dissipation strength is for all three simulations the same, $\alpha = 10$, and the collapsing time is consequently the same as well (see Fig. 12).

During the first $\tau_H$ long-range correlations resulting from the three simulations are identical. Then the $\delta(r)$ starts to separate. Indeed, the repulsive forces cause an increase of $\delta$ at small scales, compared to the simulation with the same softening length, but without repulsive forces. Yet, large scales remain unaffected.

The index $\delta(r)$ resulting from the simulation with the large softening length is larger compared to the other two simulations after one $\tau_H$. Also, the $\delta(r)$ curve is flatter after that time.

The large difference between the simulation with the long softening length and the corresponding simulation with the short softening length is astonishing, because there is a clear deviation even at large scales. This may be because the two simulations were carried out with the same particle number, $N = 160,000$. Consequently, the volume filling factors are different, namely, $V_H = 25$ and $V_H = 0.16$, meaning that for the long softening length, i.e., the large $V_H$, the mass resolution is greater than the force resolution and vice versa for the small softening length. Whereas a small volume-filling factor describes a granular phase, a large volume-filling factor describes rather a fluid phase. Thus the deviation of the velocity correlations may mainly be due to the different volume-filling factors and not due to the different softening lengths.

In order to check this, some complementary numerical experiments are carried out, whose results are presented subsequently.

6.3. Granular and fluid phase

The upper panels in Fig. 13 show the velocity-dispersion-size relation that develop in simulations with different volume-filling factors, but equal softening length, $\epsilon = 0.031$. The particle numbers are, $N = 6400$, $N = 32,000$ and $N = 160,000$. Thus, the volume-filling factors are, $V_H = 0.2$, $V_H = 1.0$ and $V_H = 5.0$, respectively. The strongest phase-space correlations appear in the simulation with the highest mass resolution, i.e., for the “fluid phase”, and weakest correlations appear in the “granular phase”. The flattest $\delta(r)$ curve results from the simulation in which force and mass resolution are equal.

This result suggests that the volume filling factor $V_H$ (and not the softening length) is the crucial parameter determining the correlation strength in simulations with equal dissipation strength, as long as a substantial part of the system forces stem from interactions of particles with relative distances larger than the softening length.

The lower panels in Fig. 13 show, for the same simulations, the evolution of the index, $D(r)$, of the mass-size relation, $M \propto r^D(r)$. The deviation from homogeneity is strongest for the “granular phase” and weakest for the “fluid phase”. Thus the order of the correlation strength in space is inverse to the order of the correlation strength in phase-space. Such an inverse order is expected in self-gravitating system with $|U| \propto T$, where $U$ is the potential energy and $T$ is the kinetic energy (Pfenniger & Combes 1994; Pfenniger 1996; Combes 1999).

The corresponding Lagrangian radii and the interval of negative specific heat are shown in Fig. 14.

6.4. Initial noise

Assuming a constant softening length, different volume filling factors $V_H$ imply different particle numbers $N$ that introduce different Poisson noise, $\sim \sqrt{N}$. Then, the statistical roughness of the initial uniform Poisson particle distribution decreases as $\sim 1/\sqrt{N}$.

Thus the initial roughness of the latter simulations (see Fig. 13) differ from each other by a factor $\sim 2.2$ and one might suppose that the different correlation strengths in these simulations are the result of the different initial, statistical roughness.

In order to check this possibility, long-range correlations, resulting from two simulations with $V_H = 1.0$ and with statistical roughness differing by a factor of $\sim 2.2$, are compared. The results are presented in Fig. 15 and the corresponding Lagrangian radii are shown in Fig. 16. The simulation with $N = 32,000$ was already presented in Fig. 13. It is now compared with a simulation with stronger initial roughness.

Despite the different initial roughness, long-range correlations in phase-space are almost identical for the two simulations.

As regards fragmentation, in addition to the parameters of the mass-size relation, the mass distributions in space were compared. We actually find a stronger fragmentation for small particle numbers. Yet $V_H$ seems to be the crucial parameter for the fragmentation strength in Fig. 13.

Consequently, the different correlation strengths in Fig. 13 cannot be accounted for by Poisson noise, but are mainly the result of the different volume-filling factors, or more precisely, due to the different ratio of force and mass resolution. Thus, dark matter clustering in high-resolution cosmological $N$-body simulations in which the force resolution is typically an order of magnitude smaller than
Fig. 11. Evolution of the velocity correlations for three simulations with global dissipation scheme and different potential regularizations. That is, the softening length and the form of the softening, respectively, change from simulation to simulation. For $\xi = 0.0$ the particle potential is a Plummer. $\xi = 2/3$ means that there are short distance repulsive forces at work. The particle number is, $N = 160000$. The solid, the dashed and the dotted vertical lines indicate the scope of application of the simulation with $(\epsilon = 0.01, \xi = 0.0)$, $(\epsilon = 0.01, \xi = 2/3)$ and $(\epsilon = 0.05, \xi = 0.0)$, respectively. The evolution of the corresponding Lagrangian radii are shown in Fig. 12.

Fig. 12. Same as Fig. 10 for three simulations with different potential regularizations, i.e., different softening lengths and forms of the softening potential, respectively. The evolution of the corresponding phase-space correlations are shown in Fig. 11.

the mass resolution, may be too strong compared to the physics of the system (Hamana et al. 2001).

6.5. Free fall of cold dissipationless systems

Here, long-range correlations are discussed that develop in cold, gravitationally unstable systems without energy dissipation. The correlation strength appearing in such dissipationless systems depends on the initial ratio between kinetic and potential energy, $a = T/U$, i.e., on the number of thermal Jeans masses given through $M/M_J = 2a^{-3/2}$.

This is shown in Fig. 17. For $a = 0.01$ the index $\delta$ of the velocity-dispersion-size relation remains zero at small scales during the entire free fall, whereas for $a = 0.0$ the
Fig. 13. **Upper panels:** Evolution of the index, $\delta(r)$, of the velocity-dispersion-size relation, $\sigma \propto r^{\delta(r)}$, resulting from three simulations with different *volume filling factor* that describe a granular phase, $V_f = 0.2$, a fluid phase, $V_f = 5.0$, and an intermediate state, $V_f = 1.0$. The particle numbers are $N = 6400$, $N = 160 000$ and $N = 32 000$, respectively. The softening length is $\epsilon = 0.031$. The solid, the dashed and the dotted vertical lines indicate the scope of application of the simulation with $V_f = 0.2$, $V_f = 1.0$ and $V_f = 5.0$, respectively. The scope is given by, $\epsilon < r < 2R_{90}$. The corresponding Lagrangian radii are presented in Fig. 14. **Lower panels:** The evolution of the index, $D(r)$, of the mass-size relation, $M \propto r^{D(r)}$. The solid, the dashed and the dotted vertical lines indicate the scope of application of the simulation with $V_f = 0.2$, $V_f = 1.0$ and $V_f = 5.0$, respectively. The scope is given by, $\epsilon < r < R_{90}/2$.

Fig. 14. Same as Fig. 10 for three simulations with different *volume filling factor*. Figure 13 shows the evolution of the corresponding long-range correlations.
Fig. 15. Upper panels: Evolution of the phase-space correlations resulting from two simulations with identical volume filling factor, $V_{ff} = 1.0$, but unequal statistical, initial roughness, $\sim 1/\sqrt{N}$. The particle number and the softening length of the first simulation (dots) are $N = 32 000$ and $\epsilon = 0.031$, respectively. The second simulation (triangles) is carried out with $N = 6400$ and $\epsilon = 0.054$. The solid and the dashed lines indicate the scope of application given by $\epsilon < r < 2R_{50}$. The evolution of the corresponding Lagrangian radii is shown in Fig. 16. Lower panels: The evolution of the spatial correlations. The scopes of application, depicted by the vertical lines, is, $\epsilon < r < R_{50}/2$.

Fig. 16. Same as Fig. 10 for two simulations with identical volume filling factor and unequal initial roughness, $\sim 1/\sqrt{N}$. The evolution of the corresponding long-range correlations is shown in Fig. 15.
Fig. 17. Upper panels: Phase-space correlations that develop during the free fall of three cold, dissipationless systems. All systems have the same volume filling factor, $V_\text{f} = 0.016$. The parameter $a$ indicates the ratio between kinetic and potential energy. The solid, the dashed and the dotted lines indicate the scope of application, $\epsilon < r < 2R_90$, of the simulation with $(a = 0.01, N = 160000), (a = 0.0, N = 160000)$ and $(a = 0.0, N = 32000)$, respectively. The evolution of the corresponding Lagrangian radii is shown in Fig. 18. Lower panels: Evolution of the spatial correlations. The scopes of application, depicted by the vertical lines, is, $\epsilon < r < R_90/2$.

index becomes $\delta > 0$ over the whole dynamical range, over a period of $\sim 0.7 \tau_\text{ff}$. That is, $350 \, M_J$ are not sufficient to develop small-scale phase-space correlations in a dissipationless system.

In order to show the dependence of the result on the initial Poisson noise, the absolutely cold simulation with $N = 160000$ is compared with a $N = 32000$-body simulation. As above, it follows that the initial roughness of the two systems differs by a factor of $\sim 2.2$.

Figure 17 shows that the non-equilibrium structures resulting from the simulations with equal $a$ but unequal particle number differ from each other during the entire free fall period. Indeed, during the first half of the free fall time the two initial conditions produce velocity correlations that differ on small and large scales. After $t = 0.5 \tau_\text{ff}$ the differences approximately disappear. However, the behavior of the spatial correlations (see lower panels of Fig. 17) is inverse, meaning that differing spatial correlations appear after $0.5 \, \tau_\text{ff}$ and persist for the rest of the free fall.

These results suggest that non-equilibrium structures appearing in cold systems or in systems with very effective energy dissipation depend more strongly on initial noise than those appearing in warm systems with less effective dissipation (see also Fig. 15).

The evolution of the Lagrangian radii during the free fall of the cold, dissipationless systems are shown in Fig. 18. Because these systems are adiabatic and self-gravitating during the whole simulation, the corresponding intervals are not plotted in the figure.
6.6. Discussion

Long-range spatial and phase-space correlations appear naturally during the collapsing phase transition in the interval of negative specific heat if the energy dissipation time is $\tau_{\text{dis}} \lesssim \tau_{\text{ff}}$ so that the time-scale of correlation growth is smaller than the time-scale of chaotic mixing, which is $\sim 1/\lambda \propto \tau_{\text{ff}}$, where $\lambda$ is the maximum Lyapunov exponent (Miller 1994).

Actually, the details of the long-range correlations depend on the applied dissipation scheme, but there are also some generic properties. That is, phase space correlations start to grow at large scales, whereas spatial correlations seem to grow from the bottom-up. Moreover, there is an upper limit for the index of the velocity-dispersion-size relation within the dynamical range, namely, $\delta \approx 1$.

Besides the dissipation strength and initial conditions, the volume filling factor $V_{\text{ff}}$ is a crucial parameter for the correlation strength. That is, phase-space correlations are strong in the fluid limit and weak for a granular phase. The behavior of the spatial correlations is exactly the other way around.

The softening length does not affect correlations within the dynamical range. Yet, sub-resolution repulsive forces affect correlations on small scales above the resolution scale. Of course, this does not hold for the onset of the correlation growth, but only after sufficient particles have attained sub-resolution distances.

The above considerations suggest that the correlations found are physically relevant and not a numerical artifact.

The long-range spatial and phase-space correlations appearing during the collapsing transition are qualitatively similar to the mass-size and the velocity-dispersion-size relation observed in the ISM (e.g. Blitz & Williams 1999; Chappell & Scalo 2001; Fuller & Myers 1992), showing that in the models, gravity alone can account for ISM-like correlations.

Furthermore, the time-scale of the correlation lifetime is the free fall time, $\tau_{\text{ff}}$, which is consistent with a dynamical scenario in which ISM-structures are highly transient (Vázquez-Semadeni 2002; Larson 2001; Klessen et al. 2000), which is related to rapid star formation and short molecular cloud lifetimes (Elmegreen 2000), that is, the corresponding time-scales are about an order of magnitude smaller than in the classical Blitz & Shu (1980) scenario.

7. Results: III. Permanent energy-flow

7.1. Nonequilibrium structures in systems subject to an energy flow

The numerical experiments presented above showed that dissipative self-gravitating systems fragment and establish long-range correlations outside of equilibrium in the interval of negative specific heat. These transient correlations persist for $1\sim 2 \tau_{\text{ff}}$ times.

Here we check if self-gravitating systems can establish persistent long-range correlations if they are maintained continuously outside of equilibrium by a permanent energy-flow. That is, the dissipated energy is continuously replenished by time-dependent potential perturbations.

Both simulations of granular and fluid phases are carried out. A typical parameter set, describing a granular system is, $N = 100000$, $\epsilon = 0.0046$, $V_{\text{ff}} = 0.001$ while $N = 160000$, $\epsilon = 0.0315$, $V_{\text{ff}} = 5$ are here typical for a fluid phase. Yet, the parameters are not fixed and the effect on the evolution is studied when parameters change. Parameters, controlling energy-flow and interaction potential, as well as their ranges, are indicated in Table 1.

The applied potential perturbations imitate massive objects passing in the vicinity on time-scales $\tau_{\text{pert}} \lesssim \tau_{\text{dyn}}$. The perturbations induce primarily ordered particle motions. Then gravitational interactions lead to a conversion of the bulk kinetic energy to random thermal motion. The energy injection due to such a forcing scheme can be quite regular until a plateau is reached.

Energy injection prevents a system from collapsing and maintains an approximately statistically steady state for $\sim 5\sim 15 \tau_{\text{dyn}}$ when energy dissipation is balanced appropriately by large scale potential perturbations. These states do not feature any persistent long-range correlations. Yet, they develop a temperature structure that is characteristic for the applied dissipation scheme. This is shown in Fig. 19, where the evolution of two granular systems subjected to an energy-flow is presented. One system dissipates its energy by a global dissipation scheme (top), the other by a local dissipation scheme (bottom).

The system with the global dissipation scheme is nearly thermalized during almost the entire simulation, whereas the local dissipation scheme leads to a permanent positive temperature gradient that is inverse compared to stars and resembles those of the ISM where dense, cool mass condensations are embedded in hotter shells.

If dissipation dominates energy injection the system undergoes in general a mono-collapse, that is, a collapsed structure is formed in which a part of the system mass is concentrated in a single dense core and the rest is distributed in a diffuse halo. However, systems with a rather fluid phase may develop several dense cores moving in a diffuse halo, when they are subjected to an appropriate energy flow. In the course of time the number of clumps varies, but a non-mono-clump structure persists for some $\tau_{\text{ff}}$ (see Fig. 20). These systems may even develop persistent phase-space correlations (see Fig. 21).

However, the clumps result not from a hierarchical fragmentation process, but are formed sequentially on the free fall time scale. Furthermore, the clumps are so dense that their evolution is strongly influenced by the applied regularization. That is, the evolution over several $\tau_{\text{ff}}$ depend on the numerical model that does not represent accurately small-scale physics.

In order to impede gravitational runaway that may hinder the formation of complex non-equilibrium structures within the given dynamical range, different measures are taken, such as short distance repulsion and the application of the dynamical friction scheme, where the friction force $F \rightarrow 0$ for $v \rightarrow \infty$. 

\[ q^2 \leq q^2_{\text{max}} \]
Yet, despite these measure, we find only either nearly homogeneous structures in systems where energy injection prevails, or systems dominated by unphysical clumps in the case of prevailing energy dissipation.

Up to now, the effect of several different dissipation schemes mainly were discussed. However, the effect of a modified forcing scheme is also checked. That is, a power-law forcing scheme is applied, which injects energy at different frequencies. This forcing scheme is a modification of those presented in Sect. 3.4 and reads, $\Phi_{\text{pert}}(\omega) \propto \omega^{\nu}$, where $\nu = -4, \ldots, 4$. Yet, such a forcing scheme also cannot induce a phase transition to complex non-equilibrium structures and the resulting mass distribution corresponds to the those described above.

7.2. Discussion

Actually the “clumpy” structure in Fig. 20 and the corresponding phase-space correlations shown in Fig. 21 do not represent real physics, nevertheless, they show that it is in principal possible to maintain spatial non-equilibrium structures and long-range phase-space correlations in a perturbed, dissipative, self-gravitating system over several dynamical times. Thus, it cannot be excluded that in the future, with a better representation of microscopic physics and forcing mechanisms at work in the ISM, models including self-gravity may produce complex non-homogeneous structures in a statistical equilibrium, i.e., persistent patterns formed by transient structures.

However, at present, models of dissipative, self-gravitating systems cannot produce such structures on the scale of Giant Molecular Clouds (e.g. Semelin & Combes 2000; Klessen et al. 2000; Huber & Pfenniger 2001a).

On larger scales, gravity gives rise to persistent non-equilibrium structures in cosmological and shearing box simulations. A common denominator of these models is that their time-dependent boundary conditions are given by a scale-free spatial flow counteracting gravity. Let us discuss this more precisely.

In cosmological and shearing box models, time-dependent boundary conditions create relative particle velocities that are inverse to gravitational acceleration and increase with particle distance, $v \propto r$. In the shearing box model, the relative azimuthal particle velocity due to the shear flow is $v_\theta \propto r_c$, where $r_c$ is the radial particle distance in cylinder coordinates. In cosmological models, the relative particle velocity induced by the Hubble flow is $v \propto r$, where $r$ is the relative particle distance in Cartesian coordinates.

These relations and consequently the corresponding flows are scale-free. The fact that the shear flow affects only the azimuthal velocity component may then account...
Fig. 20. Mass distribution of a system subjected to an energy-flow. Shown is the projection of the particle positions onto the $xy$-plane. The particle number is, $N = 32000$, and the volume filling factor is, $V_f = 1$. Energy is dissipated with a local dissipation scheme, $\beta = 150$, and short distance repulsive forces are at work, $\xi = 2/3$. The evolution of the corresponding phase-space correlations is shown in Fig. 21. The time is indicated in each panel in units of the free fall time, $\tau_f$.

Fig. 21. The evolution of the phase-space correlations in a system subjected to an energy-flow and with short distance repulsive forces (see Fig. 20). During the first $\sim 8 \tau_f$, the index of the velocity-dispersion-size relation is, $\delta = 0$, over the whole studied range, then correlations start to grow at largest scales. Finally, the index of the velocity-dispersion-size relation is, $\delta > 0$, over the whole dynamical range. In systems subjected to energy-flows, such correlations that extend over the whole dynamical range appear only if a local dissipation scheme is applied.

for the characteristic spiral-arm-like structures found in shearing box experiments, differing from those found in cosmological models, where the isotropic Hubble flow gives rise to, on average, isotropic non-equilibrium structures.

The models studied in this paper are not subject to a scale-free spatial flow counteracting gravity and persistent long-range correlations of astrophysical relevance do not appear. This may suggest that in situations where gravitational runaway is allowed, matter that has passed through a collapsing transition has to be replenished at large scales in order to attain a statistical equilibrium state of transient fragmentation.

8. Conclusions
First, equilibrium states of $N$-body models were compared with analytical models. Subsequently, the findings
resulting from this comparison were discussed and are summarized as follows:

- On the one hand, equilibrium properties of $N$-body models agree with predictions made by analytical models. An example is the energy interval of negative specific heat. One the other hand, discrepancies were found, such as the way the collapsing phase transition, separating a high-energy homogeneous phase from a low-energy collapsed phase, develops in the interval of negative specific heat. These discrepancies suggest: 1) Small scale physics becomes relevant for the system evolution when the growth of singularities triggered by gravitational instabilities is allowed. 2) Analytical models based on the Gibbs-Boltzmann entropy are not strictly applicable to non-extensive self-gravitating systems. Yet, not all of the equilibrium properties found by maximizing the Gibbs-Boltzmann entropy are expected to change if a fully consistent, generalized thermostatistical theory is applied.

Second, the collapsing transition was studied in systems with strong dissipation. The findings are:

- Dissipative self-gravitating systems develop outside of equilibrium, in the interval of negative specific heat transient long-range correlations. That is, fragmentation and nonequilibrium velocity-dispersion-size relations, with striking resemblance to those observed in the ISM, appear during the collapsing transition, when the dissipation time is shorter than the dynamical time. This suggests that nonequilibrium structures in self-gravitating interstellar gas are dynamical and highly transient.

- Besides the dissipation strength and the initial noise, the granularity turns out to be a crucial parameter for the strength of the resulting long-range correlations, substantiating the importance of a coherent mass and force resolution. That is, phase-space correlations are stronger in the fluid limit than in a granular phase. The opposite holds for spatial correlations. The inverse behavior of fragmentation strength and phase-space correlation strength is found in all simulations and is typical for self-gravitating systems.

Finally, systems subject to a permanent energy-flow were studied. We find:

- Typically driven dissipative systems evolve to a high-energy homogeneous phase or undergo a monocular collapse. Yet, model systems with a local energy dissipation can develop persistent phase-space correlations, but a persistent, hierarchical fragmented structure is not observed. This suggests that matter that has passed through a collapsing transition has to be replenished at large scales in order to maintain a hierarchical structure at molecular cloud scales.

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6.4 Complementary Discussions

6.4.1 Granularity and Criticality

The evolution of self-gravitating systems is sensible to the details of the applied dissipation scheme. Indeed, the size of clumps that form in a self-gravitating medium depends on the range over which the friction forces are active. Moreover a nonlinear dependence of the friction forces on the relative particle velocity or distance may provoke a critical behavior. Let us discuss this in greater detail.

In the local dissipation scheme the friction force has the form:

\[ F \propto \begin{cases} 
V^\zeta/r^\eta : r < \lambda \epsilon \\
0 : \text{otherwise,}
\end{cases} \tag{6.1} \]

where \( V \) and \( r \) is the relative particle velocity and distance, respectively. \( \lambda \) is a free parameter.

Typically a system with such a dissipation scheme develops a core-halo structure, where a part of the mass is concentrated in a dense core and the rest is homogeneously distributed in a halo. The amount of mass condensed in the core decreases with decreasing \( \lambda \), i.e., the core mass is smaller for more local dissipation schemes. Thus, a dissipation mechanism that works locally may produce in a spatially non-isolated system, small dense grains embedded in a homogeneous medium.

The system evolution is not only sensitive to the parameter \( \lambda \), but also to \( \zeta \) and \( \eta \). Indeed, in the nonlinear regime, i.e., for \( \zeta > 1 \) and/or \( \eta > 1 \), often a critical evolution is observed. That is, the system properties alter slowly over a period of several dynamical times, then suddenly a rapid core collapse occurs. An example of such a critical behavior is given in Fig. 6.6, where the evolution of a permanently driven system with \( \zeta = 2 \) and \( \eta = 1 \) is shown.

Such a behavior may be accounted for as follows: Dissipation schemes with \( \zeta > 1 \) and/or \( \eta > 1 \) are particularly effective in situations where particles are near and have high velocities. That is, given the mean initial particle separation and velocity, the energy dissipation is not very effective and the system evolves slowly. However, in near encounters particles dissipate energy and fall to the center of mass. As soon as enough mass is accumulated in this way or when a density fluctuation exceeds a critical value a collapsing transition is triggered. In this situation the dissipation scheme becomes very effective and enables the system to collapse on the free-fall time-scale.

To sum up, a locally effective dissipation scheme can form in an self-gravitating system a homogeneous medium in which intermittently clumps or grains are formed.

6.4.2 A Possible ISM-Shaping Scenario

Here a scenario that may account for some structure properties of the interstellar medium is proposed. The scenario is based on the findings resulting from the simulations presented in this thesis.

The shearing box simulations presented in chapter 5 show that at galactic disc scales filamentary patterns formed by transient structures can be maintained over several galactic rotations, if energy dissipation and energy injection due to differential rotation are appropriately balanced. However, if energy dissipation dominates locally, clumps form out of the filaments that typically do not disperse but lead to gravitational runaways.

The above presented findings resulting from the numerical study of self-gravitating spheres suggest that it may be difficult or even impossible to find a mechanism that may account
Figure 6.6: **Upper panel:** The potential, kinetic and total energy, $U$, $T$ and $\epsilon$, respectively, as a function of time resulting from a simulation of a system subjected to an energy-flow and with $\zeta = 2$. The Virial equation is indicated as well (Virial $\equiv \tilde{f}/2$). The sudden collapse after $t \approx 14 \tau_{\text{dyn}}$, reveals the critical system behavior. **Lower panel:** The evolution of the Lagrangian radii $\kappa$, resulting from the same simulation. The different mass fractions are: $\Delta M/M = \{5\%, 10\%, 20\%, \ldots, 80\%, 90\%, 98\%\}$.

for persistent nonequilibrium patterns on the molecular cloud scale, without introducing a continuous matter-flow (see also Semelin 1999; Semelin & Combes 2000; Klessen et al. 2000). Yet, the findings suggest that transient long-range correlations, similar to those observed in the ISM, appear naturally during a gravitational collapse.

The combination of these findings may then suggest the following scenario: At galactic disc scales gravity, differential rotation and energy dissipation form a dynamical, filamentary pattern, that is priori globally maintained due to the globally balanced energy budget.

Yet, in dense regions, inside the filaments or at intersection points of filaments, dissipation may locally dominate due to a higher inelastic collision rate, so that clumps form out of the filaments. Typically these clumps do not disperse in the course of time, but grow and lead finally to gravitational runaways. Then, due to an effective energy dissipation during the gravitational clump collapse, recursive fragmentation occurs and transient long-range correlations appear.

Since the intermittently appearing gravitational collapses transform mass from large scales down to small scales the global flow statistics can only be maintained if matter is continuously
replenished at galactic disc scales. That is, the evolution of the global flow statistics describing the structure properties of the ISM over its whole dynamical range would depend on the degree of the large-scale matter replenishment.

The above described scenario accounts for the long-range correlations of molecular cloud structures as a consequence of rapid gravitational collapses. Thus it suggest the following conclusions:

- The star-formation time-scale is about the free free-fall time-scale, $\tau_{SF} \sim \tau_f$, which is consistent with recent observations (see Elmegreen 2000 and chapter 1).

- The star-formation inefficiency, i.e., star-non-formation (see chapter 1) cannot be accounted for by a large number of gravitationally unbound clouds, rather it may be the result of a fragmentation process that does not terminate with the formation of stars, but with the formation of cold ($\approx 3K$), low-mass clumps with masses of the order of $\sim 1M_{\text{Jupiter}}$ (e.g. Pfenniger & Combes 1994).

### 6.4.3 Comments on the Analysis Technique

The velocity-size relation cannot be determined with a tree-code. Thus, all scaling relations were determined with a parallel direct-summation method. Let us discuss this in turn.

While direct-summation methods need $O(N^2)$ operations for a complete force calculation, where $N$ is the particle number, hierarchical tree methods use in general $O(N \log N)$ operations. Thus the particle interaction forces of the isolated $N$-body spheres were calculated with a tree algorithm based on the Barnes & Hut (1986, 1989) code. In this code the force on an individual particle from nearly particles is on average computed as a direct sum. The influence of remote particles is included by performing multipole expansions of clusters or cells containing many particles, truncated at a fixed relatively low order (Hernquist 1988).

In order to determine the monopole term of the expansion the code calculates the total mass $M_C$ and the center-of-mass position $r_c$ of all particles contained in a cluster. Then, considering a cluster as a particle with mass $M_C$ and position $r_c$, the mass-size relation can be determined with the tree-code.

That is, one selects a particle and enters the masses of all other particles in dependence of their distance in a histogram. This process is then repeated for all other particles. Finally, a cumulative sum over the relative particle distances yields the mass-size relation.

One might expect that by applying a similar technique also the velocity-size relation can be determined with a tree-code. Indeed, one extends the code, so that it additionally determines the center-of-mass velocity of the clusters and builds a two-dimensional histogram in which the particle masses are filled in dependence of the relative particle distance and velocity (for convenience we hereafter refer to this histogram as velocity histogram). Then the velocity-size relation can be found by a cumulative summation over the distances.

Yet, if the two dimensional velocity histogram is built with the tree code one does not obtain the correct velocity-size relation at larger scales, where the properties of particles in cluster are represented by their means. Let us demonstrate this by an example.

Consider, three particles $p_1$, $p_2$, $p_3$ with velocities $\vec{v}_{p_1} = (1, 0)$, $\vec{v}_{p_2} = (0, 1)$ and $\vec{v}_{p_3} = (0, -1)$, respectively. Assume that $p_2$ and $p_3$ are close to each other and far from $p_1$.

Then, in order to calculate the influence of the particles $p_2$ and $p_3$ on the acceleration of $p_1$ the tree code considers $p_2$ and $p_3$ as a cluster and treat the influence of them at once. The mean velocity of the cluster, consisting of $p_2$ and $p_3$ is $\vec{v}_c = (0, 0)$. Thus, the relative velocity
of the cluster and $p_1$ is $|\vec{V}|_{p_1,\text{cluster}} = 1$. Since the cluster represents two particles the value 2 is filled at $V = 1$ in the velocity histogram.

A direct-summation method treats the two particles $p_2$ and $p_3$ individually. Thus in order to build the velocity histogram, the relative particle velocity between $p_1$ and $p_2$ as well as between $p_1$ and $p_3$ are determined. They are $|\vec{V}_{p_1-p_2}| = |\vec{V}_{p_1-p_3}| = \sqrt{2}$. Since we find for both particles the same relative velocity, the value 2 is filled in the velocity histogram at $V = \sqrt{2}$.

That is, the tree-code yields an other histogram than the exact direct-summation method and can does not be used for the determination of the velocity-size relation. Thus all scaling relations were determined with a parallel code based on the direct-summation technique (Hockney & Eastwood 1981).
Part III

The Structure of the Interstellar Medium:
Observations and $\Delta$-Variance Analysis
Abstract

The structure of dust and molecular regions (cold ISM) in the vicinity of the sun, the Outer Galaxy and M51 are analyzed with the $\Delta$-variance method. Typically these structures are characterized by a power-law power-spectrum $P(k) = k^{-\beta}$ and the power spectral index can be determined with the $\Delta$-variance method. The method is wavelet based and can be applied in position space. Furthermore, the method includes an approximate correction for white noise and beam smoothing. Thus determining the power spectral index with the $\Delta$-variance method is a more robust approach than the use of conventional techniques. The here determined structure parameters, that are supplemented with published data, allow to study the structure properties of the interstellar medium over a scale range of 4 dex from sub-parsec scales up to galactic disk scales. The findings suggest a scale-free order of the ISM-structures at scales $\lesssim$ 1 pc. At larger scales the power spectral index decreases. Smaller values of $\beta$ substantiating a higher degree of fragmentation are also found in interarm regions and in the outer regions of spiral disks. Lower degrees of fragmentation may be related with higher densities. Finally, inside regions with similar mean density the structure properties are independent of the star-formation activity in a cloud.
Chapter 7

Introduction

7.1 The Need of a Statistical Description

Observations of the cold interstellar medium (ISM) reveal complex, inhomogeneous and lumpy structures at all scales from the galactic disk scales down to the resolution limit of the observing instrument. Often these structures are readily represented by discrete clouds and a description in terms of clumps, sheets and filaments forming a complex network over a variety of scales has been proposed (Scalo 1990; Green 1993).

Spectral line observations of the ISM, typically reveal line widths much broader than the thermal line width, substantiating the presence of supersonic turbulent motions and suggesting not a static but a dynamical picture of the ISM in which the global structure is formed by transient substructures (Blitz 1993).

In order to identify and to comprehend the relevant physical processes governing the shaping of these complex structures and the star formation process linked with it, a quantitative description of the ISM-structures is needed, allowing to compare different environments and scales as well as to constrain models.

The ISM-structures seem not to be completely random. Indeed, no particular “clump” size can be identified and the structures pervade the scales apparently in a self-similar manner.

Due to their transient and lumpy nature the ISM structures are described with statistical methods and since the structures may obey a scale-free order methods that are able to reveal signs of such scaling relations are most appropriate.

7.2 Power-Spectrum

One possible statistical method able to reveal signs of a hierarchical order is to determine the 2D spatial power spectrum of the observed intensity (i.e. emissivity) fluctuations. Indeed, for a self-similar structure the power-spectrum obeys a power-law,

\[ P(k) \equiv |I_k|^2 \propto k^{-\beta}, \]

(7.1)

where \( k = |\vec{k}| \) is the wave number, \( |I_k| \) are the amplitudes of the Fourier modes and \( \beta \) is the power spectral index. Subsequently we briefly discuss the meaning of the power-spectrum and some related relations.
The intensity fluctuation are defined as,

\[ I(\vec{x}) \equiv \frac{I(\vec{x}) - \langle I \rangle}{\langle I \rangle}, \]  

where \( I(\vec{x}) \) and \( \langle I \rangle \) are the observed emissivity at position \( \vec{x} \) and the mean emissivity, respectively.

Knowing the intensity fluctuations, the correlation function measuring the deviation from homogeneity at a certain scale \( r \) can be determined,

\[ \xi(r) = \langle I(\vec{x})I(\vec{x} + \vec{r}) \rangle_{\vec{x}|r=r}, \]  

where \( \vec{r} = \vec{x} - \vec{x}' \) is the distance between two points in the 2D intensity fluctuation image. Expanding the terms \( I(\vec{x}) \) and \( I(\vec{x} + \vec{r}) \) in Fourier components one finds (e.g. Peacock 1999),

\[ \xi(r) = \frac{F}{(2\pi)^2} \int |k|^2 e^{-ik\vec{r}} dk, \]  

where \( F \) is the image area and \( k \) is the wave number. Then the power spectrum, defined as the Fourier transform of the autocorrelation function is given through,

\[ P(k) \equiv |k|^2 = \frac{1}{F} \int \xi(r) e^{ikr} dr. \]  

In general the power-spectrum is not a complete statistical description, since it provides no information about the phases \( \varphi \) of the Fourier modes, \( k_k = I_k e^{ik} \), meaning that one does for instance not know if the underlying structures are rather clumpy, filamentary or mainly formed by sheets*.

Actually a scale-free self-similar structure has a power-law power-spectrum, yet since the power-spectrum provides no information about the phase correlations, the inversion of the sentence is not universally valid, i.e., a power-law power-spectrum means not necessary that the real space distribution is a fractal. Indeed, there are in principle three possibilities:

1. The power-spectrum is a power-law and the phases are correlated, but the underlying structure is not a fractal. An example of a non-fractal structure with correlated phases and Fourier modes decaying as a power-law over a certain scale range is a nearly circular, pyramidal mass distribution.

2. The power-spectrum is a power-law, the phases are correlated and the underlying structure is a fractal, i.e., the real space structure is scale-free and has well defined non-random spatial correlations.

3. The power-spectrum is a power-law and the phases are not correlated but completely random. Then the underlying structure is a fractional Brownian motion (fBm), that is, a structure with a random component and fractal properties (“random fractal”, see Mandelbrot 1982; Feder 1988; Falconer 1990) For such a structure the power-spectrum provides a complete statistical description and all the information is contained in the

* A method that for instance provides a complete statistical description is to determine the Minkowski integrals (Mecke et al. 1994; Hadwiger 1957). This method has been used to analyze the large scale structure of galaxy distributions and describes how clumpy or filamentary the structures are (Schmalzing et al. 1999; Bharadwaj et al. 2000). This may also be a promising technique to characterize the structures of the ISM.
power spectral index \( \beta \) (Peitgen & Saupe 1988). Although such structures do not show pronounced filaments, Stutzki et al. (1998) claim that molecular clouds may be well described by fBms, since their analysis of molecular cloud images indeed yield power-law power-spectra and completely random phases.

The power-spectrum has been determined for observed structures, as well as in numerical and analytical investigations.

For instance, interferometric radio observations measure directly the angular power-spectrum and several statistical studies based on the power-spectrum has been carried out in this area. For instance, Crovisier & Dickey (1983) found for HI observations of the galactic plane at 21 cm a power spectral index of \( \beta \approx 3 \) and Green (1993) deduced indices in the range \( \beta \approx 2.2-2.8 \) from a HI outer-galaxy survey.

Analytical and numerical models predict for a Kolmogorov type turbulence \( \beta = 11/3 \) (Goldreich & Sridhar 1995) and a slightly steeper slope \( \beta \sim 4 \) for shock type turbulence (Kornreich & Scalo 2000), respectively.

### 7.3 Problems of a Statistical Description

Although the 2D power-spectrum provides important information about the underlying structure, one has to act with caution when relating the statistics of the images to the statistical properties of the underlying 3D ISM-structures, since the images are deduced from observed spectral line data cubes that provide only limited information, about the real ISM-structures.

Indeed, for spectral-line observations the intensity can only be determined in dependence of the line of sight velocity component and the position in the plane perpendicular to the line of sight, i.e., only for three dimensions of the 6D phase-space. As a consequence, velocity and density fluctuations contribute to the 2D statistics. Furthermore, the statistics may be affected by nonhomogeneous excitation conditions, optical depth effects, and boundary conditions (Bensch 1999; Bensch et al. 2001). Thus relating intensity fluctuations in PPV (position-position-velocity) data cubes to the underlying density structure is a problem and one has to make simplifying assumptions and to accurately define the scope of application of the applied analysis method. Let us discuss this in turn:

- **Disentangling the velocity and density statistics:** Lazarian & Pogosyan (2000) pointed out that a priori both velocity and density fluctuations contribute to the 2D statistics since spectral line data cubes have one velocity and two spatial dimensions. Indeed, two clumps in the same line of sight, but at different distances can appear in the same velocity channel due to velocity fluctuations. Thus, in order to relate the 2D statistics to the ISM-structure one has to disentangle the effects of density and velocity fluctuations on the power-spectrum. Lazarian & Pogosyan (2000) and Lazarian et al. (2001) found that the relative influence of the density and the velocity structure depends on the range over which the velocity is integrated to create the 2D image for which the statistical analysis is carried out. If the square of the integration range is larger than the turbulent velocity dispersion of the considered scale, the integration range is called thick, meaning that the velocity information is averaged out and the power-spectrum represents the statistics of the underlying density structure (for a plain discussion on the topic see Stanimirović & Lazarian 2001).
• **Optical depth, excitation conditions and 2D projection:** Assuming homogeneous excitation conditions, the 2D power-spectrum of a velocity-integrated intensity map of an optical thin tracer describes the projected density structure of the observed tracer. If the power-spectrum is a power-law, then the 2D projection of a 3D structure is again a power-law with the same power spectral index $\beta$ (Stutzki et al. 1998; Mac Low & Ossenkopf 2000). Assuming furthermore that the tracer density is proportional to the total density the power spectral index describes the 3D density structure of the “cloud”. However, often observations are carried out with optical thick tracers and the assumption of uniform excitation conditions may not be fulfilled.

• **Boundary conditions:** In order to identify and describe accurately a hierarchical order, observations should ideally have a dynamical range of several orders of magnitude. However, in practice the dynamical range is limited by the finite resolution and finite observation time (i.e. limited map size). Indeed, observed maps typically have $\sim 1'000 - 10'000$ pixels (i.e. positions) and consequently a dynamical range of $\sim 1 - 2$ dex. Actually, large surveys have up to $\sim 1'000'000$ pixels, but these observations cover scales on which the large scale hierarchical order is affected by other systematic large scale structures such as spiral arms. Furthermore at small scales, white noise as well as the telescope beam size (resp. the point spread function in the case of optical and infrared space observations) affect the structure analysis and on large scales the analysis is influenced by edge effects. These effects further reduce the scale-range for which a statistical description is relevant.

### 7.4 The Aim of this Study

Despite the difficulties, a statistical description of the ISM-structures is needed in order to identify the relevant shaping agents by comparing different “clouds” and models.

Here, a structure analysis of dust and molecular regions using the $\Delta$-variance is presented. As the power-spectrum, the $\Delta$-variance is able to reveal signs of a scale-free order and for structures with a power-law power-spectrum the $\Delta$-variance allows to determine the power spectral index $\beta$.

The $\Delta$-variance technique is applied to intensity images mapping the ISM at different scales, in different environments and with different tracers.

That is, we determine the power spectral index of two nearby molecular clouds ($d \approx 140$ pc), L1512 (quiescent) and L1524 (low-mass star forming cloud), of molecular regions in a survey of the outer galaxy and of dust regions in M51 (Whirlpool galaxy, NGC 5194, Hubble type: Sc). Finally the data-set is supplemented with published power spectral indices of several galactic molecular clouds (Bensch et al. 2001).

The analysis are carried out for velocity-integrated maps. For the considered scales the square of the integration range is larger than the turbulent velocity dispersion. Thus the power spectral index describes the density fluctuations of the underlying structure.

The study covers a scale range of 4 dex ($\sim 0.001$ pc - 100 pc) and the amount of data allows to carry out a statistical evaluation of the power spectral indices and to compare the structure properties on different scales, at different distances from the galaxy center, of star-forming regions and of non-star-forming regions.
7.4.1 Why Using the \( \Lambda \)-Variance?

Previous studies have shown that typically structures of the ISM can be described with a power-law power-spectrum over a limited scale range.

Here structures of the ISM in different regions of spiral galaxies are analyzed with the \( \Lambda \)-variance technique since this technique is a robust method to determine the power spectral indices.

The advantage of using the power spectral index \( \beta \) for the description of a hierarchical structure is, that, unlike the fractal dimension for instance, it is invariant to a 2D projection. Thus, observations and models can be compared without taking into account projection effects.

The \( \Lambda \)-variance technique that provides the power spectral index is a wavelet based analysis method\(^1\) that is more robust with respect to disturbing effects than the direct determination of the power-spectrum itself, especially in the case of limited dynamical ranges.

The main advantages of the \( \Lambda \)-variance analysis are:

1. Unlike the power-spectrum the \( \Lambda \)-variance can be determined in position space. Thus edge effects due to periodic boundary conditions can be excluded. In particular such effects distort the power-spectrum determined in Fourier space when the size of the observed, discretely sampled map is smaller than the spatial extend of a emission source. Indeed, in such situations Fourier amplitudes at high frequencies are increased with respect to the true power-spectrum of the underlying structure due to the implicitly assumed periodic boundary conditions of the discrete Fourier transform. That is, the power outside a range limited by the Nyquist critical frequency is folded over this range, an effect known as aliasing (Press et al. 1986).

2. White noise and beam smoothing, i.e., the convolution of the telescope beam with the emission of the observed source, distort the structure parameters at small scales. This may further decrease the dynamical range and rendering an accurate characterization of a scale-free order impracticable. However, the \( \Lambda \)-variance method allows a separation between signals from the observed structures and noise. Thus it includes methods that corrects for the influence of white noise, but also for beam smearing effects.

The \( \Lambda \)-variance was introduced by Stutzki et al. (1998) as a new method to quantify the structures of the ISM. Complementary studies, have been carried out by Ossenkopf et al. (1998), Bensch (1999) and Bensch et al. (2001). That is, they applied the method to artificially generated fBm-images as well as to observed molecular clouds and studied reliability and accuracy of the method. They found that the \( \Lambda \)-variance is a robust measure of ISM-structures with respect to disturbing effects such as white noise, beam smoothing, edge and optical depth effects.

7.5 The \( \Lambda \)-Variance

7.5.1 Principle

The \( \Lambda \)-variance method is discussed in detail in Stutzki et al. (1998), Bensch (1999) and Bensch et al. (2001). Here we review the principle of this analysis method.

\(^1\)In fact, the \( \Lambda \)-variance is related to the wavelet transform (Zielinsky & Stutzki 1999).
The $\Delta$-variance analysis measures the degree of correlation at different scales by computing the convolution of the observed signal with a wavelet of variable size.

In the present context, the signal is given through the intensity fluctuations $I(x,y)$ of the observed ISM, where $x$ and $y$ are the spatial image coordinates. Then the $\Delta$-variance is defined as the variance of the convolved intensity fluctuations,

$$
\sigma^2_\Delta(L) = \frac{1}{2\pi} \text{var}(I * \bigcirc L)_{x,y}, \tag{7.6}
$$

where the convolution is defined as,

$$
I * \bigcirc L \equiv \int I(\vec{x}) \bigcirc L (\vec{x}' - \vec{x}) d^3x, \tag{7.7}
$$

and the normalized French hat wavelet of scale $L$ is,

$$
\bigcirc L (r) = \begin{cases} 
\frac{1}{\pi (L/2)^2} : & r \leq \frac{L}{2} \\
\frac{1}{\pi (L/2)^2} r : & \frac{L}{2} < r \leq \frac{3L}{2} \\
0 : & r > \frac{3L}{2}, \tag{7.8}
\end{cases}
$$

where $r = (x^2 + y^2)^{1/2}$.

The scale $L$ is referred to as lag. The French hat wavelet is an axisymmetric down-up-down function with a positive central circle surrounded by a negative annulus, so that the diameter of the circle equals the width of the annulus (Argoul et al. 1989).

For convenience let us introduce, $\Delta_L = I(x,y) * \bigcirc L$. Then the variance of the wavelet convolved intensity map is,

$$
\text{var}(\Delta_L)_{x,y} = \langle \Delta_L^2 \rangle_{x,y} - \langle \Delta_L \rangle_{x,y}^2. \tag{7.9}
$$

In order to describe the structure on different scales the $\Delta$-variance has to be determined for different lags $L$. Then, if the structure has a purely power-law power-spectrum $P(k) \propto k^{-\beta}$, the power spectral index $\beta$ can be derived from the function $\sigma^2_\Delta(L)$. Indeed, Stutzki et al. (1998) showed that the following relation holds,

$$
\sigma^2_\Delta(L) \propto L^{d_\Delta} \propto L^{\beta+2-2} \text{ for } 0 < \beta < 6. \tag{7.10}
$$

That is, the power spectral index $\beta = d_\Delta + 2$ can be derived from the slope $d_\Delta$ of the function $\sigma^2_\Delta(L)$ in a double logarithmic ($\log - \log$) plot.

The smallest lag for which the $\Delta$-variance is determined is given by the sampling interval and the largest is $\approx 1/3 \text{min}(L_x, L_y)$, where $L_x$ and $L_y$ are the side-length of the intensity image. The smallest and the largest lag fix the scope of application, i.e., the dynamical range of the $\Delta$-variance analysis method that for convenience is here referred to as the analytical range.

### 7.5.2 Correction for White Noise and Beam Smoothing

For a purely power-law power-spectrum Eq. 7.10 holds. Yet, images of the ISM are affected by white noise and beam smoothing, which distort the statistics of the image. Particularly at small scales white noise induces arbitrary structure fluctuations ($\beta \to 0, d_\Delta \to -2$), while the
convolution with the beam smoothes out the structures (i.e. $\beta$ and $d_\Delta$ are increased). Thus the influence of these effects have to be taken into account in the $\Delta$-variance analysis.

Let us first discuss the effect of white noise on the $\Delta$-variance. For this purpose consider an intensity image that is a superposition of a structure with power spectral index $\beta$ and white noise. The power spectrum of such an image is given through $P(k) = A k^{-\beta} + B k^0$ and consequently the $\Delta$-variance is,

$$\sigma^2_\Delta(L) = a L^{\beta - 2} + b L^{-2}, \quad (7.11)$$

where $A, B, a, b = \text{const.}$

The effect of the beam smoothing is more difficult to handle and an exact correction is not possible. However, assuming that the underlying structure is described by a power-law with a power spectral index $\beta$, an approximation is possible. Accordingly to this approximation the $\Delta$-variance of a beam convolved image is given through,

$$\sigma^2_\Delta(L) \propto (1 - K_\beta (L/\theta_{mb})^{-2}) L^{\beta - 2} \quad \text{for} \quad 2 < \beta < 4, \quad (7.12)$$

and for lags $L \gtrsim 2 \theta_{mb}$, where $\theta_{mb}$ is the half power beam width (HPBW) of the telescope and $K_\beta = 3.916 - 1.244 \beta + 0.119 \beta^2$.

Then in order to determine the power spectral index $\beta$ of an image distorted by white noise and beam smoothing a three-parameter $\chi^2$-square fit is done to the measured variances $\sigma^2_\Delta(L)$ using the fit function

$$\sigma^2_{\Delta, \text{fit}}(L) = a (1 - K_\beta (L/\theta_{mb}^{-2}) ) L^{\beta - 2} + b L^{-2} \quad \text{for} \quad 2 < \beta < 4, \quad (7.13)$$

where $a, b, \beta$ are the parameters to be fitted.

The above presented correction for the beam smoothing can also be applied to optical and infrared images taken with space based telescopes if the telescope point spread function (PSF) can be approximated with a Gauss function. Such an approximation is justified when the energy in the diffraction rings is small compared to the white noise and thus can be neglected. This condition is fulfilled for the optical and infrared images analyzed in this study.

When we deal with optical and infrared images the smoothing effects due to the finite mirror size is referred to as PSF-smoothing.

### 7.5.3 Correction for Large-Scale Trends

Systematic large-scale structures replacing the scale-free order may appear artificially in an image due to its limited size and particular position. In order to reduce the influence of these large-scale trends on the statics a bilinear surface is fitted and subtracted from the intensity map before the $\Delta$-variance analysis is carried out.

### 7.5.4 Error Analysis

Before we discuss the error computation it is necessary to determine the $\Delta$-variance for discretely sampled maps.

---

1. The expression $K_\beta$ cannot be computed in a closed form. Thus it is determined with a second order polynomial fit (Stutzki et al. 1998).
For this purpose consider a discretely sampled map \( I(i,j) \) with \( 1 \leq i \leq n \) and \( 1 \leq j \leq m \), where \( n \times m \) is the number of image-pixels. Then the \( \Delta \)-variance is determined as,

\[
\sigma_{\Delta}^2(L) = \frac{1}{2\pi} \left( \langle \Delta_L^2 \rangle_{i,j} - \langle \Delta_L \rangle_{i,j}^2 \right),
\]

and the statistical error is given through,

\[
\delta_{\sigma_{\Delta}^2} = \sqrt{\frac{1}{N} \sum_{i,k} \left( \frac{1}{2\pi} |\Delta_L(i,k) - \langle \Delta_L \rangle_{i,j}|^2 - \sigma_{\Delta}^2 \right)},
\]

where \( 1 \leq l \leq n/L \) and \( 1 \leq k \leq m/L \) is another set of sample positions this time separated by the lag \( L \) in order to have an independent set of wavelet convolved intensities. The number of sample positions is \( N = n/L \times m/L \).

The term \( \delta^2_{\sigma_{\Delta}} \) may be considered as the variance of \( \sigma_{\Delta}^2 \) (variance of the \( \Delta \)-variance).

The errors of the \( \Delta \)-variance determined for the different lags enter the the \( \chi^2 \)-square fit (see Eq. 7.13) as weighting factors and the fit yields the standard deviations of the fit parameter \( \beta \). Finally, the accuracy of \( \beta \) with respect to the limited dynamical range are estimated by comparing the fit parameters with known structure parameters of artificial clouds.

Then the combination of the standard deviation from the fit and the estimated error due to the limited dynamical range yield the final error of \( \beta \).

### 7.5.5 Summary

Let us briefly list the principle steps of the structure analysis using the \( \Delta \)-variance:

- For the selected field, the velocity is integrated over the entire velocity range.
- A bilinear surface is fitted and subtracted from the velocity integrated map.
- For different lags covering the dynamical range of the map the \( \Delta \)-variance \( \sigma_{\Delta}^2(L) \) and its error are determined via Eq. 7.14 and Eq. 7.15, respectively.
- In order to derive the power spectral index \( \beta \) from the \( \sigma_{\Delta}^2(L) \), computed for the noisy map, a three parameter fit is done using the fit function determined in Eq. 7.13. The final error of \( \beta \) takes into account the uncertainty due to the fit and the limited dynamical range. Since the velocity integration range is thick (see “disentangling the velocity and density statistics” on page 121), the index \( \beta \) describes the underlying density-statistics.

### 7.6 Relation Between \( \beta \) and Other Structure Parameters

Sylos Labini et al. (1998) calculated the power spectral index for fractals with dimension \( D_{\text{MS}} \) and found that the generalization

\[
D_{\text{MS}} \approx \beta \quad \text{for} \quad 1.7 < D_{\text{MS}} < 2.3
\]
is accurate to better than 10\%. Here the fractal dimension $D_{MS}$ is the index of the mass-size relation that describes to what degree the structure fills the embedding space. Since, the power spectral index is invariant to 2D projection the latter relation can be applied directly to the spectral indices $\beta$ determined with the $\Delta$-variance analysis.

For fBm-images Stutzki et al. (1998) found the relation,

$$d_{PA} = \frac{(6 - \beta)}{2}$$

(7.17)

where $d_{PA}$ is the perimeter-area relation.

According to these relations, a small $\beta$ means that the corresponding mass distribution fills the embedding space very inhomogeneously and that its surface is very irregular. Thus, a smaller $\beta$ means that the structure is rather fragmented, while a larger $\beta$ substantiates the presence of a rather smooth structure.

### 7.7 Outline

In the next chapter we present the observations and the structure analysis of molecular and dust regions in the Milky Way and M51, respectively. That is: 1) Our observations of L1512 and L1524 with the KOSMA telescope are presented. Their structure is analyzed with the $\Delta$-variance. 2) The structure of molecular regions in the Outer-Galaxy are analyzed. The survey of the Outer-Galaxy was conducted by Heyer et al. (1998). 3) An analysis of dust regions in the spiral galaxy M51 is carried out. The data-set is provided by Polletta (2001). Finally, the results are supplemented with published findings and summarized. The conclusions are presented on page 156.
Chapter 8

Application of the $\Delta$-Variance Analysis-Method

8.1 L1512 and L1524

The two molecular regions L1512 and L1524 in the vicinity of the sun ($d \approx 140$ pc) were selected in order to compare a dark, pre-star-forming cloud (L1512) harboring potential sites of future star formation with a low-mass star-forming cloud (L1524).

With this study we want to address the question whether the star-formation process is an entirely local process, i.e., the formation of a proto-stellar core occurs independently from the evolution of the rest of the cloud, or if it requires the molecular cloud as a whole to undergo a structural transition in order to become a star-forming cloud. Thus the map is not centered at the dense core L1524 itself, but covers an arbitrary piece of the cloud westward to the main peak. Hereafter the region is referred to as L1524 (Haro 6-10).

8.1.1 Sources

Here, we briefly present the two selected nearby molecular regions:

- **L1512**: The dark cloud L1512 is located at the edge of the Taurus-Auriga star-forming complex, close to the galactic plane ($b \approx -5^\circ$). The cloud contains an opaque core with an extension of the order of $\sim 0.1$ pc. The core is one of the few cores in the complex that is not associated with an IRAS infra-red point source. The width of the $^{32}$HC$_3$N lines are the narrowest ever observed in a dense core. Since, the temperature in the core is estimated to be $11.6$ K, the contribution of the non-thermal motion to the observed line-width is only $\sim 0.04$ kms$^{-1}$ (Fuller & Myers 1993), suggesting that the core has dissipated away almost all of its turbulent energy (i.e. non-thermal support against self-gravity) and may be about to become a star-forming cloud. Thus the cloud is referred to as pre-star-forming cloud. The distance of L1512 from the sun is $d \approx 140$ pc. For a more extended discussion on L1512 see Falgarone et al. (1998) and Heithausen et al. (1999).

- **L1524 (Haro 6-10)**: The low-mass star-forming cloud L1524 is not located at the edge, but within the Taurus-Auriga complex. The selected region of L1524 contains the $H\alpha$ star that is associated with the Herbig-Haro objects 6-10 and coincides with a IRAS
point source (IRAS 04263+2426). Furthermore, it is associated with a variable infrared source having forbidden lines, which is characteristic for Herbig-Haro objects that are in turn typical for low-mass star forming regions. Pastor et al. (1991) derived from CS observations of L1524 (Haro 6-10) a mass and a velocity gradient that is compatible with gravitationally bound motions. Although the CO line shapes appear complex, there is no conclusive evidence for a CO outflow distorting the structure of the selected region (Edwards & Snell 1984; Levreault 1988). The distance of L1524 from the sun is \( d \approx 140 \) pc. For a more detailed discussion on L1524 see Pastor et al. (1991). An extended map of L1524 with its neighboring clouds is presented in Nercessian et al. (1988).

8.1.2 Observations

The observations of L1512 and L1524 (Haro 6-10) were carried out with the 3m sub-millimeter KOSMA\(^*\) telescope at Gornegrat, Switzerland (Winnewisser et al. 1986, 1990; Degiacomi et al. 1995; Kramer et al. 1998a). \(^{12}\)CO \( J = 3 - 2 \) rotational transitions were observed in October and November 2000, and in March 2001. A 345 GHz SIS\(^\dagger\) receiver was used with a noise temperature between 100 and 140 K. As backend, a high resolution acousto-optical spectrometer with a resolution of 0.056 km s\(^{-1}\) at 345 GHz was used.

All the data were taken with the efficient on-the-fly (OTF) observing mode\(^\ddagger\) (Beuther et al. 2000). A spacing of 30\(^\circ\) corresponding to the Nyquist sampling interval of the KOSMA telescope beam at 345 GHz was employed. Overall, the source was covered 2-5 times.

The \((0,0)\) position of the L1512 and L1524 (Haro 6-10) map is \( \alpha_{1950} = 5^h00^m 54^s.500 \), \( \delta_{1950} = 32^\circ 29' 54'' .00 \) and \( \alpha_{1950} = 4^h26^m 22^s.100 \), \( \delta_{1950} = 24^\circ 26' 25'' .00 \), respectively. The pointing was checked to be better than 20\(^\circ\) by observing planets (Jupiter).

Observations of Jupiter were also used to determine the half power beam width (HPBW). The main beam efficiency \( B_{\text{eff}} \) was derived using disk brightness temperatures and the forward efficiency \( F_{\text{eff}} \) was derived from skydips. The observing parameters are summarized in Tab. 8.1.

The sky transmission was estimated by measuring the radiation temperature of the blank sky. Then the intensities were corrected for the main beam efficiency. Here, the intensities are presented on the main beam temperature scale \( T_{\text{mb}} = (F_{\text{eff}}/B_{\text{eff}})T_A^* \) (Rohlf & Wilson 2000).

Polynomial baselines up to third order and sinusoidal baselines were fitted and subtracted from the observed spectra. All data reduction was done using the GILDAS\(^\dagger\) package.

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\(^*\)KOSMA: Kölner Observatorium für Submillimeter und Millimeter Astronomie.

\(^\dagger\)SIS: Superconducting-Insulating-Superconducting layer

\(^\ddagger\)OTF observing mode: After observing the OFF-position the telescope scans the source (ON-position) in the direction of right ascension or in declination at a constant speed while continuously integrating the received signal over a certain period. Here, the signal is integrated over 15 sec during which the telescope moves 30\(^\circ\). After ten integration periods, i.e., after the telescope has scanned a line with a length of 5\(^\circ\), it moves again to the OFF-position and a new cycle starts.

\(^\dagger\)GILDAS, the "Grenoble Image and Line Data Analysis Software", is a collection of software developed by the Observatoire de Grenoble and IRAM, oriented towards radioastronomy applications.
Table 8.1: Observing parameters

<table>
<thead>
<tr>
<th>Source</th>
<th>Transition</th>
<th>Frequency</th>
<th>HPBW(a)</th>
<th>(B_{\text{eff}})</th>
<th>(F_{\text{eff}})</th>
<th># pixels(^d)</th>
<th>(\langle r_{\text{atm}} \rangle)(^e)</th>
<th>(\Delta T_{\text{mb}})(^f)</th>
</tr>
</thead>
<tbody>
<tr>
<td>L1512</td>
<td>(^{12})CO 3 – 2</td>
<td>345.796</td>
<td>88</td>
<td>72</td>
<td>90</td>
<td>1500</td>
<td>0.33</td>
<td>0.36</td>
</tr>
<tr>
<td>L1524 (Haro 6-10)</td>
<td>(^{12})CO 3 – 2</td>
<td>345.796</td>
<td>88</td>
<td>72</td>
<td>90</td>
<td>2400</td>
<td>0.41</td>
<td>0.51</td>
</tr>
</tbody>
</table>

\(^a\)Half power beam width
\(^b\)Main beam efficiency
\(^c\)Forward efficiency
\(^d\)Number of map pixels (i.e. observed positions)
\(^e\)Mean atmospheric zenith opacity
\(^f\)Mean rms noise per channel

8.1.3 Results

Here we are concerned with the density-structure of the two clouds. Thus we apply the \(\Delta\)-variance analysis method to the observed velocity-integrated intensity maps. The maps and the corresponding \(\Delta\)-variance analysis of L1512 and L1524 (Haro 6-10), respectively are shown in Fig. 8.1 and Fig. 8.2, respectively.

The figures clearly show that the \(\Delta\)-variance is a convenient tool to separate white noise from the underlying structure. Indeed, at larger scales the underlying structure dominates the intensity fluctuations of the image and the \(\Delta\)-variance follow a power-law power-spectrum \(\sigma_{\Delta}^2(L) \propto L^{d_{\Delta}}\). However, towards smaller lags there is a turnover with a minimum near the telescope beam size, meaning that on these scales white noise starts to dominate the image structure. At smallest scales the index asymptotically approaches \(d_{\Delta} = -2\). Thus the power spectral index approaches \(\beta = d_{\Delta} + 2 = 0\), which is the power spectral index of purely white noise.

Since the correction for the beam smearing at small scales is only approximative the \(\chi^2\)-fit is only done for scales larger than the half power beam width (see Eq. 7.13). Also, largest scales are not considered for the fit, if the \(\Delta\)-variance shows signs of a turnover due to edge effects, such as it is the case for the largest lag in the analysis of L1512. Thus the \(\chi^2\)-fit that yields the power spectral index \(\beta\) is typically done for intermediate lags.

Here the errors of the power spectral indices \(\beta\) are large (\(\approx 35\%\)) since the maps of L1512 and L1524 (Haro 6-10) have very limited fit-intervals, namely 0.3 dex and 0.4 dex, so that it is not possible to draw final conclusions about the structure of the observed molecular regions. Nevertheless, the results seem to confirm what has found with other analysis tools, namely, that the structure parameters of molecular regions are similar in quiescent and star-forming regions (Kramer et al. 1998b; Heithausen et al. 1998). This may suggest that the last steps of gas compression leading to star-formation are local processes that are not preceded by a global structure change of the cloud. Furthermore, the found power spectral indices are consistent with published indices being in the range \(\sim 2.5 - 3.0\) for scales of the order of \(\sim 0.1\) pc (Bensch 1999; Bensch et al. 2001).

However, in order to obtain more conclusive results the \(\Delta\)-variance has to be applied to larger data-sets. That is, the here presented maps have to be extended and the study should be complemented with observations on other scales. For L1512, IRAM\(^*\) observations were

\(^*\)IRAM: Institut de RadioAstronomie Millimétrique
already carried out resolving scales of the order of $\sim 0.01$ pc. The $\Delta$-variance analysis yields for these scales power spectral indices of $\beta \sim 3$ (Bensch 1999). Thus the extension of the maps and the complementation of the study with high resolution observations of L1521 (Haro 6-10) would be an interesting future project that may help to better understand the transition from a pre-star-forming cloud to a star-forming cloud.

### 8.2 The Outer Galaxy

In the previous section the $\Delta$-variance was applied to nearby molecular clouds. Together with published data, the structure parameters derived with the $\Delta$-variance cover a scale range of the order of $\sim 0.01 - 1$ pc. Here we apply the $\Delta$-variance to a more extended data-set (various molecular complexes with analytical ranges of 1.2-2.2 dex) and analyze the structures for larger scales ($\sim 0.1 - 10$ pc). That is, the structure of molecular regions extracted from the FCRAO$^{\text{II}}$ CO Survey of the Outer Galaxy are studied (Heyer et al. 1998).

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$^{\text{II}}$FCRAO: Five College Radio Astronomy Observatory.
8.2. THE OUTER GALAXY

8.2.1 The Data-Set

The FCRAO CO survey of the Outer Galaxy is comprised of $1.7 \times 10^6$ $^{12}$CO ($J = 1-0$) spectra and covers the area between the Galactic longitudes, 102°.49 and 141°.54, and latitudes, −3°.03 and 5°.41. The local standard of rest velocity $V_{\text{LSR}}$ ranges from −153 to +40 km s$^{-1}$ and thus includes the Local arm ($-20 < V_{\text{LSR}} < +4$ km s$^{-1}$) and the Perseus arm ($-61 < V_{\text{LSR}} < -33$ km s$^{-1}$). The half-power beamwidth at 115 GHz is 45" and the velocity resolution is 0.98 km s$^{-1}$. The survey provides an unbiased data-set that has, with 2.8 dex and 3.5 dex in galactic longitude and latitude, respectively, one of the highest spatial dynamical range of the molecular interstellar medium ever observed, thus enabling to study scaling relations from the spiral arm scale down to sub-parsec scales.

Brunt & Heyer (2002) extracted from the Outer Galaxy Survey an ensemble of molecular regions with a broad range of cloud morphologies in order to study their velocity structure. Here the $\Delta$-variance analysis is carried out for the same molecular fields.

The ensemble consists of 23 individual fields. Nine of the fields are within the Perseus spiral arm. The remaining fields have centroid velocities with respect to the local standard of rest greater than $-20$ km s$^{-1}$ and are collectively referred to as "local" fields. The ensemble contains several giant molecular clouds associated with well-known HII regions and OB associations, namely, W5, W3, Cas A, NGC 7538, Sh 156, Sh 152 in the Perseus arm and Cep OB3, Sh 140 in the Local arm. Also, some of the Local arm fields are associated with optical HII regions that have not been studied in detail so far.

The fields cover isolated or "related" regions of molecular emission and were selected...
visually from the inspection of spatially restricted images integrated over limited velocity ranges.

As a consequence of the visual selection, some fields 1) are truncated at the boundaries or 2) may contain clouds at different distances that share a common velocity channel due to velocity fluctuations. However, since the Δ-variance analysis is carried out in position space, in general the effect of truncating a region does not cause a significant problem and since we study velocity integrated images with thick integration intervals (see chapter 7), the second effect does not influence the analysis of the density structure as well.

In order to extend the dynamical range and to include larger scales, complementary to the individual fields, 7 composite fields that comprise several of the individual fields and 8 large scale fields are analyzed. The large scale fields are integrated over the entire velocity range of the Local and the Perseus arm, respectively, and extend over the entire latitudinal range of the survey.

For convenience, the individual and composite fields are referred to as follows: P1-P9 for individual fields within the Perseus arm, L1-L14 for individual fields within the Local arm and C1-C7 for composite fields. The large scale fields in the Perseus arm are called Perseus left, Perseus middle, Perseus right and Perseus total and those in the Local arm are referred to as Local left, Local middle, Local right and Local total, respectively. The fields Perseus total and Local total cover the entire Perseus arm and Local arm, respectively.

The spectroscopic and angular limits \((v_{\text{min}} : v_{\text{max}}, \ l_{\text{max}} : l_{\text{min}}, \ b_{\text{min}} : b_{\text{max}})\) as well as the number of spatial pixels of the fields, analyzed in this study are listed in Appendix B in Tab. B.1 (individual and composite fields) and in Tab. B.3 (large scale fields). An overview of the location of the selected fields is given subsequently.

Kinematic distances, allowing to derive absolute spatial scales, were derived from the velocity and position centroids of each field by assuming a flat rotation curve with \(V_0 = 220 \ \text{km s}^{-1}\) and \(R_0 = 8.5 \ \text{kpc}\), where \(R_0\) is the solar Galactocentric distance and \(V_0 = V(R_0)\) is the rotation velocity.

### 8.2.2 Results

#### Individual and Composite Fields

The locations of the individual and composite fields in velocity and position space are shown in Fig. 8.3.

In order to investigate the density structure the Δ-variance analysis is applied to the velocity integrated spectral line images of the selected fields.

The Δ-variance \(\sigma_\Delta^2(L)\) and the power spectral indices \(\beta\) resulting from the structure analysis of the fields located in the Perseus arm and the Local arm are shown in Fig. 8.4 and Fig. 8.5, respectively.

The analytical range of the individual and composite fields is \(\sim 1.5 \ \text{dex}\) and \(\sim 2.0 \ \text{dex}\), respectively. However, as previously the \(\chi^2\)-fit, from which the power spectral index \(\beta\) is derived, is only done for intermediate lags, since the fit-interval is limited at small scales by white noise and/or beam smoothing effects and at large scales by a turnover of the scale-free order to large-scale systematic structure.

Here the size of the fit-interval varies between 0.37 dex and 1.41 dex. Fit-intervals that are clearly smaller than the available analytical range appear particularly in the Perseus arm. The mean fit-interval of the individual fields located in the Perseus arm and the Local arm
Figure 8.3: The boxes show the individual fields in the Perseus arm (P1-P9) and the Local arm (L1-L14), respectively, as well as the composite fields (C1 C7) that are analyzed in this study. **Top:** Longitude $v_{LSR}$ image of the mean antenna temperature over all latitudes. **Middle:** Image of $^{12}$CO ($J = 1-0$) integrated intensity over the Perseus arm velocity range. **Bottom:** Image of $^{12}$CO ($J = 1-0$) integrated intensity over the Local arm velocity range.
Figure 8.4: $\Delta$-variance $\sigma^2_\Delta(L)$ of the velocity integrated individual (P1-P9) and composite (C1-C2) fields located in the Perseus arm. The solid line depicts the $\chi^2$-fit to the data using the $\Delta$-variance model of Eq. 7.13. The dashed line shows the $\Delta$-variance corrected for white noise and beam smoothing. The power spectral index $\beta$ derived from the fit is indicated in each panel.
Figure 8.5: The same as Fig. 8.4 for the individual (L1-L14) and composite (C3-C7) fields located in the Local arm. The figure shows clearly that the $\Delta$-variance follows a power-law in all fields over a scale range lying between 0.8 dex and 1.4 dex.
are 0.6 dex and 1.0 dex, respectively. The mean fit-interval of the composite fields is 1.0 dex as well. The smallest and the largest scales (of all here considered fields), showing signs of a scale-free order are \( \sim 0.4 \) pc and \( \sim 60 \) pc, respectively. The fit-intervals and the power spectral indices \( \beta \) are summarized in Appendix B in Tab. B.2.

There is no correlation between the size of the fit-interval and the power spectral index \( \beta \). The indices vary between 2.09 and 3.24, meaning that molecular cloud complexes with rather fragmented structures \( (\beta \sim 2) \) as well as rather smooth structures \( (\beta \sim 3) \) are present in the Outer Galaxy Survey.

In order to check if there is a relation between the different structure morphologies and environments present in the survey, a statistical evaluation of the the power spectral indices is carried out. That is, the distribution of the indices is studied for fields in the Local arm and the Perseus arm, for fields thought to be related with star-formation and for all other fields.

Here fields thought to be related with star-formation are those that are associated with identified HII regions and OB associations, which for convenience hereafter are referred to as star-formation fields, while the rest of the fields are referred to as other fields. Since a detection of low mass stars in the more distant clouds is not possible, here star-formation rather refers to medium and/or high mass star formation.

The histograms showing the distribution of \( \beta \) in the different environments are presented in Fig. 8.6. The field P2 is not taken into account in the histograms due to its large error.

The histograms show that the mean spectral index \( \langle \beta \rangle \) of the star-forming fields and the other fields are almost identical \( (\beta \approx 2.6) \), which substantiates once more that the structure properties, as described by the \( \Delta \)-variance of \(^{12}\text{CO} \) maps, are on average identical in star-forming and quiescent clouds.

Furthermore, the mean power spectral index in the Perseus arm \( (\beta = 2.46) \) is significantly smaller than that in the Local arm \( (\beta = 2.69) \). Indeed, the statistical significance level, determined with the student’s t-test (Press et al. 1986), for the hypothesis that the two observed distributions do have different means is \( p = 0.95 \). A value close to one \( (p \gtrsim 0.95) \) implies a high confidence level in the hypothesis meaning that the observed difference is very significant.

Since the mean distance from the Galactic center to the selected fields in the Perseus arm is 2.6 kpc greater than to the local fields, this may suggest that the structure of the interstellar medium is more fragmented for regions farther away from the galaxy center.

We will refer to the dependence of the power spectral index on the distance from the galaxy center in Sect. 8.3, where we analyze the structure of dust fields in the galaxy M51 and in Sect. 8.4, where the here determined structure parameters are supplemented with published data and summarized.

**Large Scale Fields**

The locations of the large scale fields in velocity and position space are shown in Fig. 8.7. The spectroscopic and spatial boundaries are indicated in Appendix B in Tab. B.3.

The \( \Delta \)-variance \( \sigma_{\Delta}^2(L) \) and the spectral indices resulting from the structure analysis of the large scale fields are shown in Fig. 8.8. The power spectral indices \( \beta \) and the fit-intervals are summarized in Appendix B in Tab. B.4.

Since the large scale fields are too extended to identify them with a single distance, so that absolute spatial scales cannot be derived, the lags are here indicated in units of arcmin.
Figure 8.6: Normalized distribution of the power spectral indices $\beta$ resulting from the analysis of the selected individual fields in the outer-galaxy survey. In each panel the mean power spectral index $\langle \beta \rangle$ and the standard deviations are indicated. Also indicated are the size of the data-set and the mean fit-interval. $N(\beta)$ is the number of indices within the interval $[\beta - \Delta \beta/2, \beta + \Delta \beta/2]$, where the binning is $\Delta \beta = 0.1$. Top: All Fields in the Perseus arm, the Local arm and those of the entire survey, respectively. Middle: Fields with identified HII regions and OB associations in the Perseus arm, in the Local arm and of the entire survey, respectively. Bottom: Fields without identified HII regions and/or OB associations.
Figure 8.7: The boxes show 6 of the 8 large scale fields (Perseus left, Perseus middle, Perseus right, Local left, Local middle and Local right) analyzed in this study. The remaining two large scale fields (Perseus total and Local total) are composite fields that comprise the 3 large scale fields of the Perseus arm and Local arm, respectively. That is, the fields, Perseus total and Local total, cover (spectroscopically and spatially) the entire Perseus arm and Local arm, respectively. Top: Longitude-$\nu_{LSR}$ image of the mean antenna temperature over all latitudes. Middle: Image of $^{12}$CO ($J = 1 - 0$) integrated intensity over the Perseus arm velocity range. Bottom: Image of $^{12}$CO ($J = 1 - 0$) integrated intensity over the Local arm velocity range.
The analytical range of the large scale fields is $\sim 2.3$ dex. Besides the left and the middle field in the Perseus arm, that do not show extended molecular emission, the fit-interval is in all fields about 1 dex. The smallest and the largest scales, found in the selected large scale fields, showing a scale-free order are $\sim 2$ arcmin and $\sim 40$ arcmin, respectively.

The varying structure morphologies of the individual sub-structures are averaged out in the large scale fields and, besides the power spectral index of the field with the weakest molecular emission (Perseus middle), the indices of the individual large scale fields cover a narrow range between $\beta \sim 2.3$ and $\beta \sim 2.6$.

Again we find that the indices are smaller for the fields in the Perseus arm. An exception is the left field in the Perseus arm. Yet, for this field the error of the index is large due to the limited fit-interval. However, in order to provide clear evidence for the dependence of $\beta$ on the distance from the galaxy center, a larger data-set covering different distances from the galaxy center have to be analyzed. Among other things this problem is addressed in the next section, where the structure of dust fields in M51 is analyzed.

8.3 M51

In the previous section the $\Delta$-variance was applied to molecular regions in the Outer-Galaxy. The resulting structure parameters cover a scale range of the order of $\sim 0.1 - 10$ pc. Here, we analyze the structure of dust regions in the galaxy M51. That is, the $\Delta$-variance analysis is applied to $V-H$ (optical - near infrared) color images of M51. The data-set allows to determine the structure properties of the ISM for scales of the order of $\sim 10 - 100$ pc and to check if the power spectral index varies with the distance from the galaxy center.
8.3.1 The Data-Set

With a distance of 9.6 Mpc, M51 (Whirlpool galaxy, NGC 5194, Hubble type: Sc) is the nearest grand design spiral galaxy. M51 and its companion NGC 5195 build an interacting system and as a consequence of this interaction star-formation in M51 is very pronounced, mainly along the spiral arms.

M51 was observed with the Hubble Space Telescope (HST) at optical wavelength with the Wide Field and Planetary Camera (WFPC2) using the F55W (V band, $\lambda_{\text{eff}} = 0.5151 \mu m$) filter and in the near IR with Camera 3 of the Near Infrared Camera Multi-Object Spectrograph (NICMOS) using the F160W (H band, $\lambda_{\text{eff}} = 1.1035 \mu m$) filter (Polletta 2001).

Here we want to investigate the dust structure of M51, that can be traced by the attenuation of light from background stars. However, since near IR images primarily trace the stellar distribution and optical wavelength images trace both the stellar component and the dust, the distribution of the dust is better traced by optical-IR colors than solely by the individual optical or near IR image (Martini & Pogge 1999; Polletta 2001).

Dust features are best traced with color images combining the shortest available optical wavelength with the longest available IR wavelength. In fact, the available HST image of M51 with the shortest optical and those with the longest IR wavelength is the $B$ and the $K$ band image, respectively. However, since the signal to noise ratio of the V and H band image is better (rms-noise = 0.23 mag) we analyze here the $V - H$ color image of M51.

The $V - H$ color image tracing the dust features has a size of 6.6 x 6.8 kpc (d=9.6 Mpc). The resolution is 9.4 pc (0.2”). Consequently, the dynamical range of the image is $\sim 2.8$ dex and the pixel number is $\sim 0.5 \times 10^6$.

Since we are not concerned with the grand design global structure of the spiral galaxy but with the possible scale-free order of the dust structure in sub-regions, we extract an ensemble of dust regions from the $V - H$ color image of M51.

The ensemble consists of 33 visually selected fields, 24 of them are located in spiral arm regions and the central area (s1-24), respectively, and the remaining 9 are located in interarm regions (i1-i9).

The angular limits $x_{\text{offset}}^{\text{min}} : x_{\text{offset}}^{\text{max}}$, $y_{\text{offset}}^{\text{min}} : y_{\text{offset}}^{\text{max}}$ as well as the number of spatial pixels of the individual fields, analyzed in this study are indicated in Appendix B in Tab. B.5. An overview of the selected fields is given in Fig. 8.9. The contours of the $^{12}\text{CO} (J = 1 - 0)$ emission overlaid on the $V - H$ color image show that besides the interarm field 18 all selected fields are associated with regions of detected CO emission. The $^{12}\text{CO}$ contour lines clarify the run of the spiral arms and provide indications for regions of “related” molecular emission. They were used for the selection of the individual dust fields.

The $^{12}\text{CO} (J = 1 - 0)$ emission was mapped with the Owens Valley millimeter array at 2.5” resolution (Aalto et al. 1999).

8.3.2 Results

The $\Delta$-Variance Analysis

The $\Delta$-variance $\sigma^2_\Delta (L)$ and the power spectral indices $\beta$ of the selected fields located in spiral arm regions and the central area (s1-s24) are shown in Fig. 8.10. In Fig. 8.11 the same is shown for the interarm fields (i1-i9).

Since the correction for the PSF-smoothing is only valid for $2 < \beta < 4$ (see Eq. 7.13), for values $\beta < 2$ only a liner fit to the data is done.
Figure 8.9: $^{12}$CO ($J = 1 - 0$) column density contours superimposed on the $V - H$ color image of M51. The lowest CO contour level and the contour steps are 3$\sigma$. The 3$\sigma$ level of the $V - H$ image is 0.7 mag. The boxes depict the selected individual fields located in spiral arm regions and the central area (s1-s24) as well as the fields located in interarm regions (i1-i9). In the $V - H$ image regions with strongest absorption, i.e., regions with highest dust column density are red. Since the resolution of the CO map ($2.5''$) is a factor 13 worse than that of the optical image ($0.2''$) only the structure of the fields extracted from the $V - H$ color image are analyzed.
Figure 8.10: $\Delta$-variance $\sigma_3^2(L)$ of the velocity integrated individual fields located in M51 in spiral arm regions and the central region (s1-s24), respectively. For $2 < \beta < 4$, the solid line depicts the $\chi^2$-fit to the data using the $\Delta$-variance model of Eq. 7.13 and the dashed line shows the $\Delta$-variance corrected for white noise and PSF-smoothing. For $\beta < 2$ the solid line depicts the linear $\chi^2$-fit to the data. The power spectral indices $\beta$ derived from the fits are indicated in each panel.
Figure 8.11: The same as Fig. 8.10 for the fields located in interarm regions of M51 (i1-i9).
The mean analytical range and the mean fit-interval of the selected fields is \( \sim 1.3 \) dex and \( \sim 0.6 \) dex, respectively. The largest fit-interval is found in the field s9. For this field the fit-interval substantiates a power-law power-spectrum over 1 dex from \( \sim 30 \) pc up to \( \sim 300 \) pc.

Below scales of the order of \( \sim 30 \) pc the structure is distorted by the telescope point-spread function (PSF) and at larger scales of the order of 100-300 pc the structure is dominated by large scale systematic structure such as spiral arms.

Indeed, at small scales the \( \Delta \)-variance \( \sigma_{\Delta}^2 \) plunges down. This indicates the presence of smooth small-scale structures in the image and is typical for the PSF-smoothing.

In order to illustrate the influence of the systematic spiral arm structure at large scales, let us consider the field s20. This field extends over one of the two main spiral arms and the analysis of the field shows a rise of \( \sigma_{\Delta}^2 \) above 100 pc. If the length of the field is reduced so that the entire field is located inside the arm, i.e., so that it does not cover the "border" of the arm, the increase of \( \sigma_{\Delta}^2 \) for lags larger than 100 pc disappears.

On the other hand, the analysis of field s17 shows a decrease of \( \sigma_{\Delta}^2 \) at large scales. Yet, the drop of \( \sigma_{\Delta}^2 \) disappears when the field is extended towards the left beyond the "border" of the spiral arm at \( x_{\text{offset}} = -45 \) arcsec.

This shows how the systematic spiral arm structure affects the structure analysis at large scales. However, the \( \Delta \)-variance \( \sigma_{\Delta}^2 \) at intermediate scales is independent of the image size\(^{**}\). That is, the power spectral index \( \beta \) derived from the fit-interval at intermediate scales, is not influenced by systematic large scale trends, and is thus a reliable measure of the fragmentation strength of the underlying structure.

The power spectral indices and the fit-intervals of the individual fields in M51 are summarized in Appendix B in Tab. B.6.

\( \beta \) in Spiral Arm and Interarm Fields

In order to compare the structure of fields located in spiral arm regions with those of interarm regions, a statistical evaluation is carried out.

That is, the distributions of the power spectral indices derived from spiral arm and interarm fields, respectively, are compared. However, spiral arm fields are on average less far away from the galaxy center \( R_{\text{G C}} \) than interarm fields. In order to compare the structure properties of spiral arm fields and interarm fields independent of the distance from the galaxy center the innermost, but also the outermost regions of M51, where no interarm fields are present, are excluded from the statistical evaluation. That is, only fields within \( 1.6 \leq R_{\text{G C}} \leq 3.5 \) are taken into account. The distributions of the power spectral indices \( \beta \) are shown in Fig. 8.12.

The mean power spectral index derived from the fields located in spiral arm and interarm regions is \( \langle \beta \rangle = 2.09 \) and \( \langle \beta \rangle = 1.69 \), respectively. The significance level, determined with the student's t-test, for the hypothesis that the two means are different is \( p = 0.97 \).

These results substantiate that the structure in interarm regions is more lumpy and fragmented than in spiral arm regions.

\(^{**}\)The \( \Delta \)-variance at intermediate scales is independent of the image size as long as the pixel number is sufficient \( ( \geq 2000 ) \), which is here the case for all selected fields.
Figure 8.12: Normalized distribution of the power spectral indices $\beta$ resulting from the analysis of dust fields located within $1.6 \leq R_{GC} \leq 3.5$ kpc, where $R_{GC}$ is the distance from the galaxy center of M51. In each panel the mean power spectral index $\langle \beta \rangle$ and the standard deviations are indicated. Also indicated are the size of the data-set and the mean fit-interval. Left panel: The fields located in interarm regions (i1-i9). Right panel: Fields located in spiral arm regions (s1-s5, s8, s17 and s19-s24).

Relation Between $\beta$ and the Distance From the Galaxy Center

Here, the dependence of the power spectral index on the distance from the galaxy center is studied. For this purpose, only the fields located in spiral arm regions and the central area (s1-s24) are considered. Interarm fields are not considered since they are biased. Indeed, they are mainly located in the outer regions and have systematically small indices as we have shown above.

The plot in the left panel of Fig. 8.13 shows the index $\beta$ of the fields s1-s24 as a function of the distance $R_{GC}$ from the galaxy center (GC). The plot shows that the index $\beta$ decreases with the distance from the GC. A linear $\chi^2$-fit to the data yields a slope of $\alpha = 0.12 \pm 0.05$.

The largest selected fields are rather located in the inner regions of M51. Since the dynamical range of larger fields extends to larger scales, the observed decrease of $\beta$ with increasing distance from the GC may be the result of scaling effects. In order to check this possibility, $\beta(R_{GC})$ is shown in the right panel of Fig. 8.13 only for fields with fit-intervals covering similar scales, namely, 28 – 65 pc and 28 – 93 pc, respectively.

The slope $\alpha = 0.15 \pm 0.11$ derived from this sub-ensemble of fields is consistent with the slope derived from the entire ensemble (s1-s24), but the error is larger since there are fewer data points.

However, the two plots in Fig. 8.13 suggest that the dust distribution in M51 on scales of the order of $\sim 10 – 100$ pc is more fragmented in the outer regions of the disk than in the inner regions.

$\beta$ in the Outer Galaxy and in M51

The mean power spectral index and the mean fit-interval derived from 22 fields\(^{\dagger}\) of the Outer Galaxy survey is $\langle \beta \rangle = 2.61 \pm 0.28$ and 2-18 pc, respectively.

\(^{\dagger}\)Field P2 is not considered due to its large error.
Figure 8.13: The power spectral index $\beta$ as a function of the distance from the galaxy center $R_{GC}$. **Left panel:** The indices and the $\chi^2$-fit (solid line) to the data derived from all fields located in spiral arm regions and the central area (s1-s24), respectively. Above the panel the slope of the fitted straight line $\alpha$ and its error are indicated. **Right panel:** The same as in the left panel for a sub-ensemble of fields that have similar fit-intervals, namely 28–65 pc (s7, s14 and s17-s19) and 28–93 pc (s3-s6, s8, s15 and s20), respectively.

The mean index resulting from the analysis of the 24 fields located in spiral arm regions of M51 and its central area is $\langle \beta \rangle = 2.17 \pm 0.20$. The mean fit-interval is 28-128 pc.

The probability, calculated with the student’s t-test, that the means of the two observed distributions are significantly different is one.

This suggests that on average the structure is not described by an universal power-law power-spectrum, but that their is a tendency towards a flatter power-spectrum and consequently towards more fragmented structures at larger scales.

### 8.3.3 Discussion

**Does the $V - H$ Color Image Represent the Structure of the Cold ISM?**

Assuming that the type of dust is on average everywhere the same and that the cold gas and dust component are well mixed the following relation is valid (Binney & Merrifield 1998),

$$
\Sigma_H \propto E(V - H) = (V - H) - (V - H)_{int},
$$

where $\Sigma_H$ is the gas surface density and $E(V - H)$ is the color excess (reddening). $(V - H)$ and $(V - H)_{int}$ are the observed and the intrinsic colors, respectively.

The smallest and the largest scales in M51, involved in the determination of the power spectral index $\beta$, are 25 pc and 300 pc, respectively. Thus, if the averaged $(V - H)_{int}$ is constant over these scales, the power spectral index determined with the $\Delta$-variance correctly describes the underlying gas-density structure.

If the averaged $(V - H)_{int}$ is not constant over these scales we expect that the regions with different stellar populations introduce systematic structures in the field, such as at the edge of star-forming clouds or spiral arms. These systematic structures terminate the scale-free order in the image and introduce thus a turnover in the $\Delta$-variance. Thus scales at which the image is affected by a varying $(V - H)_{int}$ can be recognized in the analysis and are not taken into account in the description of the structure properties.
Is the Image Affected by Optical Depth Effects?

(V-H) color images are distorted by optical depth effects as soon as all the light in the shorter waveband is absorbed.

Subsequently the magnitude \((V - H)_{99}\) corresponding to a dust density that absorbs 99% of the light in the \(V\) band is calculated. Then, we determine the number of pixels in the \((V - H)\) color image of M51 with a magnitude larger than \((V - H)_{99}\) in order to check if the structure analysis is distorted by optical depth effects.

If 99% of the light in the \(V\) band is absorbed the following relation is valid,

\[
F_{\text{obs}} = 0.01F_{\text{em}} = 10^{-A_V/2.5}F_{\text{em}},
\]

where \(F_{\text{obs}}\) and \(F_{\text{em}}\) are the observed and emitted flux, respectively, and \(A_V\) is the visual extinction.

The visual extinction and the color excess are related via \(A_V = 1.2E(V - H)\) (Martini & Pogge 1999; Mathis 1990). Then, by assuming an extreme value \((V - H)_{\text{int}} = -0.8\), which is the color magnitude for a main sequence star of spectral type O, one finds \((V - H)_{99} = 3.4\). In the \((V - H)\) color image 1% of the pixels have a magnitude larger than 3.4, which is an upper limit for the number of pixels that are affected by optical depth effects. If one takes \((V - H)_{\text{int}} = 1.6\), which corresponds to a solar type main sequence star, one finds that only a single image pixels of the \(\approx 0.5 \times 10^6\) pixels may be affected by optical depth effects.

Since the few individual pixels with magnitudes larger than \((V - H)_{99} = 3.4\) are widely scattered optical depth effects are in general only point effects meaning that the structure analysis is not distorted by these effects.

Global Density and Fragmentation Strength

Toskai et al. (2002) determined for spiral arm and interarm regions in M51 the ratio \(R_{12/13} = ^{12}\text{CO}/^{13}\text{CO} (J = 1 - 0)\) that can be used as an indicator of the gas density.

They found that at scales of the order of \(\sim 100\) pc the gas density in the spiral arms is at least a factor 3 greater than in the interarm regions.

We found that the mean power spectral indices in spiral arm regions is \(\langle \beta \rangle > 2\), whereas those in interarm regions is \(\langle \beta \rangle < 2\).

This suggests that the passage of the spiral density wave not only increases the gas and dust density but also reorganizes the ISM-structures so that they become smoother, i.e., less fragmented.

The increase of the gas and dust density together with the “smoothing” of the ISM-structure may then change the properties of molecular clouds in a way that they are potentially able to form stars.

However, since inside the spiral arm the structure properties of the ISM are independent of the star-formation strength, smooth structures \(\beta \gtrsim 2\) and spiral arm densities may be a condition that actually favors star-formation, but is not sufficient to trigger it alone. This suggests that global as well as local properties have to be fulfilled, so that star-formation can finally occur.

Fractal Dimension

In chapter 7 the relation \(D_{\text{MS}} \approx \beta\) for \(1.7 < D < 2.3\), where \(D_{\text{MS}}\) is the fractal dimension as derived from the mass-size relation, was introduced.
Thus, if the dust structure in interarm regions of M51 is described by a fractal, its dimension is smaller than 2, since we found for these regions \( \beta < 2 \). Furthermore, on the same condition the fractal dimension may fall below 2 for \( R_{GC} \gtrsim 3 \) kpc.

### 8.4 Completion with Other Surveys

Among other things the investigations carried out so far suggest that the power spectral index decreases (i.e. deviation from homogeneity and fragmentation increases) with 1) increasing scales and 2) increasing distance from the galaxy center.

In order to check these findings the above presented data-sets are supplemented with further structure parameters. These structure parameters were determined elsewhere and are partly published and partly not yet published.

#### 8.4.1 Supplementing Data-Sets

Here we briefly present the supplementing data-sets:

- **The IRAM key-project data-set:** This data-set comprises the published power spectral indices derived from the high and low resolution observations of 3 nearby \( (d \lesssim 150 \) pc), quiescent molecular clouds carried out with different telescopes (IRAM 30m, FCRAO 14m, KOSMA 3m, CFA* 1.2m) in the framework of the IRAM key-project. The three molecular clouds are observed in different rotational transitions of \(^{12}\)CO and \(^{13}\)CO. Thus the mean power spectral index of the different tracers is compared with the structure parameters of the other observations. The mean power spectral index of all 3 clouds and all tracers is \( \beta = 3.0 \pm 0.3 \) and the mean fit-interval is 0.1 – 0.7 pc. Details of the individual clouds and the power spectral indices are published in Bensch (1999) and Bensch et al. (2001), and summarized in Appendix B in Tab. B.7.

- **The BU/FCRAO Galactic Ring data-set:** The data-set comprises the unpublished structure parameters of 16 fields extracted from the BU\(^{\dagger}\)/FCRAO Galactic Ring survey\(^\dagger\) (Simon et al. 2001). The here considered map of the BU/FCRAO survey of the Inner Galaxy and the 5 kpc Galactic Ring consists of 450'000 individual \(^{13}\)CO \( (J = 1 - 0) \) spectra and covers an area in the first galactic quadrant between the Galactic longitudes, -1° and 0.5°, and latitudes, -50° and 40°. The \( V_{LSR} \) ranges from -5 to 80 km s\(^{-1}\). The mean power spectral index and the mean fit-interval of the velocity integrated, selected fields is \( \beta = 2.6 \pm 0.3 \) and 3.8 – 16.9 pc, respectively. The individual indices are summarized in Appendix B in Tab. B.8.

- **The Bell Labs data-set:** The data-set comprises the published power spectral indices of 4 molecular clouds (Perseus/NGC 1333, Orion A, Orion B, Mon OB1/NGC 224 and Mon R2) with solar distances between \( \sim 0.5 \) and \( \sim 1 \) kpc. The indices result from the analysis of velocity integrated \(^{13}\)CO \( (J = 1 - 0) \) maps observed with the 7m AT&T Bell Laboratories mm-wave telescope. The mean power spectral index of the 4 clouds is \( \beta = 2.7 \pm 0.2 \) and the mean fit-interval is 0.5 – 5.2 pc. The individual indices are

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*CFA: Harvard-Smithsonian Center for Astrophysics

\(^\dagger\)BU: Boston University

\(^\dagger\)http://www.bu.edu/grs
8.4. COMPLETION WITH OTHER SURVEYS

published in Bensch (1999) and Bensch et al. (2001), and summarized in Appendix B in Tab. B.9.

8.4.2 Results

First, the dependence of $\beta$ on the scale and the distance from the galaxy center (GC) is studied in the Milky Way. For this purpose the Outer Galaxy survey and the observation of L1512 and L1524 are supplemented with the above presented data-sets.

The power spectral indices of the combined data-set as a function of the scale $\langle S \rangle$ and the distance from the galaxy center $R_{GC}$ are shown in Fig. 8.14. Here the scale is $\langle S \rangle = (i_l + i_u)/2$, where $i_l$ and $i_u$ are the lower and the upper limit of the fit interval, respectively.

Despite the large data-set the covering of the $\langle S \rangle - R_{GC}$-plane is incomplete. Nevertheless, Fig. 8.14 confirms the trend that the degree of fragmentation and inhomogeneity increases with increasing scale and increasing distance from the galaxy center. Indeed, the power spectral index is mainly increased (red) for small scales and small $R_{GC}$, while small indices (blue) appear at larger scales and farther away from the GC. However the figure shows also that there are local fluctuations which may distort the general trend. An example for that is the field P7 located in the Perseus arm, whose index is increased $\beta = 2.82 \pm 0.09$ given the “large” scale of the fit-interval ($\langle S \rangle \approx 20$ pc) and the “large” distance from the galaxy center ($R_{GC} \approx 11$ kpc).

In order to extend the scale range and to point out the dependence of $\beta$ on the scale, the above considered data-set, describing the structure properties of the cold ISM in the Milky Way, is supplemented with the power spectral indices resulting from the analysis of spiral arm fields in M51.

The power spectral indices $\beta$ derived from molecular and dust regions in the Milky Way and M51, as a function of the scale are shown in Fig. 8.15.

The data suggest that for scales smaller than $\mathcal{O}(1)$ pc the structure can be described on average by an uniform power-law power-spectrum with a power spectral index $\langle \beta \rangle \approx 3.0$. For larger scales the index decreases according to $\langle \beta \rangle \sim \langle S \rangle^\alpha$, where a linear $\chi^2$-fit to the data yields $\alpha = -0.07 \pm 0.01$, so that on scales of the order of $\sim 100$ pc the index $\beta$ may fall below two.

However these results are only approximative since the data-set involves different galaxies with different sizes and fields with different distances from the galaxy center.

Indeed at small scales ($\lesssim 1$ pc) the data represent the structure properties of the ISM in the Milky Way for $8 \lesssim R_{GC} \lesssim 9$ kpc. At scales of the order of $\sim 10$ pc the data approximate the mean Galactic structure properties in the range $5 \lesssim R_{GC} \lesssim 12$ kpc. Finally, the M51 data-set describes on average the structures at scales of $\mathcal{O}(100)$ pc for $0 \lesssim R_{GC} \lesssim 3.5$ kpc.

Although certain scales may be biased, often the effect due to the different $R_{GC}$ is averaged out, suggesting that the observed trend ($\langle \beta \rangle \sim \langle S \rangle^\alpha$) is real.

In Fig. 8.15 the dispersion of $\beta$ at a certain scale is partly due to different distances from the GC and partly due to fluctuations that are independent of $R_{GC}$ and also independent of the star-formation rate in the cloud.

The molecular cloud MCLD 123.5+24.9 located in the Polaris Flare was observed with different telescopes (IRAM 30m, FCRAO 14m, KOSMA 3m, CfA 1.2m) having different resolutions and consequently, covering different scale ranges. While the IRAM telescope has

$^5$At 115 GHz, i.e., for $^{12}\text{CO} (J = 1 - 0)$ the half power beam width of the IRAM, FCRAO and CfA telescope is $\theta \approx 0.35'$, 0.75' and 8.7', respectively.
Figure 8.14: The power spectral index $\beta$ as a function of the scale $\langle S \rangle$ and the distance from the Galactic center $R_{\text{GC}}$. The dots depict the scale and the distance $R_{\text{GC}}$ of the different Galactic molecular clouds presented in this study. The contour plot results from the power spectral indices of these clouds. The arrow depicts the field P7 that is located in the Perseus arm and whose $\beta$ is large given the location in the diagram.

- **Nearby molecular clouds** (Polaris Flare, L1512, L134A), IRAM key-project, IRAM 30m, $^{12}\text{CO}$ and $^{13}\text{CO}$.
- **Nearby molecular clouds** (Polaris Flare, L1512, L1524), KOSMA 3m, $^{12}\text{CO}$.
- **Nearby molecular cloud** (Polaris Flare), FCRAO 14m, $^{13}\text{CO}$.
- Molecular regions in the **Inner Galaxy** and the 5 kpc galactic ring. BU/FCRAO galactic ring survey, FCRAO 14m, $^{13}\text{CO}$.
- **Different clouds** with distances $d = 0.15 - 1$ kpc (Orion A, Orion B, NGC 1333, NGC 2264, Mon R2), AT&T Bell Labs survey, Bell Labs mm-wave telescope 7m, $^{13}\text{CO}$.
- **Local arm** molecular regions, FCRAO survey of the Outer Galaxy, FCRAO 14m, $^{12}\text{CO}$.
- **Nearby molecular cloud** (Polaris Flare), CfA 1.2 m, $^{12}\text{CO}$.
- **Perseus arm** molecular regions, FCRAO survey of the Outer Galaxy, FCRAO 14m, $^{12}\text{CO}$. 
Figure 8.15: The power spectral index $\beta$ as a function of the scale $\langle S \rangle$. The data-set comprises structure parameters of molecular and dust regions in the Milky Way and M51. The solid lines indicate the linear $\chi^2$-fits to the data. The slopes $\alpha$ of the fitted straight lines are indicated as well. The dashed lines depict the interval $2 < \beta < 3$ and the arrows mark the observation of the Polaris Flare with different resolutions, i.e., at different scales. For clarification these data-points are replotted in the lower-left corner of the figure.

- **Nearby molecular clouds** (Polaris Flare, L1512, L134A), IRAM key-project, IRAM 30m, $^{12}$CO and $^{13}$CO.
- **Nearby molecular clouds** (Polaris Flare, L1512, L1524), KOSMA 3m, $^{12}$CO.
- **Molecular regions in the Inner Galaxy** and the 5 kpc galactic ring, BU/FCRAO galactic ring survey, FCRAO (14 m), $^{13}$CO.
- **Different clouds** with distances $d = 0.15 - 1$ kpc (Orion A, Orion B, NGC 1333, NGC 2264, Mon R2), AT&T Bell Labs survey, Bell Labs mm-wave telescope 7m, $^{13}$CO.
- **Local arm** molecular regions, FCRAO survey of the Outer Galaxy, FCRAO 14m, $^{12}$CO.
- **Nearby molecular cloud** (Polaris Flare), CfA 1.2m, $^{12}$CO.
- **Perseus arm** molecular regions, FCRAO survey of the Outer Galaxy, FCRAO 14m, $^{12}$CO.
- Dust regions located in the spiral arms of M51, Optical and near infrared imaging of M51, HST 2.4m, V-H.
the highest resolution and maps the cloud at scales of the order of $\sim 0.01$ pc, the CfA has the lowest resolution and maps the entire Polaris Flare at scales of the order of $\sim 10$ pc.

The corresponding observations are plotted in Fig. 8.15 and marked with an arrow. For a better illustration they are replotted separately in the lower-left corner of the figure.

The structure analysis of MCLD 123.5+24.9 and the Polaris Flare further substantiate the previous results. Indeed, even if the structures inside the same cloud complex are considered we find that for scales smaller than $\sim 1$ pc the observed structures are consistent with an uniform power-law and that at larger scales the power spectral index $\beta$ decreases.

### 8.4.3 Discussion

**Are the Results Affected by Optical Depth Effects?**

In this study several low-$J$ $^{12}\text{CO}$ maps were analyzed. Since these are optically thick lines the question arises if the structure analysis carried out here is distorted by optical depth effects. This should not be the case since molecular clouds are macro-turbulent at the here considered scales. Let us discuss this in turn.

The observed line profiles of the optically thick low-$J$ $^{12}\text{CO}$ transitions are almost Gaussian, rather than flat topped as it would be expected due to optical depth effects.

This can be explained by the macro-turbulent nature of molecular clouds. Indeed, the intrinsic linewidth of the cloud clumps is much smaller than the interclump velocity dispersion (Stutzki 1993; Tauber et al. 1991). Thus, several clumps in the same line of sight are observed in different velocity channels. As a consequence, even an optically thick line basically counts the number of clumps along each line of sight and is consequently a measure for the amount of matter as long as the medium is macro-turbulent, which is at least the case down to scales of the order of $\sim 0.01$ pc (Fuller & Myers 1992).

This picture is confirmed by this study. To show this, let us compare the power spectral indices of maps observed in optically thick (low-$J$ $^{12}\text{CO}$) and optically thin (low-$J$ $^{13}\text{CO}$) lines.

That is, we plot the power spectral indices $\beta$ derived only from $^{13}\text{CO}$ maps 1) as a function of the scale $\langle S \rangle$ and the distance from the galaxy center $R_{GC}$ (see Fig. 8.16) and 2) only as a function of the scale $\langle S \rangle$ (see Fig. 8.17). Then, a comparison with the corresponding plots that also include the $^{12}\text{CO}$ data (cf. Fig. 8.14 and Fig. 8.15), shows that the power spectral indices derived from the observation of optically thin and optically thick lines are on average the same.

Since we do not have $^{13}\text{CO}$ data of the Outer Galaxy, the trend that the degree of fragmentation increases with the distance from the galaxy center is not clearly visible in Fig. 8.16. However, we assume that the trend seen in the $^{12}\text{CO}$ data-set (cf. Fig. 8.14) is not affected by optical depth effects since the densities in the $^{12}\text{CO}$ Outer Galaxy data-set are lower than in the IRAM data-set, where the analysis of optically thin and optically thick lines yields the same structure properties.

To sum up, the here presented findings are not the result of optical depth effects.

**Density and Fragmentation Strength**

In Sect. 8.3 a relation between the gas (resp. dust) density and the degree of fragmentation was found. That is, the ISM-structures are smoother in the dense spiral arms and more fragmented in the less dense interarm regions.
Figure 8.16: The power spectral index $\beta$ as a function of the scale $\langle S \rangle$ and the distance from the Galactic center $R_{GC}$. The indices $\beta$ are derived from maps observed in low-$J^{13}$CO transitions. These lines are for the here considered densities optically thin.

- **L1512**, IRAM 30m, mean of an observation in $^{13}$CO ($J = 1 - 0$) and $^{13}$CO ($J = 2 - 1$).
- **Polaris Flare**, FCRAO 14m, $^{13}$CO ($J = 1 - 0$).
- Molecular regions in the **Inner Galaxy** and the 5 kpc galactic ring, FCRAO 14m, $^{13}$CO ($J = 1 - 0$).
- **Different clouds** (Orion A, Orion B, NGC 1333, NGC 2264, Mon R2), Bell Labs 7m, $^{13}$CO ($J = 1 - 0$).

The analysis of the structure properties in the Milky Way and M51 shows that the degree of fragmentation increases with increasing scale and increasing distance from the galaxy center $R_{GC}$.

Both of these trends are consistent with the assumption that there is a relation between the mean density of the ISM and the degree of fragmentation. Let us discuss this in turn.

For larger distances from the galaxy center the spiral density waves become weaker. Thus farther away from the galaxy center where the structure is more fragmented the ISM is also more diffuse. That the ISM becomes more diffuse for larger $R_{GC}$ is supported by the observation of the ratio $R_{12/13} = ^{12}$CO/$^{13}$CO ($J = 1 - 0$) in the Milky Way and other galaxies (e.g. Toskai et al. 2002; Solomon et al. 1979).

For a hierarchical structure whose mass distribution can be approximated with a power-law mass-size relation $M \sim S^{D_{MS}}$, where $S$ is the scale, the density decreases with increasing scales $\rho \sim 1/S^{D_{MS}}$ if $D_{MS} < 3$. Thus, the decreasing degree of fragmentation with increasing scales may also be related to a decreasing density.

To sum up, the above considerations suggest that an increasing degree of fragmentation is related to a decreasing density of the ISM.
Figure 8.17: The power spectral index $\beta$ as a function of the scale $\langle S \rangle$. The indices $\beta$ are derived from maps observed in low-$J^{13}$CO transitions. Furthermore, the indices $\beta$ derived from the analysis of dust regions in M51 are shown. The solid lines indicate the linear $\chi^2$-fits to the molecular data of the Milky Way. The slope of the fitted straight lines $\alpha$ are indicated as well. The dash-dotted line depicts the fit to the molecular data of the Milky Way and the dust data of M51 for scales larger than 1 pc. The slope of the dash-dotted line is the same as those of the corresponding solid line. This shows that the conclusion that the structure is more fragmented for larger scales is independent of the dust data-set.

- **L1512**, IRAM 30m, mean of an observation in $^{13}$CO ($J = 1 - 0$) and $^{13}$CO ($J = 2 - 1$).
- **Polaris Flare**, FCRAO 14m, $^{13}$CO ($J = 1 - 0$).
- Molecular regions in the Inner Galaxy and the 5 kpc galactic ring, FCRAO 14m, $^{13}$CO ($J = 1 - 0$).
- **Different clouds** (Orion A, Orion B, NGC 1333, NGC 2264, Mon R2), Bell Labs 7m, $^{13}$CO ($J = 1 - 0$).
- Dust regions located in the spiral arms of M51.

### 8.5 Conclusions

We applied the $\Delta$-variance technique to determine the power spectral index $\beta$ of the interstellar medium at different scales and in different environments of the Milky Way and M51. The main conclusions are summarized as follows:

1. At scales $\lesssim 1$ pc the structure of the individual molecular regions may be described by a uniform power-law with $\beta \approx 3$ for local molecular clouds. This substantiates the scale-free hierarchical order of the ISM up to scales of the order of $\sim 1$ pc. At larger scales the power spectral index decreases on average. Thus at scales $\gtrsim 1$ pc the structure is no longer strictly scale-free over more than $\sim 1$ dex. However, the hierarchical order may continue, but with a smaller index $\beta$ indicating that the structure becomes more fragmented at larger scales.
2. Lower values of $\beta$ substantiating a higher degree of fragmentation and inhomogeneity are also found in interarm regions and in the outer regions of spiral disks. Typically, in inner disks up to spiral arm scales the power spectral index is in interarm regions $\lesssim 2$ while it is $\gtrsim 2$ in spiral arms. Interarm regions and outer disk regions are zones with globally less intense star-formation and lower gas (i.e. dust) densities. Thus a molecular region that has an intense global star-formation rate (compared to other regions within the galaxy) at the spiral arm scale may be related with a high density and a smooth structure of the ISM. However, since the structure within a spiral arm region is independent of the star-formation activity, smooth structures with $\beta \gtrsim 2$ and spiral arm densities may only be a condition that actually favors star-formation but is not sufficient to trigger it.

3. The power spectral indices of different molecular regions at a particular scale and at a particular distance from the galaxy center are not identical (within the error limits), but show a dispersion that cannot be accounted for by the different star-formation activities of the regions. Thus besides the scale and the distance from the galaxy center other factors that are not related with the star formation process influence the structure properties in the ISM.

4. The largest power spectral indices presented in this study describe the structures of nearby molecular clouds ($d \lesssim 150$ pc) on scales of the order of $\sim 0.01 - 1$ pc. The mean power spectral index for these clouds is $\beta = 2.98 \pm 0.39$. This is less than the values predicted by Kolmogorov type turbulence $\beta = 3.7$ and shock type turbulence $\beta = 4$. For larger scales and larger distances from the GC, the deviation from model predictions increases. This shows that hydrodynamic turbulence models typically produce less fragmented structures than they are observed in the ISM.

Acknowledgments:

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Summary of the Thesis (English Version)

In this thesis we are concerned with the structure formation in self-gravitating systems out of equilibrium. In particular our interest is focused on nonequilibrium structures in the cold interstellar medium.

In order to better understand the relevance of gravity for the formation of long-range correlations at galactic disc scales and at molecular cloud scales, numerical simulations of shearing discs and spherical $N$-body systems are carried out. The corresponding numerical models also allow us to study the effect of perturbations, such as different dissipation and forcing schemes, on the evolution of gravitationally unstable systems.

Besides numerical simulations, a further approach, helping us to better understand the shaping process at work in the ISM, is to carry out a detailed structure-analysis. Thus we determined and compared the power-spectrum of the ISM at different scales, in different galactic environments and even in different galaxies.

Subsequently, the different projects, findings and conclusions are summarized.

Project 1: Lumpy Structures in Self-Gravitating Disks

P1.1 Motivation

Observations of the interstellar medium, spiral disks and cosmic structures reveal similar characteristics. The structures in these systems are very lumpy and inhomogeneous. Moreover these structures do not seem to be comply random, but obey in part certain scaling-laws.

Local Models of self-gravitating disks show how such inhomogeneous structures can be maintained at the kpc-scale (e.g. Toomre & Kalnajs 1991; Toomre 1990). Their basic physical ingredients are self-gravity, dissipation and differential rotation.

In order to explore in greater detail the nonequilibrium structures, resulting from gravitational instabilities in a system maintained outside of equilibrium by differential rotation and dissipation, local simulations of self-gravitating disks are carried out in 2D and 3D. In particular we want to check if self-gravitation in combination with time-dependent bound-
Figure P1.1: The dotted frame represents a section of the disk, being infinite in two directions, seen from above. The two meshes are affine coordinate systems on which the mass distribution is periodic. The forces are computed in both meshes and then added with weighting factors. Such a procedure avoids discontinuities in the time evolution of the force field (see chapter 5). The dark and the light box represent the local computation box in affine coordinates. a: The initial state of the two meshes \( t = 0 \). b: The meshes at \( t = L_y/(2L_x\Omega_0) \). c: When the meshes reach these inclinations \( t = L_y/(L_x\Omega_0) \), they jump back to the positions shown in (a) and the process starts again without introducing discontinuities in the dynamics.

ary conditions and a slight dissipation can produce inhomogeneous, lumpy and eventually self-similar structures, resembling those observed in molecular clouds and flocculent spirals.

In order to increase the performance with respect to previous models in which the gravitational forces are calculated with direct-summation methods, a technique is developed that allows to use a particle-mesh FFT-method\(^\text{FFT}\) for the force calculation (Hockney & Eastwood 1981). That is, time-dependent affine coordinate systems are used with respect to which the system remains periodic at all times, so that the forces can be calculated in the Fourier space. The time-dependent affine coordinate systems are shown in Fig. P1.1.

**P1.2 Results and Conclusions**

**Persistent Patterns and Transient Structures**

The structures resulting from the local simulations of self-gravitating disks are homogeneous, wavy, filamentary or clumpy, depending on the relative strength of the competing gravitational and dissipative forces. The transition between the different states is smooth.

Since the disks are heated up due to gravitational instabilities leading to a conversion of

\(^\text{FFT}:\) Fast Fourier transform
the bulk kinetic energy (shear-flow) into random thermal motion, disk inhomogeneities cannot grow and tend to smear out, if energy dissipation is too weak. However, with an appropriate choice of the dissipation strength, persistent wavy or, in the case of stronger dissipation, filamentary patterns formed by transient structures appear (see Fig. P1.2).

As long as the structure is mainly wavy or filamentary, self-gravitation and dissipation ensure a statistical equilibrium, where repeated transient structures are formed. These structures, appearing on the kpc scale, are spiral arm like and thus on average non-isotropic.

When the dissipative factors begin to dominate the evolution process, clumps form inside the filaments (see Fig. P1.3). Typically, the clumps do not disperse, meaning that their growth is not limited in time, which leads then to gravitational runaways. Thus, clumpy structures do in general not evolve to a statistical equilibrium.

Kim & Ostriker (2001) carried out magneto-hydrodynamic simulations of gaseous shearing disks. The long-range correlations appearing in these simulations are qualitatively similar to those resulting from our N-body model, suggesting that the fundamental agents of structure formation at the kpc scale are gravity, differential rotation and dissipation, and that magnetic fields and gas pressure are second order effects.

**Scaling Relations**

Actually, some of our simulations show mass-size or velocity-dispersion-size relations, that can be approximated by power-laws for a scale-range of about 1 dex, but the dynamical range is two small to draw final conclusions about the existence of a scale-free phase-space structures at the kpc-scale.
Figure P1.3: The same as in Fig. P1.2 for a simulation with “strong” energy dissipation. The cooling time is \( \tau_{\text{cool}} = 14 \tau_{\text{osc}} \).

However, also simulations with a larger dynamical range may fail to produce scale-free structures over an extended scale-range. Indeed, in a hierarchically organized medium the error propagation may be super-exponential since the error evolution at largest scales is not determined by the dynamical time of these scales, but by the much smaller dynamical time of the smallest scales. Systems with dimensions lower than about 2.2 are particularly concerned. Thus, this point is relevant for particle simulations of gravitationally unstable systems with dimensions \( D \sim 2 \), such as thin disks and cosmological structures.

**Velocity-Dispersion Ellipsoid**

The anisotropy of the velocity-dispersion ellipsoid, resulting from our simulations, has systematically the same order \( \sigma_R > \sigma_\phi > \sigma_z \) and relative amplitudes as observed in the Galaxy and in \( N \)-body simulations of spirals. Since the models are deliberately a simplified representation of reality, we learn from this that this order may be due to a very general property of galactic disks, to be substantially self-gravitating in \( z \), and to rotate differentially with a similar shear rate set by a constant rotation curve.

**Project 2: Long-Range Correlations in Self-Gravitating Systems**

A spatially isolated, spherical \( N \)-body model was used in order to study three physical situations, namely, micro-canonical systems, the gravitational collapse of strongly dissipative systems and systems that are subject to an energy-flow.

Actually the here studied models are scale-free, however, due to their initially, extended spherical geometry the results are particularly relevant for systems at sub-galactic scales, such
Figure P2.1: Comparison of theoretical predictions and simulated systems. **Left:** Inverse temperature $\beta = 1/T$ versus energy $\varepsilon$ for models with softened potentials. The solid line indicates the theoretical result and the dotted curve depicts the evolution of the simulated system. The circle indicates the initial state of the simulations. The range of negative specific heat corresponds to the range where the slope of $\beta(\varepsilon)$ is positive. **Right:** The dotted lines describe the simulated evolution of the Lagrangian radii $\kappa$. Each curve depicts the radius of a sphere containing a certain mass fraction. The different mass fractions are: $\Delta M/M = \{5\%, 10\%, 20\%, \ldots, 80\%, 90\%, 95\%\}$. The solid line depicts the theoretical 95%-Lagrangian radius. The dashed vertical line depicts the moment when the system becomes fully self-gravitating. The dash-dotted and the dotted line mark the interval of negative specific heat.

as molecular clouds.

**P2.1 Micro-Canonical Systems**

**Motivation**

In order to study with the same experiment the equilibrium properties of systems in dependence of their energy, a weak dissipation scheme is applied.

In this way the gravitating systems are maintained in quasi-equilibrium states, which allows us to compare numerical simulations with findings, resulting from theoretical studies of self-gravitating, micro-canonical systems. In particular, the simulations are compared with predictions resulting from a micro-canonical approach using a continuum approximation (mean field approximation, see chapter 3). Such an approximation is necessary in order to make complex many-body systems mathematically tractable. However, due to the approximation one looses any information about the system granularity and correlated particle motions.

Yet, these information are present in the $N$-body simulations. Thus we can study the effect of granularity and correlated particle motions on the system evolution.

**Results and Conclusions**

On the one hand, equilibrium properties of $N$-body models agree with predictions made by analytical models using the mean-field approximation. An example is the energy interval of negative specific heat. On the other hand, discrepancies are found, such as the way the collapsing phase transition, separating a high-energy homogeneous phase from a low-energy collapsed phase, develops in the interval of negative specific heat (see Fig. P2.1). These discrepancies suggest that small scale physics and correlated motions become relevant for the
Figure P2.2: Evolution of the index $\delta(r)$ of the velocity-dispersion-size relation $\sigma_v \propto r^{\delta(r)}$ during the collapsing transition for three simulations with global dissipation scheme and different dissipation strength. The correlations result from simulations that were carried out with 160'000 particles. The solid, the dashed and the dotted vertical lines indicate the scope of application of the simulation with $\alpha = 1.0$, $\alpha = 5.0$ and $\alpha = 9.0$, respectively, where $\alpha$ indicates the dissipation strength. This corresponds to $\tau_{\delta_0} = 2.0$, $0.4$, $0.2$ $\tau_\Pi$. The lower cutoff is given by the softening length, that is here $\epsilon = 0.01$, and the upper cutoff by $2R_{00}$, where $R_{00}$ is the radius of the sphere centered at the origin which contains 90% of the mass. The time is indicated above each panel.

system evolution when the growth of singularities triggered by gravitational instabilities is allowed.

If instead of micro-canonical ensembles, (grand-) canonical ensembles are used, discrepancies between theory and simulations are expected as well, since these ensembles are not strictly consistent with self-gravitating, i.e., nonextensive systems.

However, not all of the equilibrium properties found by maximizing the Gibbs-Boltzmann entropy are expected to change if a fully consistent, generalized thermostatistical theory is applied that includes granularity and correlated particle motions.

P2.2 Gravitational Collapse of Strongly Dissipative Systems

Motivation

Elmegreen (2000) suggested that the lifetime of star-forming clouds are a few free-fall times. Furthermore, observations of star-forming and quiescent clouds reveal similar structure prop-
erties. This may then suggest that the ISM-structures are formed during gravitational collapses at the free-fall time-scale.

In order to check this possibility, different effective energy-dissipation-schemes that provoke a gravitational contraction on the free-fall time-scale are applied in the models. This allows us then to study if such rapidly contracting systems can develop long-range phase-space correlations, such as those observed in the ISM and how they do depend on the different dissipation schemes.

**Results and Conclusions**

The findings resulting from the study of the collapsing transition in systems with strong dissipation are:

1. Dissipative self-gravitating systems develop outside of equilibrium, in the interval of negative specific heat transient long-range correlations. That is, fragmentation and nonequilibrium velocity-dispersion-size relations, with striking resemblance to those observed in the ISM, appear during the gravitational contraction, when the dissipation time $\tau_{\text{dis}}$ is shorter than the free fall time $\tau_f$ (see Fig. P2.2). This suggests that nonequilibrium structures in self-gravitating interstellar gas are *dynamical and highly transient*.

2. Besides the dissipation strength and the initial noise, the granularity turns out to be a crucial parameter for the strength of the resulting long-range correlations, substantiating the importance of a coherent mass and force resolution. That is, phase-space correlations are stronger in the fluid limit than in a granular phase. The opposite holds for spatial correlations. The inverse behavior of fragmentation strength and phase-space correlation strength is found in all simulations and is typical for self-gravitating systems.

**P2.3 Systems Subject to an Energy-Flow**

**Motivation**

Scenarios have been suggested in which large amounts interstellar matter form hierarchically fragmented structures in dynamical equilibrium, so that, even though local properties are transient, the global flow statistics can be maintained without the need of newly supplied matter.

In order to check if such persistent flow patterns can be formed in the models at molecular cloud scales we study the structure evolution in systems that are permanently maintained out of equilibrium by energy-flows.

**Results and Conclusions**

Physically realistic phase-space correlations, similar to those observed in molecular clouds cannot be maintained in the spherical $N$-body models. Attempts to produce a persistent flow statistics at molecular cloud scales with alternative models have failed as well (e.g. Semelin & Combes 2000; Klessen et al. 2000). This suggests that matter that has passed through a collapsing transition has to be replenished at larger scales in order to maintain a hierarchical structure at molecular cloud scales.
Project 3: Structure Analysis Using the $\Delta$-Variance

P3.1 Motivation

A robust and unbiased statistical description of the structures observed in the cold ISM is needed in order to identify the relevant shaping agents and to constrain the models by comparing the structure at different scales and in different environments.

Here the structure of dust and molecular regions in the vicinity of the sun, the Outer Galaxy and M51 are analyzed with the $\Delta$-variance method. Typically these structures are characterized by a power-law power-spectrum $P(k) = k^{-\beta}$ and the power spectral index can be determined with the $\Delta$-variance method. The method is wavelet based and can be applied in position space. Furthermore, the method includes an approximate correction for white noise and beam smoothing. Thus determining the power spectral index with the $\Delta$-variance method is a more robust approach than the use of conventional techniques. The here determined structure parameters, that are supplemented with published data, allow to study the structure properties of the interstellar medium over a scale range of 4 dex from sub-parsec scales up to the galactic disk scales.

P3.2 Results and Conclusions

The findings suggest an on average scale-free order of the ISM-structures at scales $\lesssim 1$ pc. At larger scales the power spectral index decreases (see Fig. P3.1). Smaller values of $\beta$ substantiating a higher degree of fragmentation are also found in interarm regions and in the outer regions of spiral disks (see Fig. P3.2). A correlation between the density of the ISM and the fragmentation degree of its structure is found. That is, lower degrees of fragmentation may be related with higher densities. Finally, inside regions with similar mean density the structure properties are independent of the star-formation activity in a cloud.

The largest power spectral indices presented in this study describe the structures of nearby molecular clouds ($d \lesssim 150$ pc) on the scales of the order of $\sim 0.01 - 1$ pc. The mean power spectral index for these clouds is $\beta = 2.98 \pm 0.39$. This is less than the values predicted by Kolmogorov type turbulence $\beta = 3.7$ (Goldreich & Sridhar 1995) and shock type turbulence $\beta = 4$ (Komreich & Scalo 2000). For larger scales and larger distances from the galactic center (GC), the deviation from model predictions increases. This shows that hydrodynamic turbulence models typically produce less fragmented structures than they are observed in the ISM.

Concluding Thoughts

Here we advance some ideas already developed in previous chapters.

It has been suggested that the lifetime of molecular clouds is at least an order of magnitude longer than a gravitational cloud collapse would take in the absence of any further support.

Furthermore, observations of the ISM typically reveal a turbulent velocity field, substantiating the dynamical nature of the ISM-structures.

Thus it has been thought that a permanent energy injection driving the turbulent motion, prohibits the gravitational collapse and maintains the statistics of the flow constant.

According to such a scenario the scaling-laws observed in the ISM, describe global flow patterns formed by transient structures. These patterns are maintained as long as the energy
Figure P3.1: The power spectral index $\beta$ as a function of the scale $\langle S \rangle$. The data-set comprises structure parameters of molecular and dust regions in the Milky Way and M51. The solid lines indicate the linear $\chi^2$-fits to the data. The slope of the fitted straight line $\alpha$ is indicated as well. The dashed lines depict the interval $2 \leq \beta \leq 3$ and the arrows mark the observation of the Polaris Flare with different resolutions, i.e., at different scales. For clarification these data-points are replotted in the lower-left corner of the figure.

- Molecular regions in the Milky Way.
- Dust regions in M51.

Figure P3.2: The power spectral index $\beta$ as a function of the scale $\langle S \rangle$ and the distance from the galaxy center $R_{GC}$. The dots depict the scale and the distance $R_{GC}$ of the different Galactic molecular clouds presented in this study. The contour plot results from the power spectral indices of these clouds.
dissipated by inelastic collisions is replenished.

In order to check this, numerical simulations at galactic disk scales and molecular cloud scales were carried out.

At galactic disk scales persistent filamentary patterns formed by transient structures appear in the shearing box models.

However, at molecular cloud scales long-range correlations similar to those observed in the ISM cannot be maintained. For instance in the above presented spherical models, such correlations appear only during the recursive fragmentation process of a gravitational collapse, suggesting that at these scales a constant flow-statistics can only be maintained if the system is additionally subject to a matter-flow.

Besides in shearing box simulations, persistent long-range correlations appear also in cosmological simulations. The presence of nonequilibrium structures at cosmological scales may not be surprising, since the age of the universe is smaller than its dynamical time. However, we want to focus on another point that may be relevant for the formation of persistent long-range correlations in self-gravitating systems, namely, the presence of an anti-gravitational flow. Let us discuss this in turn.

In cosmological and shearing box models, time-dependent boundary conditions create relative particle velocities that are inverse to the gravitational acceleration and increase with the relative particle distance \( v \propto r \). Indeed, in the shearing box model, the relative azimuthal particle velocity due to the shear flow is \( \propto r \), where \( r_c \) is the radial particle distance in cylinder coordinates, and in cosmological models, the relative particle velocity induced by the Hubble flow is \( v_r \propto r \), where \( r \) is the relative particle distance in Cartesian coordinates.

The fact that the shear flow affects only the azimuthal velocity component may then account for the characteristic spiral-arm-like structures found in shearing box experiments, differing from the on average isotropic cosmological structures.

At molecular cloud scales there is no such spatial flow counteracting gravity which may explain why at these scales hierarchically fragmented structures cannot be maintained in the models.

These findings may then suggest the following scenario: At galactic disc scales gravity, differential rotation (shear) and energy dissipation form dynamical, filamentary patterns, that are a priori globally maintained due to the globally balanced energy budget.

Yet, in dense regions, dissipation may locally dominate due to a higher inelastic collision rate, so that clumps form out of the filaments. Typically, these clumps do not disperse in the course of time, but grow. A gravitational collapse is triggered as soon as the clump mass exceeds the critical Jeans mass. Then, during the subsequent clump collapse, recursive fragmentation occurs and transient long-range correlations appear.

Since the intermittently appearing gravitational collapses transform mass from large scales down to small scales the global flow statistics can only be maintained if matter is continuously replenished at galactic disc scales.

Thus, the evolution of the global flow statistics describing the structure properties of the ISM over its whole dynamical range depends on the properties of the matter-flow through the scales, i.e., on the large-scale matter replenishment and the dissipative factors.

In these scenario besides gravity the dissipative factors are an important shaping agent of the ISM determining both the time-scale of the large scale morphological evolution as well as the lifetime of molecular clouds, that may be an order of magnitude smaller than classically assumed (Blitz & Shu 1980).
Actually, gravity and the dissipative factors dominantly influence the structure formation over the whole dynamical range, yet, the details of the shaping process at the kpc-scale, where gravity, shear and dissipation form filamentary patterns, is not the same as at molecular cloud scales, where the long-range correlations may result from a recursive fragmentation process.

Thus the ISM-structures at the kpc-scale and the molecular cloud scale may actually be similar (lumpy, fragmented) but are not expected to follow the same scaling-relations.

**Perspectives**

**Numerical Simulations**

**Dynamical Range**

In order to study the structure evolution of the ISM at galactic disk scales as well as at molecular cloud scales simulations with principally two different models have been carried out. However, it would be very attractive to study the whole scale-range involved in the ISM shaping process with a single model, that allows then to check the above described scenario in which gravitational clump collapses involving recursive fragmentation appear intermittently in dens filamentary structures.

This could be achieved by using adaptive mesh refinement techniques. Based on such techniques $N$-body codes as well as hydrodynamic codes have been developed (Yahagi & Yoshii 2001; Teyssier 2002). These techniques are particularly effective in situations, where most of the system space is filled by a low density medium and high densities appear only in a small volume fraction. Thus the technique is appropriate to simulate the evolution of cosmological as well as ISM-structures (Truelove et al. 1998). Indeed, for instance 3D cosmological simulations based on a adaptive mesh refinement technique can attain a dynamical range of 4 dex (Teyssier 2002).

**Angular Momentum**

In this thesis the equilibrium configuration of a self-gravitating system enclosed in a finite sphere was studied as a function of the total energy using $N$-body simulations. The simulations were compared with theoretical findings and some discrepancies were pointed out.

Besides energy a further conserved quantity that is crucial for astrophysical systems is the angular momentum. The properties of spherically confined systems have been studied as a function of the total energy and the total angular momentum 1) in the fluid limit by employing a micro-canonical mean field approach (Votyakov et al. 2002) and 2) for small particle numbers $N \lesssim 20$ by employing a “purely” micro-canonical approach (Fliegans & Gross 2001). These studies predict collapsed phases with two clusters, but also the possibility of ring and disk formation.

It would be interesting to check these results with $N$-body simulations, that do, contrary to the mean field approach, not neglect granularity and correlations and can be carried out with large particle numbers ($N \gg 10^4$).

Furthermore, it would be interesting to check if the equilibrium properties change when the particles itself are allowed to possess an angular momentum. In the models this could for instance be achieved by using barbell particles instead of point particles.

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*Finite volume with $N \to \infty$, where $N$ is the particle number.*
Observations and Structure Analysis

The Structure of a Star-Forming Cloud and a Pre-Star-Forming Cloud

Observations carried out with the KOSMA telescope (HPBW = 88°) of the pre-star-forming cloud L1512 and the low-mass star-forming cloud L1524 were presented in this thesis. Yet, the maps are too small to draw final conclusions about the structure properties in the corresponding clouds. Thus in order to obtain more conclusive results, the maps should be extended and be completed with observations at other scales. As mentioned in chapter 7, for L1512, IRAM observations resolving scales of the order of ~ 0.01 were already carried out. The \( \Delta \)-variance analysis yields for these scales power spectral indices of \( \beta \sim 3 \) (Bensch 1999). Thus the extension of the maps complemented with high-resolution observation of L1524 and the subsequent structure analysis would be an interesting project that may help to better understand the transition from a pre-star-forming cloud to a star-forming cloud.

Minkowski Integrals

The \( \Delta \)-variance applied in this thesis is based on a wavelet with circular symmetry and is thus only sensitive to the azimuthally averaged power-spectrum. The power-spectrum in turn provides only information about the correlation of the amplitudes of the Fourier modes but not of their phases. That is, the power-spectrum is not a complete statistical description and one does for instance not know if the underlying structure is rather clumpy, filamentary or formed by sheets.

Alternative statistical measures often used for the analysis of ISM-structures, such as the mass-size relation or the perimeter-area relation are actually appropriate for the identification of a scale-free order but provides only very limited geometrical informations. Furthermore, they are typically applied after the image has been decomposed into discrete clumps. In order to carry out the decomposition some a priori assumptions about the clump geometry have to be made. These assumptions may oversimplify the structure properties of the ISM and thus distort the analysis.

A statistical descriptor that does not make a priori assumptions of the structure and provides topological as well as geometrical informations is given by the set of the four Minkowski integrals (e.g. Mecke et al. 1994). The integrals have been used to analyze the large scale structure of galaxy distributions and provide information about the connectivity and shape of the underlying structure. Codes calculating the Minkowski integrals of point images (e.g. galaxy distributions) are available (Mecke et al. 1994; Schmalzing & Buchert 1997). The modification and subsequent application of the technique for the description of ISM-structures may be a promising future project.
Résumé de la thèse (Version française)

Alles Sprachliche kann immer missverstanden werden. Dass wir uns so oft verstehen, beruht größtenteils auf dem guten Willen: auf dem Wunsch zu verstehen; auf einer selbstkritischen Einstellung zum allgegenwärtigen Problem, ob man richtig verstanden hat; und auf dem Resultat dieser Einstellung, das dann “Einfühlung” genannt wird.

Popper 1994

Dans cette thèse nous nous sommes intéressés à la formation de structures dans des systèmes auto-gravitants hors équilibre. En particulier notre intérêt se tourne vers les structures de non-équilibre dans le milieu interstellaire (MIS) froid.

Afin de mieux comprendre l’importance de la gravitation dans la formation des corrélations à longue portée aux échelles des disques galactiques et aux échelles des nuages moléculaires, des simulations numériques de disques de cisaillement et des systèmes à $N$-corps sphériques sont effectués. Les modèles numériques correspondants nous permettent également d’étudier l’effet des perturbations sur l’évolution des systèmes gravitationnellement instables, telles que des différents procédés de dissipations et mécanismes propulseurs.

En plus des simulations numériques, une autre approche qui nous aide à comprendre mieux les processus de formation de structure actives dans le MIS, est d’effectuer une analyse de structure détaillée. Ainsi nous avons déterminé et comparé le spectre de puissance du MIS à différentes échelles, dans différents environnements galactiques et même dans différentes galaxies.

Ultérieurement, les différents projets, résultats et conclusions sont récapitulés.

Projet 1: Structures grumeleuses dans des disques auto-gravitants

P1.1 Motivation

Les observations du milieu interstellaire, des disques spiraux et des structures cosmiques indiquent des caractéristiques semblables. Les structures dans ces systèmes sont très grumeleuses et non-homogènes. En outre ces structures ne semblent pas être complètement aléatoire, mais obéissent en partie à certaines lois d’échelles.

Les modèles locaux des disques auto-gravitants montrent comment de telles structures non-homogènes peuvent être maintenues aux échelles du kpc (e.g. Toomre & Kalnajs 1991; Toomre 1990). Leurs ingrédients physiques de base sont l’auto-gravitation, la dissipation et la rotation différentielle.
Figure P1.1: Le cadre pointillé représente une section du disque, étant infinie dans deux directions, vues d'en haut. Les deux mailles sont des systèmes de coordonnées inclinées sur lesquels la distribution de masse est périodique. Les forces sont calculées dans les deux mailles et puis additionnées avec des facteurs pondérés. Un tel procédé évite des discontinuités dans l'évolution temporelle du champ de force (cf. chapitre 5). La boîte foncée et la boîte claire représentent la boîte locale de calcul en coordonnées inclinées. a: L'état initial des deux mailles (t = 0). b: Les mailles à t = L_y/(2L_xΩ_0). c: Quand les mailles atteignent ces inclinations (t = L_y/(L_xΩ_0)), elles sautent de nouveau aux positions montrées en (a) et le processus commence de nouveau sans introduire des discontinuités dans la dynamique.

Afin d'explorer plus en détail les structures de non-équilibre, résultant des instabilités gravitationnelles dans un système maintenu hors équilibre par la rotation différentielle et la dissipation, des simulations locales des disques auto-gravitants sont effectuées en 2D et 3D. En particulier, nous voulons vérifier si l'auto-gravitation combinée avec des conditions aux bords dépendant du temps et une légère dissipation peut produire des structures non-homogènes, grumeleuses et éventuellement auto-similaires, ressemblant à celles observées dans les nuages moléculaires et des galaxies spirales floconneuses (floculent spirals).

Afin d'augmenter la performance par rapport aux modèles précédents dans lesquels les forces de la gravité sont calculées avec des méthodes d'addition directe, on développe une technique qui permet d'employer une méthode FFT** particule-maille pour le calcul des forces (Hockney & Eastwood 1981). C'est-à-dire, des systèmes de coordonnées inclinées, dépendant du temps sont employés par rapport auxquels le système demeuré périodique à tout moment, de sorte que les forces puissent être calculées dans l'espace de Fourier. Les systèmes de coordonnées inclinées, dépendant du temps sont montrés à la Fig. P1.1.

**FFT: Fast Fourier transform: Transformation de Fourier rapide.
P1.2 Résultats et conclusions

Motifs persistants et structures transitoire

Les structures résultant des simulations locales des disques auto-gravitants sont homogènes, ondulées, filamentueuses ou grumeleuses, selon l'ampleur relative des forces gravitationnelles et dissipatives en concurrence. La transition entre les différents états est lisse.

Puisque les disques sont réchauffés, en raison des instabilités gravitationnelle menant à une conversion de l'énergie cinétique, présente dans le flux de cisaillement, en mouvement thermique aléatoire, les inhomogénéités ne peuvent pas se développer et ont tendance à s'effacer, si la dissipation d'énergie est trop faible. Cependant, avec un choix approprié de l'intensité de la dissipation des motifs persistants ondulés ou, dans le cas d'une dissipation plus forte, filamentueux, formés par des structures transitoires, apparaissent (cf. Fig. P1.2).

Aussi longtemps que la structure est principalement ondulée ou filamentueuse, l’auto-gravitation et la dissipation assurent un équilibre statistique, où des structures transitoires répétées sont formées. Ces structures, apparaissant sur l'échelle du kpc, sont comme des bras spiraux et donc en moyenne non-isotropes.

Quand les facteurs dissipatifs commencent à dominer le processus d'évolution, des grumeaux se forment à l'intérieur des filaments (cf. Fig. P1.3). Typiquement, les grumeaux ne se dispensent pas, ce signifie que leur croissance n'est temporellement pas limitée. Ceci mène alors aux effondrements gravitationnels. C'est pourquoi, les structures grumeleuses en général n'évolue pas vers un équilibre statistique.

Kim & Ostriker (2001) ont effectué des simulations magnéto-hydrodynamiques des disques de cisaillement gazeux. Les corrélations à longue portée apparaissant dans ces simulations sont qualitativement semblables à ceux qui résultent de notre modèle à N-corps, suggérant que les agents fondamentaux de la formation de structure aux échelles du kpc sont la gravitation, la rotation différentielle ainsi que la dissipation, et que les champs magnétiques et la pression du gaz sont des effets du second degré.

Relations d'échelles

Certes, certaines de nos simulations montrent des relations de masse-taille ou de dispersion de vitesse-taille, qui peuvent être approchées par des lois de puissance pour une intervalle d'échelle d'environ 1 dex. Mais l'intervalle dynamique est trop petit pour tirer des conclusions finales sur l'existence de structures dans l'espace de phase invariantes d'échelle aux échelles de l'ordre du kpc.

Cependant, les simulations calculées sur un intervalle dynamique plus large pourraient également ne pas produire des structures invariantes d'échelle sur un intervalle d'échelles étendu. En effet, dans un milieu hiérarchiquement organisé la propagation des erreurs peut être super-exponentielle puisque l'évolution des erreurs aux plus grandes échelles n'est pas déterminée par l'échelle de temps dynamique de ces grandes échelles, mais par l'échelle de temps dynamique des plus petites échelles qui est beaucoup plus courte. Des systèmes avec des dimensions plus petites qu'environ 2,2 sont particulièrement concernés. Ce point est important pour des simulations à N-corps de systèmes gravitationnellement instables avec des dimensions $D \sim 2$, tel que les disques minces et les structures cosmologiques.
Figure P1.2: Evolution des positions des particules vues depuis le dessus du disque galactique. Les structures résultent d’une simulation 3D avec une dissipation d’énergie “faible”. Le temps de refroidissement est $\tau_{\text{cool}} = 16 \tau_{\text{osc}}$, où $\tau_{\text{osc}}$ est la période du mouvement épicycloïdal (Binney & Tremaine 1994). La simulation a été effectuée avec 131'040 particules. Seulement chaque quatrième particule est montrée. L’intervalle dynamique est de 1,8 dex. Au-dessus de chaque panneau le nombre de rotations galactiques est indiqué.

Figure P1.3: De même que la Fig. P1.2 pour une simulation avec une dissipation d’énergie “forte”. Le temps de refroidissement est $\tau_{\text{cool}} = 14 \tau_{\text{osc}}$. 
**Ellipsoïde de dispersion de vitesse**

L'anisotropie de l'ellipsoïde de dispersion de vitesse, résultant de nos simulations, a systématiquement le même ordre \( \sigma_r > \sigma_\phi > \sigma_z \) et les mêmes amplitudes relatives que ceux observés dans la Galaxie et dans des simulations N-corps des galaxies spirales. Puisque les modèles sont délibérément une représentation simplifiée de la réalité, ceci nous apprend que cet ordre peut être dû à une propriété très générale des disques galactiques, qui sont substantiellement auto-gravitants dans la direction \( z \), et qui tournent différemment avec un taux de cisaillement donné par une courbe de rotation constante.

**Projet 2: Corrélations à longue portée dans des systèmes auto-gravitants**

Un modèle N-corps sphérique spatialement isolé a été employé afin d'étudier trois situations physiques, à savoir, les systèmes micro-canoniaques, l'effondrement gravitationnel des systèmes fortement dissipatifs et des systèmes qui sont sujets à un flux d'énergie.

En fait les modèles étudiés ici sont invariant d'échelles, cependant, en raison de leur géométrie initialement sphérique et étendue, les résultats ont une importance particulière pour des systèmes aux échelles sous-galactiques, tels que les nuages moléculaires.

**P2.1 Systèmes micro-canoniaques**

**Motivation**

Afin d'étudier avec la même expérience les propriétés d'équilibre des systèmes en fonction de leur énergie, un procédé de dissipation faible est appliqué.

De cette façon, les systèmes gravitants sont maintenus dans des états de quasi-équilibre, ce qui nous permet de comparer des simulations numériques aux résultats déduite des études théoriques des systèmes auto-gravitants micro-canoniaques. En particulier, les simulations sont comparées aux prévisions résultant d'une approche micro-canionique utilisant une approximation de continu (la limite du champ moyen, cf. chapitre 3). Une telle approximation est nécessaire afin de rendre les systèmes complexes à grand nombre de particules mathématiquement résolvables.

Cependant, en raison de l'approximation on perd toutes les informations sur la granularité du système et les corrélations entre les particules. Pourtant, ces informations sont présentes dans les simulations N-corps. Donc nous pouvons étudier l'effet de la granularité et des mouvements des particules corrélés à l'évolution du système.

**Résultats et conclusions**

D'une part, les propriétés d'équilibre des modèles N-corps sont consistants aux prévisions faites par les modèles analytiques en utilisant la limite du champ moyen. Un exemple est l'intervalle d'énergie de la chaleur spécifique négative. D'autre part, on trouve des divergences, telles que la manière avec laquelle la transition de phase d'effondrement, séparant une phase homogène à haute énergie d'une phase effondrée à basse énergie, se développe dans l'intervalle de la chaleur spécifique négative (cf. Fig. P2.1). Ces divergences suggèrent que la physique à petite échelle et les mouvements corrélés deviennent importants pour l'évolution du système.
Figure P2.1: Comparaison des prévisions théoriques avec des systèmes simulés. **A gauche:** Température inverse $\beta = 1/T$ en fonction de l'énergie $\varepsilon$ pour des modèles avec des potentiels adoucis. La ligne en trait-plein indique le résultat théorique et la courbe en pointillé décrit l'évolution du système simulé. Le cercle correspond à l'état initial des simulations. L'intervalles de la chaleur spécifique négative correspond à l'intervalle où la pente de $\beta(\varepsilon)$ est positive. **A droite:** Les lignes en pointillé décrivent l'évolution simulée des rayons lagrangiens $\kappa$. Chaque courbe décrit le rayon d'une sphère contenant une certaine fraction de masse. Les différentes fractions de masse sont: $\delta M/M = \{5\%, 10\%, 20\%, \ldots, 80\%, 90\%, 95\%\}$. La ligne en trait-plein décrit le rayon lagrangien théorique de 95%. La ligne verticale en traitéillé correspond au moment où le système devient entièrement auto-gravitant. Les lignes verticales en trait-pointillé et en pointillé marquent l'intervalle de la chaleur spécifique négative.

quand la croissance des singularités déclenchées par des instabilités gravitationnelles sont permises.

Si au lieu des ensembles micro-canoniques, des ensembles (grand)canoniques sont utilisés, des anomalies entre la théorie et des simulations sont également prévues, puisque ces ensembles ne sont pas strictement consistants avec des systèmes auto-gravitants, c.-à-d., des systèmes nonextensives.

Cependant, on s'attend que ne pas toutes les propriétés d'équilibre trouvées en maximisant l'entropie de Gibbs-Boltzmann changent, si une théorie statistique entièrement cohérente et généralisée est appliquée qui inclut la granularité et les corrélations des particules.

P2.2 Effondrement gravitationnel des systèmes fortement dissipatifs

Motivation

Elmegreen (2000) a suggéré que la vie des nuages de formation stellaire est de quelques fois le temps de chute libre. En outre, les observations des nuages de formation stellaire et des nuages à l'état de repos révèlent des propriétés de structure semblables. Ceci pourrait indiquer que les structures du MIS se soient formées lors d'effondrements gravitationnels à l'échelle de temps de chute libre.

Afin de contrôler cette possibilité, différents procédés de dissipation d'énergie efficace qui provoquent une contraction gravitationnelle à l'échelle de temps de chute libre sont appliqués dans les modèles. Ceci nous permet alors d'étudier si de tels systèmes en contraction rapide peuvent développer des corrélations à longue portée dans l'espace, comme celles observées dans le MIS, et comment elles dépendent des différentes méthodes de dissipation.
Figure P2.2: Évolution de l’indice $\delta(r)$ de la relation dispersion de vitesse-taille $\sigma_v \propto r^{\delta(r)}$ pendant la transition d’effondrement pour trois simulations avec un procédé de dissipation global et des intensités de dissipation différentes. Les corrélations sont basées sur des simulations effectuées avec 160'000 particules. Les lignes verticales en trait-plin, en traitillé et en pointillé indiquent le rayon d’application de la simulation avec $\alpha = 1.0$, $\alpha = 5.0$ et $\alpha = 9.0$ respectivement, où $\alpha$ indique l’intensité de dissipation. Ceci correspond à des temps $\tau_{\text{dis}} = 2.0$, $0.4$, $0.2$ $\tau_\text{H}$. La limite inférieure est donnée par la longueur d’adoucissement (softening length), qui est ici $\epsilon = 0.01$, et la limite supérieure par $2R_{90}$, où $R_{90}$ est le rayon d’une sphère centrée à l’origine et contenant 90% de la masse. Le temps est indiqué au-dessus de chaque diagramme.

Résultats et conclusions

Les résultats de l’étude de la transition d’effondrement dans des systèmes avec une forte dissipation sont:

1. Les systèmes auto-gravitants et dissipatifs développent, hors équilibre, dans l’intervalle de chaleur spécifique négative des corrélations transitoires à longue portée. En d’autres termes, de la fragmentation et des relations dispersion de vitesse-taille de non-équilibre, ayant une ressemblance saisissante à celles observées dans le MIS, apparaissent au cours de la contraction gravitationnelle, lorsque le temps de dissipation $\tau_{\text{dis}}$ est plus court que le temps de chute libre $\tau_\text{H}$ (cf. Fig. P2.2). Ceci suggère que les structures de non-équilibre du gaz interstellaire auto-gravitant sont dynamiques et fortement transitoires.

2. En plus de l’intensité de la dissipation et du bruit initial, la granularité s’avère être un paramètre crucial en ce qui concerne l’intensité des corrélations à longue portée,
justifiant l’importance d’une résolution en masse et en force cohérente. Autrement dit, 
les corrélation dans l’espace de phase sont plus fortes dans la limite fluide que dans une 
phase granulaire. L’opposé est vrai pour les corrélation spatiales. Le comportement 
inverse de l’intensité de la fragmentaion et de l’intensité des corrélation dans l’espace de 
phase est observé dans toutes les simulations et est typique des systèmes auto-gravitants.

P2.3 Systèmes sujet à un flux d’énergie

Motivation

Des scénarios dans lesquels de grandes quantités de matière interstellaire forment des struc-
tures hiérarchiquement fragmentées en équilibre dynamique ont été proposés. Bien que les 
propriétés locales soient transitoires, la statistique globale du flux peut être maintenue sans 
apport de matière nouvelle.

Afin de vérifier si de tels motifs de flux persistants peuvent être formés dans les modèles aux 
échelles des nuages moléculaires, nous étudions l’évolution de la structure dans des systèmes 
qui sont maintenus hors équilibre en permanence par des flux d’énergie.

Résultats et conclusions

Des corrélation physiquement réalistes dans l’espace de phase, semblables à celles observées 
dans les nuages moléculaires ne peuvent pas être maintenues dans les modèles N-corps 
 sphériques. Les tentatives visant à produire une statistique de flux persistante aux échelles 
des nuages moléculaires à l’aide de modèles alternatifs ont également échoué (cf. par exemple 
Semelin & Combes 2000; Klessen et al. 2000). Ceci suggère que la matière qui a traversé une 
transition de collapse doit être compensée à des échelles plus grandes afin de maintenir une 
structure hiérarchique aux échelles des nuages moléculaires.

Projet 3: Analyse de structure en utilisant la Δ-variance

P3.1 Motivation

Une description statistique robuste et sans biais des structures observées dans le MIS froid 
est nécessaire si nous voulons identifier les agents de formation importants et contraindre les 
modèles en comparant la structure à différentes échelles et dans différents environnements.

Ici, la structure des régions moléculaire et de la poussière à proximité du soleil, de la 
galaxie externe et de M51 sont analysées avec la méthode de Δ-variance. Ces structures 
sont typiquement caractérisées par un spectre de puissance en loi de puissance (power-law 
 power-spectrum) \( P(k) = k^{-\beta} \) et l’indice de puissance spectral (power spectral index) peut être 
déterminé par la méthode de Δ-variance. Cette méthode est basée sur des ondelettes et peut 
bé être appliquée dans l’espace des positions. Elle inclut en outre une correction approximative 
pour le bruit blanc (white noise) et le lissage de faisceau (beam smoothing). De ce fait, la 
détermination de l’indice de puissance spectral par la méthode de Δ-variance est une approche 
plus robuste que l’utilisation des techniques conventionnelles. Les paramètres de structure 
déterminés ici, complétés par des données publiées, permettent d’étudier les propriétés de 
la structure du milieu interstellaire sur un intervalle d’échelle de 4 dex, allant des échelles 
inférieures au parsec jusqu’aux échelles des disques galactiques.
Figure P3.1: L’indice de puissance spectral $\beta$ en fonction de l’échelle $\langle S \rangle$. L’ensemble de données comporte des paramètres de structure des régions moléculaires et de la poussière de la voie lactée et du M51. Les lignes solides indiquent le "$\chi^2$-fit" linéaire aux données. La pente de la droite adaptée $\alpha$ est également indiquée. Les lignes traitillées dépeignent l'intervalle $2 \leq \beta \leq 3$ et les flèches marquent l’observation du Polaris Flare à différentes résolutions, c.-à-d., à différentes échelles. Pour la clarification ces points de donnée sont retracés dans la figure en bas à gauche.

- Régions moléculaire dans la **Voie Lactée**.
- Régions de la poussière dans M51.

### P3.2 Résultats et conclusions

Les résultats suggèrent que l’ordre des structures du milieu interstellaire est en moyenne invariant d’échelle pour les échelles $\lesssim 1$ pc. Aux échelles plus grandes, l’indice de puissance spectral diminue (cf. Fig. P3.1). Des valeurs $\beta$ plus petites, témoignant d’un degré de fragmentation plus élevé, sont également trouvées dans les régions inter bras et dans des régions externes des disques spiraux (cf. Fig. P3.2). On trouve une corrélation entre la densité du MIS et le degré de fragmentation de sa structure. Des degrés de fragmentation plus faibles pouvant être associés à des densités plus élevées. Finalement, à l’intérieur des régions ayant des densités moyennes semblables, les propriétés de la structure sont indépendantes de l’activité de formation stellaire dans un nuage.

Les indices de puissance spectraux les plus grands présentés dans cette étude décrivent les structures des nuages moléculaires voisins ($d \lesssim 150$ pc) sur des échelles de l’ordre de $\sim 0,01 \sim 1$ pc. L’indice de puissance spectral moyen de ces nuages est $\beta = 2,98 \pm 0,39$. Cette valeur est inférieure à celles prévues par la turbulence $\beta = 3,7$ de type Kolmogorov (Goldreich & Sridhar 1995) et la turbulence de type choc $\beta = 4$ (Kornreich & Scalo 2000). Pour des échelles plus grandes et des distances au centre galactique (CG) plus importantes, la déviation par rapport aux prévisions des modèles augmente. Ceci montre que les modèles de turbulence hydrodynamique produisent typiquement des structures moins fragmentées que celles observées dans le MIS.
Figure P3.2: L’indice de puissance spectral $\beta$ en fonction de l’échelle $<S>$ et de la distance du centre de la galaxie $R_{GC}$. Les points dépeignent l’échelle et la distance $R_{GC}$ des différents nuages moléculaires galactiques présentés dans cette étude. Le tracage des courbes de niveau (contour plot) résulte des indices de puissance spectraux de ces nuages.

**Raisonnement final**

Ici nous avançons quelques idées déjà développées aux chapitres précédents.

Il a été suggéré que la durée de vie des nuages moléculaires soit au moins un ordre de grandeur plus grande que celle d’un effondrement gravitationnel de nuage en l’absence d’un support supplémentaire.

En outre, les observations du MIS indiquent typiquement un champ de vitesse turbulent, prouvant la nature dynamique des structures du MIS. C’est pourquoi il a été supposé qu’une injection permanente d’énergie soutient le mouvement turbulent et empêche l’effondrement gravitationnel, ce qui maintient la statistique de flux constante.

Selon un tel scénario, les lois d’échelle observées dans le MIS, décrivent des motifs globaux du flux formés par des structures transitaires. Ces motifs sont maintenus aussi longtemps que l’énergie dissipée par des collisions inélastiques est compensée.

Afin de vérifier ceci, des simulations numériques aux échelles des disques galactiques et de nuage moléculaire ont été effectuées.

Aux échelles des disques galactiques, des motifs filamentueux et persistants, formés par des structures transitaires, apparaissent dans des modèles de boîte de cisaillement (shearing box models).

Cependant, aux échelles des nuages moléculaires, des corrélations à longue portée semblables à celles observées dans le MIS ne peuvent pas être maintenues. Par exemple, dans les modèles sphériques présentés ci-dessus, de telles corrélations apparaissent seulement durant le processus de fragmentation récursif d’un effondrement gravitationnel, suggérant qu’à ces échelles une statistique de flux constante ne puissent être maintenue que si le système est sujet à un flux supplémentaire de matière.

A part dans des simulations de boîte de cisaillement, les corrélations à longue portée persistantes apparaissent également dans des simulations cosmologiques. La présence des
structures de non-équilibre à l'échelle cosmologique n'est pas très étonnante, car l'âge de l'univers est plus petit que son temps dynamique. Cependant, nous voulons mettre l'accent sur un autre point qui pourrait être important pour la formation des corrélations à longue portée persistantes dans des systèmes auto-gravitants, à savoir la présence d'un flux anti-gravitationnel. Discutons ceci dans l'ordre.

Dans les modèles cosmologiques et de boîte de cisaillement, les conditions aux bords dépendantes du temps créent des vitesses relatives de particules qui sont opposées à l'accélération gravitationnelle et qui augmentent avec la distance relative de particules $v \propto r$. En effet, dans des modèles de boîte de cisaillement, la vitesse des particules relative azimutale, due au flux de cisaillement, est $v_\theta \propto r_c$, où $r_c$ est la distance radial des particules en coordonnées cylindriques, et dans des modèles cosmologiques, la vitesse relative des particules induite par le flux de Hubble est $v_r \propto r$, où $r$ est la distance relative des particules en coordonnées cartésiennes.

Le fait que le flux de cisaillement affecte seulement la composante de la vitesse azimutale pourrait expliquer les structures ressemblant aux bras spiraux, qui sont observée dans des expériences de boîte de cisaillement, et qui se distinguent des structures cosmologiques en moyenne isotropes.

Aux échelles des nuages moléculaires, il n'y a pas de tel flux spatial contrecarrant la gravité. Ceci pourrait expliquer pourquoi, à ces échelles, des structures hiérarchiquement fragmentées ne peuvent pas être maintenues dans les modèles.

Ces résultats pourraient alors suggérer le scénario suivant: Aux échelles des disques galactiques la gravité, la rotation différentielle (cisaillement) et la dissipation d'énergie forment des motifs dynamiques et filamentueux, qui sont, a priori, globalement maintenus en raison du bilan énergétique globalement équilibré.

Cependant, dans des régions denses, la dissipation peut être localement dominante en raison d'un taux de collision inélastique plus élevée, de sorte que des grumeaux se forment à l'intérieur des filaments. Typiquement, ces grumeaux ne se dispersent pas au cours du temps, mais croissent (en masse). Un effondrement gravitationnel est déclenché dès que la masse d'un grumeau excède la masse critique de Jeans. Puis, pendant l'effondrement ultérieur du grumeau, la fragmentation récursive se produit et les corrélations à longue portée transitoires apparaissent.

Puisque les effondrements gravitationnels, apparaissants par intermittence, transfèrent la masse des grandes échelles aux petites échelles, la statistique globales du flux ne peux être maintenue que si la matière est compensée à l'échelle de disque galactique.

Pour cela, l'évolution de la statistique globale du flux, dérivant les propriétés de structure du MIS sur tout l'intervalle dynamique, dépend des propriétés du flux de matière à travers les échelles, c.-à-d., sur le réapprovisionnement de matière aux grandes échelles et les facteurs dissipatifs.

Dans ce scénario, mise à part la gravité, les facteurs dissipatifs sont un agent important de formation de structure dans le MIS, déterminant non seulement l'échelle de temps de l'évolution morphologique à grande échelle, mais aussi la durée de vie des nuages moléculaires, qui pourrait être un ordre de grandeur plus courte que classiquement supposé (Blitz & Shu 1980).

Certes, la gravité et les facteurs dissipatifs influencent de manière prépondérante la formation de structure sur l'intervalle dynamique entier; pourtant les détails du processus de formation à l'échelle du kpc, où la gravitation, le cisaillement et la dissipation forment des motifs filamentueux, ne sont pas identique à ceux aux échelles des nuages moléculaires, où les
correlations à longue portée peuvent résulter d’un processus récursif de fragmentation.

Pour cette raison, les structures du MIS à l’échelle du kpc et aux échelles des nuages moléculaires peuvent en effet être semblable (grumeleuses, fragmentées) mais on ne s’attend pas qu’elles suivent les mêmes relations d’échelles.

**Perspectives**

**Simulations numériques**

**Intervalle dynamique**

Afin d’étudier l’évolution de structure du MIS aux échelles des disques galactiques ainsi qu’aux échelles des nuages moléculaires, des simulations avec principalement deux modèles différents ont été effectués. Cependant, il serait très attrayant d’étudier tout l’intervalle dynamique impliqué dans le processus de formation de structure dans le MIS avec un seul modèle, qui permettrait alors de vérifier le scénario décrit ci-dessus, dans lequel des effondrements gravitationnels, impliquant la fragmentation récursive, apparaissent par intermittence dans des structures filamentueuses denses.

Ceci pourrait être réalisé en utilisant des techniques de raffinement adaptatifs des mailles (adaptive mesh refinement). Basés sur une telle technique, des codes N-corps ainsi que des codes hydrodynamiques ont été développés (Yahagi & Yoshii 2001; Teyssier 2002). Ces techniques sont particulièrement efficaces dans des situations où la majeure partie de l’espace du système est constitué d’un milieu de faible densité et où les fortes densités n’apparaissent que dans une petite fraction du volume. Donc, cette technique est appropriée pour simuler l’évolution des structures cosmologiques ainsi que des structures du MIS (Truelove et al. 1998). En effet, par exemple les simulations 3D cosmologiques basées sur une technique d’affinement adaptatives des mailles peuvent atteindre un intervalle dynamique de 4 dex (Teyssier 2002).

**Impulsion angulaire**

Dans cette thèse, la configuration d’équilibre d’un système auto-gravitant, enfermé dans une sphère finie, a été étudiée en fonction de l’énergie totale en utilisant des simulations N-corps. Les simulations ont été comparées aux résultats théoriques et quelques divergences ont été mises en évidence.

A part l’énergie, une autre quantité conservée, qui est cruciale pour les systèmes astrophysiques est l’impulsion angulaire. Les propriétés des systèmes sphériquement confinés ont été étudiées en fonction de l’énergie totale et de l’impulsion angulaire totale 1) dans la limite de fluide en utilisant une approche micro-canonical de champs moyen (Votyakov et al. 2002) et 2) pour des petits nombres de particules $N \lesssim 20$ en utilisant une approche micro-canonical “pure” (Fliegeens & Gross 2001). Ces études prévoient des phases effondrées avec deux amas, ainsi que la possibilité de formation d’un anneau ou d’un disque.

Il serait intéressant de vérifier ces résultats avec des simulations N-corps qui, contrairement à l’approche de champs moyen, ne négligent pas la granularité et les corrélations moyennes et qui peuvent être effectué avec des grands nombres de particules ($n \gg 10'000$).

En outre, il serait intéressant de contrôler si les propriétés d’équilibre changent lorsque les particules elles-mêmes peuvent posséder une impulsion angulaire. Dans les modèles, ceci

---

**Volume fini avec $N \rightarrow \infty$, où $N$ est le nombre de particules.**
pourrait par exemple être réalisé en utilisant des particules de haltère au lieu des particules ponctuelles.

**Observations et analyse de structure**

**La structure d’un nuage de formation stellaire et d’un nuage de pré-formation stellaire**

Dans cette thèse on a présenté des observations, effectuées avec le télescope KOSMA (HPBW = 88") du nuage de pré-formation stellaire L1512 et du nuage de formation stellaire de faible masse L1524. Cependant, les champs sont trop petits pour tirer des conclusions définitives à propos des propriétés de structure dans les nuages correspondants. Afin d’obtenir des résultats plus concluants, les champs devraient être élargis et complétés avec des observations à d’autres échelles. Comme mentionné dans le chapitre 7, pour le cas de L1512, des observations d’IRAM résolvant des échelles de l’ordre de \( \sim 0,01 \) ont déjà été effectuées. L’analyse \( \Delta \)-variance fournit, pour ces échelles, des indices de puissance spectraux \( \beta \sim 3 \) (Bensch 1999). Par conséquent l’élargissement des cartes, le complément avec des observations à haute résolution de L1524 et l’analyse suivante de structure constituerait un projet intéressant, qui pourrait aider à mieux comprendre la transition d’un nuage de pré-formation stellaire à un nuage de formation stellaire.

**Intégrales de Minkowski**

La \( \Delta \)-variance appliquée dans cette thèse est basée sur une ondelette de symétrie circulaire; elle est par conséquent sensible uniquement à la moyenne du spectre de puissance sur les azimuts. Le spectre de puissance, quant à lui, ne fournit que des informations au sujet de la corrélation des amplitudes des modes de Fourier, mais pas de leurs phases. C’est-à-dire, que le spectre de puissance n’est pas une description statistique complète, et on ne sait pas, par exemple, si la structure sous-jacente est plutôt grumeleux, filamentuse ou en "feuilles".


Un descripteur statistique, qui ne fait pas d’hypothèses a priori sur la structure et qui fournit des informations topologiques ainsi que géométriques, est donné par l’ensemble des quatre intégrales de Minkowski (voir par exemple Mecke et al. 1994). Ces intégrales ont été employées pour analyser la structure à grande échelle de la distribution des galaxies; elle fournissent des informations au sujet de la connectivité et de la forme de la structure sous-jacente. Des codes calculant les intégrales de Minkowski d’images ponctuelles (par exemple des distributions de galaxie) sont disponibles (Mecke et al. 1994; Schmalzing & Buchert 1997). La modification, puis l’application de la technique de description des structures du MIS peuvent constituer un projet futur prometteur.
Appendixes
Appendix A

Thermodynamic Limit for Power-Law Interactions

The total energy density $\varepsilon$ of a $D$-dimensional system with interaction potential $\propto r^{-\alpha}$ converges if the interaction potential is short-ranged, $\alpha > D$, and diverges if the interaction potential is long-ranged, $0 \leq \alpha \leq D$. Consequently the thermodynamic limit exists only for short-range interaction potentials. In order to demonstrate this we follow here the proof of Goldenfeld (1992).

Consider an interaction potential between two particles separated by a distance $r$ of the form,
\[
U(r) = A/r^\alpha .
\] (A.1)

For a spherical system of radius $R$, the energy $E$ is given by,
\[
E(R) = \frac{1}{2} \int_{\Omega} d^D r \, d^D r' \rho(\vec{r}) \ U(|\vec{r} - \vec{r}'|) \ \rho(\vec{r}') ,
\] (A.2)

where $\rho(\vec{r})$ is the “charge” density at $\vec{r}$ and $\Omega$ is a $D$-dimensional sphere of radius $R$. For a uniform system $\rho(\vec{r}) = \rho$ and with the above interaction potential the integral becomes,
\[
E(R) = \frac{1}{2} A \rho^2 \int_{\Omega} d^D r \, d^D r' \frac{1}{|\vec{r} - \vec{r}'|}\alpha .
\] (A.3)

Making the change of variables $\vec{r} = R\vec{x}$ as well as $\vec{r}' = R\vec{y}$, and $\Omega$ becoming the unit sphere $\Omega_{us}$, one can simply extract the dependence of $R$:
\[
E(R) = \frac{1}{2} A \rho^2 \int_{\Omega_{us}} R^D d^D x \, R^D d^D y \frac{1}{|R\vec{x} - R\vec{y}|}\alpha
\] (A.4)
\[
= \frac{1}{2} A \rho^2 R^{2D-\alpha} C ,
\] (A.5)

where $C$ is a term independent of $R$,
\[
C = \int_{\Omega_{us}} d^D x \, d^D y \frac{1}{|\vec{x} - \vec{y}|}\alpha .
\] (A.6)
Thus
\[
\varepsilon \equiv \frac{E(R)}{V(R)} = \frac{A\rho^2 CR^{2D-\alpha}}{2V_{us}R^D} \propto R^{D-\alpha},
\] (A.7)

where \(V_{us}\) is the volume of a unit sphere in \(D\) dimensions. In the limit \(R \to \infty\), we see that the thermodynamic limit is only well defined if \(\alpha > D\). The case \(\alpha = D\) shall be discussed more below.

In fact, one should be more careful since \(C\) may not converge for \(\alpha > D\). The problem arises in principle when \(|\vec{r} \to \vec{r}'|\), because the singularity of the integrand in Eq. A.3 is not integrable. This does not represent a problem if one assumes for instance that the particles have a hard core repulsion at radius \(a\), so that the interaction \(U(r) \propto r^{-\alpha}\) only applies for \(r > a\).

Let us examine the integral in Eq. A.6 for this situation. Making the change of variables
\[
\bar{x} = x - y \\
\bar{r} = (x - y)/2
\] (A.8) (A.9)

one can find (Goldenfeld 1992),
\[
C = V_d \int_{a/R}^1 S_d u^{D-1} \frac{du}{u^\alpha} = \frac{V_{us}S_{us}}{D-\alpha} \left(1 - (a/R)^{D-\alpha}\right), \quad \text{for } D \neq \alpha,
\] (A.10)

where \(S_{us}\) is the surface area of the unit sphere in \(D\) dimensions. Then, the energy density is,
\[
\varepsilon = \frac{A\rho S_{us}R^{D-\alpha}}{2(D-\alpha)} \left(1 - (a/R)^{D-\alpha}\right), \quad \text{for } D \neq \alpha.
\] (A.11)

With this final, improved calculation of the existence of the thermodynamic limit, the case \(\alpha = D\) can be examined. In equation A.10, the inner integral yields \(\log(R/a)\), so that as \(R \to \infty\) for a fixed \(a\), the bulk energy per unit volume, i.e., the energy density \(\varepsilon\) diverges. Thus, the thermodynamic limit exists only for \(\alpha > D\).
Appendix B

Structure Analysis Using the $\Delta$-variance
Outer Galaxy: Boundaries of the individual and composite fields.

<table>
<thead>
<tr>
<th>Field</th>
<th>(v_{\text{min}}) [km/s]</th>
<th>(v_{\text{max}}) [km/s]</th>
<th>(l_{\text{max}}) [deg.]</th>
<th>(l_{\text{min}}) [deg.]</th>
<th>(b_{\text{min}}) [deg.]</th>
<th>(b_{\text{max}}) [deg.]</th>
<th># pixel</th>
<th>Dist. [kpc]</th>
<th>Comments</th>
</tr>
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<tbody>
<tr>
<td>P1</td>
<td>-52.0</td>
<td>-30.1</td>
<td>139.09</td>
<td>135.97</td>
<td>0.54</td>
<td>2.40</td>
<td>29480</td>
<td>3.7</td>
<td>W5</td>
</tr>
<tr>
<td>P2</td>
<td>-60.2</td>
<td>-30.1</td>
<td>134.43</td>
<td>132.65</td>
<td>-0.24</td>
<td>1.53</td>
<td>16348</td>
<td>4.3</td>
<td>W3</td>
</tr>
<tr>
<td>P3</td>
<td>-56.1</td>
<td>-32.5</td>
<td>125.22</td>
<td>121.61</td>
<td>-1.63</td>
<td>0.75</td>
<td>44720</td>
<td>4.0</td>
<td></td>
</tr>
<tr>
<td>P4</td>
<td>-55.3</td>
<td>-25.2</td>
<td>112.94</td>
<td>110.25</td>
<td>-3.03</td>
<td>-1.70</td>
<td>18624</td>
<td>3.9</td>
<td></td>
</tr>
<tr>
<td>P5</td>
<td>-64.2</td>
<td>-39.0</td>
<td>112.61</td>
<td>110.59</td>
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<td>3.46</td>
<td>21316</td>
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<td></td>
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<td>-69.9</td>
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<td>109.68</td>
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<td>0.46</td>
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<td>109.65</td>
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<td>6820</td>
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<td>5.41</td>
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</tr>
<tr>
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<td>130.38</td>
<td>126.82</td>
<td>2.97</td>
<td>5.41</td>
<td>45056</td>
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</tr>
<tr>
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<td>127.87</td>
<td>124.31</td>
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</tr>
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<td>122.78</td>
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</tr>
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<tr>
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</tr>
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<td>108.12</td>
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<td>5.41</td>
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<td>-3.3</td>
<td>127.87</td>
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<td>188856</td>
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<tr>
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<td>115.72</td>
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Table B.1: Boundaries, number of spatial pixels and distances of the individual (P1-P9, L1-L14) and composite (C1-C7) fields, extracted from the FCRAO Outer Galaxy survey. The pixel number indicates the number of observed positions in the velocity integrated field. While P1, P2, P4, P6, P7, P9, L12 and L13 are associated with well-studied HII regions and OB associations, L4 and L5 are associated with optical HII regions that have not been studied in detail so far. The identified OB associations and HII regions, associated with the individual fields, are indicated in the last column. For the composite fields the comprised individual fields are indicated in brackets.
Outer Galaxy: $\beta$ of the individual and composite fields.

<table>
<thead>
<tr>
<th>Field</th>
<th>$\beta$</th>
<th>Fit-interval [pc]</th>
<th>[']</th>
<th>[ dex]</th>
</tr>
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<td>P1</td>
<td>2.22 ± 0.13</td>
<td>2.70-27.93</td>
<td>2.51-25.95</td>
<td>1.01</td>
</tr>
<tr>
<td>P2</td>
<td>2.62 ± 1.70</td>
<td>3.14-7.33</td>
<td>2.51-5.86</td>
<td>0.37</td>
</tr>
<tr>
<td>P3</td>
<td>2.31 ± 0.20</td>
<td>1.94-5.84</td>
<td>1.67-5.02</td>
<td>0.48</td>
</tr>
<tr>
<td>P4</td>
<td>2.57 ± 0.38</td>
<td>2.85-6.65</td>
<td>2.51-5.86</td>
<td>0.37</td>
</tr>
<tr>
<td>P5</td>
<td>2.09 ± 0.57</td>
<td>3.65-8.52</td>
<td>2.51-5.86</td>
<td>0.37</td>
</tr>
<tr>
<td>P6</td>
<td>2.54 ± 0.07</td>
<td>3.80-39.25</td>
<td>2.51-25.95</td>
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</tr>
<tr>
<td>P7</td>
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<td>2.51-25.95</td>
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</tr>
<tr>
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<td>3.80-8.86</td>
<td>2.51-5.86</td>
<td>0.37</td>
</tr>
<tr>
<td>P9</td>
<td>2.57 ± 0.37</td>
<td>3.80-8.96</td>
<td>2.51-5.86</td>
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</tr>
<tr>
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<td>2.51-5.86</td>
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</tr>
<tr>
<td>C2 (P5-P9)</td>
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<td>2.46 ± 0.07</td>
<td>0.49-3.89</td>
<td>1.67-13.39</td>
<td>0.90</td>
</tr>
<tr>
<td>L2</td>
<td>2.79 ± 0.17</td>
<td>0.58-3.90</td>
<td>5.02-33.48</td>
<td>0.82</td>
</tr>
<tr>
<td>L3</td>
<td>2.46 ± 0.05</td>
<td>0.39-5.65</td>
<td>3.35-48.55</td>
<td>1.16</td>
</tr>
<tr>
<td>L4</td>
<td>2.84 ± 0.12</td>
<td>0.73-5.36</td>
<td>2.51-18.41</td>
<td>0.87</td>
</tr>
<tr>
<td>L5</td>
<td>2.91 ± 0.07</td>
<td>0.98-13.05</td>
<td>4.19-56.08</td>
<td>1.13</td>
</tr>
<tr>
<td>L6</td>
<td>2.63 ± 0.09</td>
<td>1.10-8.03</td>
<td>2.51-18.41</td>
<td>0.87</td>
</tr>
<tr>
<td>L7</td>
<td>3.24 ± 0.51</td>
<td>1.58-5.96</td>
<td>10.88-41.01</td>
<td>0.58</td>
</tr>
<tr>
<td>L8</td>
<td>2.89 ± 0.06</td>
<td>0.78-12.08</td>
<td>1.67-25.95</td>
<td>1.19</td>
</tr>
<tr>
<td>L9</td>
<td>2.59 ± 0.08</td>
<td>0.49-3.04</td>
<td>3.35-20.92</td>
<td>0.80</td>
</tr>
<tr>
<td>L10</td>
<td>2.25 ± 0.03</td>
<td>0.66-14.68</td>
<td>2.51-56.08</td>
<td>1.35</td>
</tr>
<tr>
<td>L11</td>
<td>3.07 ± 0.07</td>
<td>0.78-12.08</td>
<td>1.67-25.95</td>
<td>1.19</td>
</tr>
<tr>
<td>L12</td>
<td>2.57 ± 0.10</td>
<td>0.73-4.63</td>
<td>2.51-15.90</td>
<td>0.80</td>
</tr>
<tr>
<td>L13</td>
<td>2.54 ± 0.17</td>
<td>0.66-4.82</td>
<td>2.51-18.41</td>
<td>0.87</td>
</tr>
<tr>
<td>L14</td>
<td>2.49 ± 0.14</td>
<td>0.80-5.09</td>
<td>2.51-15.90</td>
<td>0.80</td>
</tr>
<tr>
<td>C3 (L2,L3)</td>
<td>2.44 ± 0.06</td>
<td>0.49-5.65</td>
<td>4.19-48.55</td>
<td>1.06</td>
</tr>
<tr>
<td>C4 (L4,L5)</td>
<td>2.58 ± 0.04</td>
<td>0.88-22.57</td>
<td>3.35-86.21</td>
<td>1.41</td>
</tr>
<tr>
<td>C5 (L7,L9)</td>
<td>2.57 ± 0.08</td>
<td>0.61-3.77</td>
<td>4.19-25.95</td>
<td>0.79</td>
</tr>
<tr>
<td>C6 (L13,L14)</td>
<td>2.39 ± 0.13</td>
<td>0.73-6.82</td>
<td>2.51-23.44</td>
<td>0.97</td>
</tr>
<tr>
<td>C7 (L12,L14)</td>
<td>2.43 ± 0.07</td>
<td>0.73-5.36</td>
<td>2.51-18.41</td>
<td>0.87</td>
</tr>
</tbody>
</table>

Table B.2: The power spectral index $\beta$ of the individual fields located in the Perseus arm (P1-P9) and the Local arm (L1-L14), respectively, as well as of the composite fields (C1-C7). Moreover, the fit-interval of the $\chi^2$-fit is indicated. For the composite fields the comprised individual fields are indicated in brackets.
Outer Galaxy: Boundaries of the large scale fields.

<table>
<thead>
<tr>
<th>Field</th>
<th>$v_{\text{min}}$ [km/s]</th>
<th>$v_{\text{max}}$ [km/s]</th>
<th>$l_{\text{max}}$ [deg.]</th>
<th>$l_{\text{min}}$ [deg.]</th>
<th>$b_{\text{min}}$ [deg.]</th>
<th>$b_{\text{max}}$ [deg.]</th>
<th># pixels</th>
</tr>
</thead>
<tbody>
<tr>
<td>Perseus left</td>
<td>-61.0</td>
<td>-33.0</td>
<td>141.54</td>
<td>128.52</td>
<td>-3.03</td>
<td>5.41</td>
<td>566004</td>
</tr>
<tr>
<td>Perseus middle</td>
<td>-61.0</td>
<td>-33.0</td>
<td>128.52</td>
<td>115.51</td>
<td>-3.03</td>
<td>5.41</td>
<td>565398</td>
</tr>
<tr>
<td>Perseus right</td>
<td>-61.0</td>
<td>-33.0</td>
<td>115.51</td>
<td>102.49</td>
<td>-3.03</td>
<td>5.41</td>
<td>565398</td>
</tr>
<tr>
<td>Perseus total</td>
<td>-61.0</td>
<td>-33.0</td>
<td>141.54</td>
<td>102.49</td>
<td>-3.03</td>
<td>5.41</td>
<td>1696800</td>
</tr>
<tr>
<td>Local left</td>
<td>-20.0</td>
<td>4.0</td>
<td>141.54</td>
<td>128.52</td>
<td>-3.03</td>
<td>5.41</td>
<td>566004</td>
</tr>
<tr>
<td>Local middle</td>
<td>-20.0</td>
<td>4.0</td>
<td>128.52</td>
<td>115.51</td>
<td>-3.03</td>
<td>5.41</td>
<td>565398</td>
</tr>
<tr>
<td>Local right</td>
<td>-20.0</td>
<td>4.0</td>
<td>115.51</td>
<td>102.49</td>
<td>-3.03</td>
<td>5.41</td>
<td>565398</td>
</tr>
<tr>
<td>Local total</td>
<td>-20.0</td>
<td>4.0</td>
<td>141.54</td>
<td>102.49</td>
<td>-3.03</td>
<td>5.41</td>
<td>1696800</td>
</tr>
</tbody>
</table>

Table B.3: Boundaries and number of spatial pixels of the large scale fields extracted from the Outer Galaxy survey. The pixel number indicates the number of observed positions in the velocity integrated field.

Outer Galaxy: $\beta$ of the large scale fields.

<table>
<thead>
<tr>
<th>Field</th>
<th>$\beta$</th>
<th>Fit-interval</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>[°]</td>
<td>[dex]</td>
</tr>
<tr>
<td>Perseus left</td>
<td>2.44 ± 0.80</td>
<td>2.51-5.86 0.37</td>
</tr>
<tr>
<td>Perseus middle</td>
<td>2.07 ± 0.04</td>
<td>1.67-5.86 0.55</td>
</tr>
<tr>
<td>Perseus right</td>
<td>2.31 ± 0.04</td>
<td>2.51-41.01 1.21</td>
</tr>
<tr>
<td>Perseus total</td>
<td>2.18 ± 0.03</td>
<td>2.51-41.01 1.21</td>
</tr>
<tr>
<td>Local left</td>
<td>2.47 ± 0.03</td>
<td>2.51-25.95 1.01</td>
</tr>
<tr>
<td>Local middle</td>
<td>2.58 ± 0.03</td>
<td>2.51-25.95 1.01</td>
</tr>
<tr>
<td>Local right</td>
<td>2.36 ± 0.05</td>
<td>2.51-20.92 0.92</td>
</tr>
<tr>
<td>Local total</td>
<td>2.46 ± 0.02</td>
<td>1.67-23.44 1.15</td>
</tr>
</tbody>
</table>

Table B.4: The power spectral index $\beta$ of the selected “large” scale fields. Moreover, the fit-interval of the $\chi^2$-fit is indicated.
### M51: Boundaries of the individual fields.

<table>
<thead>
<tr>
<th>Field</th>
<th>$x_{\text{offset}}^{\min}$ [arcsec.]</th>
<th>$x_{\text{offset}}^{\max}$ [arcsec.]</th>
<th>$y_{\text{offset}}^{\min}$ [arcsec.]</th>
<th>$y_{\text{offset}}^{\max}$ [arcsec.]</th>
<th># pixels</th>
</tr>
</thead>
<tbody>
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<td>-66.6</td>
<td>-49.7</td>
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</tr>
<tr>
<td>s2</td>
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<td>-19.8</td>
<td>-66.6</td>
<td>-51.6</td>
<td>6308</td>
</tr>
<tr>
<td>s3</td>
<td>-1.9</td>
<td>9.8</td>
<td>-66.6</td>
<td>-54.1</td>
<td>3717</td>
</tr>
<tr>
<td>s4</td>
<td>15.0</td>
<td>33.2</td>
<td>-59.3</td>
<td>-44.5</td>
<td>6808</td>
</tr>
<tr>
<td>s5</td>
<td>17.4</td>
<td>38.1</td>
<td>-45.1</td>
<td>-31.6</td>
<td>7245</td>
</tr>
<tr>
<td>s6</td>
<td>8.2</td>
<td>18.4</td>
<td>-37.7</td>
<td>-21.7</td>
<td>4264</td>
</tr>
<tr>
<td>s7</td>
<td>-12.6</td>
<td>5.1</td>
<td>-33.2</td>
<td>-6.3</td>
<td>12104</td>
</tr>
<tr>
<td>s8</td>
<td>23.9</td>
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<td>-31.6</td>
<td>-15.8</td>
<td>5840</td>
</tr>
<tr>
<td>s9</td>
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<td>-27.7</td>
<td>-6.3</td>
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</tr>
<tr>
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<td>20.6</td>
<td>40.3</td>
<td>-15.8</td>
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</tr>
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<td>-6.3</td>
<td>3.9</td>
<td>3484</td>
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<td>-2.9</td>
<td>6.7</td>
<td>1617</td>
</tr>
<tr>
<td>s15</td>
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<td>9.4</td>
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</tr>
<tr>
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</tr>
<tr>
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<td>35.9</td>
<td>5254</td>
</tr>
<tr>
<td>s18</td>
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<td>33.2</td>
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<td>44.9</td>
<td>58.5</td>
<td>4620</td>
</tr>
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<td>45.8</td>
<td>59.7</td>
<td>4615</td>
</tr>
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<td>47.1</td>
<td>64.3</td>
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</tr>
<tr>
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<td>53.8</td>
<td>73.2</td>
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</tr>
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<td>-15.7</td>
<td>-33.1</td>
<td>7652</td>
</tr>
<tr>
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<td>-32.8</td>
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<td>i3</td>
<td>45.8</td>
<td>55.4</td>
<td>-33.1</td>
<td>-17.4</td>
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</tr>
<tr>
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<td>-42.0</td>
<td>-20.8</td>
<td>2.9</td>
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</tr>
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<td>29.5</td>
<td>38.7</td>
<td>2961</td>
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<td>50.1</td>
<td>6808</td>
</tr>
<tr>
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<td>-40.4</td>
<td>43.3</td>
<td>61.4</td>
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</tr>
</tbody>
</table>

Table B.5: Boundaries and pixel number of the individual fields located in M51 in spiral arm regions and the central area (s1-s24) as well as in interarm regions (i1-i9).
M51: $\beta$ of the individual fields.

<table>
<thead>
<tr>
<th>Field</th>
<th>$\beta$</th>
<th>Fit-interval</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>[pc]</td>
</tr>
<tr>
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<td>$1.99 \pm 0.15$</td>
<td>27.93-148.93</td>
</tr>
<tr>
<td>s2</td>
<td>$1.95 \pm 0.19$</td>
<td>27.93-121.01</td>
</tr>
<tr>
<td>s3</td>
<td>$1.85 \pm 0.32$</td>
<td>27.93-93.08</td>
</tr>
<tr>
<td>s4</td>
<td>$2.29 \pm 0.32$</td>
<td>27.93-93.08</td>
</tr>
<tr>
<td>s5</td>
<td>$1.91 \pm 0.22$</td>
<td>27.93-93.08</td>
</tr>
<tr>
<td>s6</td>
<td>$2.20 \pm 0.40$</td>
<td>27.93-93.08</td>
</tr>
<tr>
<td>s7</td>
<td>$2.47 \pm 0.42$</td>
<td>27.93-65.16</td>
</tr>
<tr>
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<td>$2.15 \pm 0.33$</td>
<td>27.93-93.08</td>
</tr>
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<td>27.93-288.56</td>
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<td>$2.10 \pm 0.37$</td>
<td>27.93-121.01</td>
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<td>$2.22 \pm 0.13$</td>
<td>27.93-176.86</td>
</tr>
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<td>s12</td>
<td>$2.30 \pm 0.26$</td>
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<td>$2.20 \pm 0.22$</td>
<td>27.93-148.93</td>
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<td>$2.30 \pm 0.38$</td>
<td>27.93-65.16</td>
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<td>s15</td>
<td>$2.28 \pm 0.23$</td>
<td>27.93-93.08</td>
</tr>
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<td>$2.44 \pm 0.23$</td>
<td>27.93-176.86</td>
</tr>
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<td>$2.32 \pm 0.78$</td>
<td>27.93-65.16</td>
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<tr>
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<td>$1.90 \pm 0.42$</td>
<td>27.93-65.16</td>
</tr>
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<td>s19</td>
<td>$1.93 \pm 0.26$</td>
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<td>$2.39 \pm 0.15$</td>
<td>27.93-204.79</td>
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<tr>
<td>s22</td>
<td>$1.90 \pm 0.20$</td>
<td>27.93-176.86</td>
</tr>
<tr>
<td>s23</td>
<td>$2.22 \pm 0.20$</td>
<td>27.93-148.93</td>
</tr>
<tr>
<td>s24</td>
<td>$2.29 \pm 0.12$</td>
<td>27.93-260.64</td>
</tr>
</tbody>
</table>

Table B.6: The power spectral index $\beta$ of the individual fields located in M51 in spiral arm regions and the central area (s1-s24) as well as the fields located in interarm regions. Moreover, the fit-interval of the $\chi^2$-fit is indicated.
The IRAM key-project data-set.

<table>
<thead>
<tr>
<th>Name</th>
<th>Transition</th>
<th>telescope</th>
<th>$\beta$</th>
<th>Fit-interval</th>
</tr>
</thead>
<tbody>
<tr>
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<td></td>
<td></td>
<td></td>
<td>[pc]</td>
</tr>
<tr>
<td>Polaris Flare/</td>
<td>$^{12}$CO J = 1 – 0</td>
<td>CfA 1.2m</td>
<td>2.61 ± 0.16</td>
<td>0.75-6.01</td>
</tr>
<tr>
<td>MCLD 123.5+24.9</td>
<td>$^{12}$CO J = 2 – 1</td>
<td>3kmsa 3m</td>
<td>3.05 ± 0.22</td>
<td>0.13-0.60</td>
</tr>
<tr>
<td></td>
<td>$^{13}$CO J = 1 – 0</td>
<td>Fcrao 14m</td>
<td>3.22 ± 0.15</td>
<td>0.06-0.48</td>
</tr>
<tr>
<td></td>
<td>$^{12}$CO J = 1 – 0</td>
<td>Iram 30m</td>
<td>3.29 ± 0.22</td>
<td>0.02-0.07</td>
</tr>
<tr>
<td></td>
<td>$^{12}$CO J = 2 – 1</td>
<td>Iram 30m</td>
<td>3.25 ± 0.23</td>
<td>0.02-0.07</td>
</tr>
<tr>
<td>L1512</td>
<td>$^{12}$CO J = 1 – 0</td>
<td>Iram 30m</td>
<td>2.53 ± 0.41</td>
<td>0.02-0.04</td>
</tr>
<tr>
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<td>Iram 30m</td>
<td>3.06 ± 0.46</td>
<td>0.02-0.04</td>
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<td>Iram 30m</td>
<td>3.24 ± 0.36</td>
<td>0.02-0.04</td>
</tr>
<tr>
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<td>Iram 30m</td>
<td>2.76 ± 0.63</td>
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<td>Iram 30m</td>
<td>3.15 ± 0.40</td>
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</tr>
<tr>
<td></td>
<td>$^{12}$CO J = 2 – 1</td>
<td>Iram 30m</td>
<td>3.18 ± 0.45</td>
<td>0.02-0.04</td>
</tr>
</tbody>
</table>

Table B.7: The power spectral index and the fit-interval of 3 quiescent molecular clouds observed in different transitions in the framework of the IRAM key-project. The distance to the Polaris Flare and L1512 is $d \approx 150$ pc. Those to L134A is $d \approx 140$ pc.

The BU/FCRAO Galactic Ring data-set.

<table>
<thead>
<tr>
<th>Name/Position $^a$</th>
<th>Distance [kpc]</th>
<th>Comment</th>
<th>$\beta$</th>
<th>Fit-interval</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td>[pc]</td>
<td>[dex]</td>
</tr>
<tr>
<td>(l=48.6/b=0.0)</td>
<td>10.0</td>
<td>star-forming</td>
<td>2.40 ± 0.45</td>
<td>3.20-10.65</td>
</tr>
<tr>
<td>(l=48.7/b=-0.3)</td>
<td>2.8</td>
<td>quiescent</td>
<td>2.50 ± 0.33</td>
<td>1.80-9.24</td>
</tr>
<tr>
<td>L673/683</td>
<td>0.3</td>
<td>quiescent</td>
<td>3.23 ± 0.21</td>
<td>0.16-0.83</td>
</tr>
<tr>
<td>(l=47.7/b=-0.7)</td>
<td>5.7</td>
<td>quiescent</td>
<td>2.67 ± 0.32</td>
<td>3.67-18.82</td>
</tr>
<tr>
<td>(l=47.2/b=0.3)</td>
<td>5.8</td>
<td>star-forming</td>
<td>3.0/2.5 $^b$</td>
<td>1.86-13.70</td>
</tr>
<tr>
<td>(l=47.8/b=-0.3)</td>
<td>5.9</td>
<td>star-forming</td>
<td>2.40 ± 0.14</td>
<td>3.80-30.76</td>
</tr>
<tr>
<td>GRSMC45.60+0.30$^d$</td>
<td>1.8</td>
<td>quiescent</td>
<td>2.79 ± 0.12</td>
<td>0.97-5.94</td>
</tr>
<tr>
<td>GRSMC45.46+0.05$^d$</td>
<td>6.0</td>
<td>star-forming</td>
<td>2.76 ± 0.24</td>
<td>3.22-19.80</td>
</tr>
<tr>
<td>(l=44.7/b=-0.1)</td>
<td>4.0</td>
<td>star-forming</td>
<td>2.44 ± 0.15</td>
<td>1.72-13.19</td>
</tr>
<tr>
<td>GRMC43.30-0.33$^d$</td>
<td>3.0</td>
<td>quiescent</td>
<td>2.88 ± 0.23</td>
<td>1.93-12.77</td>
</tr>
<tr>
<td>W49$^d$</td>
<td>11.4</td>
<td>star-forming</td>
<td>2.45 ± 0.99</td>
<td>6.12-23.08</td>
</tr>
<tr>
<td>(l=42/b=-0.5)</td>
<td>6.3</td>
<td>star-forming</td>
<td>2.12/2.75 $^c$</td>
<td>5.38-38.95</td>
</tr>
<tr>
<td>(l=42.4/b=-0.3)</td>
<td>2.5</td>
<td>quiescent</td>
<td>2.43 ± 0.21</td>
<td>1.34-10.64</td>
</tr>
<tr>
<td>(l=41.5/b=0.0)</td>
<td>11.5</td>
<td>star-forming</td>
<td>2.00 ± 0.15</td>
<td>6.17-60.48</td>
</tr>
<tr>
<td>(l=41.0/b=-0.1)</td>
<td>3.0</td>
<td>star-forming</td>
<td>2.68 ± 0.11</td>
<td>1.61-18.50</td>
</tr>
<tr>
<td>(l=40.7/b=-0.2)</td>
<td>10.5</td>
<td>star-forming</td>
<td>2.54 ± 0.28</td>
<td>5.64-27.97</td>
</tr>
</tbody>
</table>

$^a$Name or Position of the selected molecular region.

$^b$Fit done for 2 regimes: 1) $\beta = 3.0 \pm 0.7$ for $1.86 \leq L \leq 4.33$ pc  
2) $\beta = 3.0 \pm 0.9$ for $6.23 \leq L \leq 13.70$ pc

$^c$Fit done for 2 regimes: 1) $\beta = 2.12 \pm 0.15$ for $5.38 \leq L \leq 14.76$ pc  
2) $\beta = 2.75 \pm 0.49$ for $12.84 \leq L \leq 38.95$ pc

$^d$Data set published in Simon et al. (2001); remainder: unpublished data

Table B.8: The power spectral index $\beta$ and the fit interval of some fields extracted from the BU/FCRAO Galactic Ring survey. The survey maps the Inner Galaxy and the 5 kpc Galactic Ring in the $^{13}$CO J = 1 – 0 transition. The observations are carried out with the FCRAO 14m telescope.
The Bell Labs data-set.

<table>
<thead>
<tr>
<th>Name</th>
<th>Distance [pc]</th>
<th>β</th>
<th>Fit.interval [pc]</th>
<th>[dex]</th>
</tr>
</thead>
<tbody>
<tr>
<td>Perseus/NGC 1333</td>
<td>350</td>
<td>3.07 ± 0.10</td>
<td>0.31-2.52</td>
<td>0.91</td>
</tr>
<tr>
<td>Orion A</td>
<td>450</td>
<td>2.54 ± 0.05</td>
<td>0.39-7.50</td>
<td>1.28</td>
</tr>
<tr>
<td>Orion B</td>
<td>415</td>
<td>2.68 ± 0.12</td>
<td>0.36-6.92</td>
<td>1.28</td>
</tr>
<tr>
<td>Mon OB1/NGC 2264</td>
<td>800</td>
<td>2.54 ± 0.12</td>
<td>0.93-7.14</td>
<td>0.89</td>
</tr>
<tr>
<td>Mon R2</td>
<td>950</td>
<td>2.76 ± 0.12</td>
<td>0.69-2.16</td>
<td>1.28</td>
</tr>
</tbody>
</table>

Table B.9: The spectral power index β and the fit interval of 4 molecular clouds located in the second and third Galactic quadrant. The clouds were observed in the $^{13}$CO $J = 1 - 0$ transition with the 7m AT&T Bell Laboratories mm-wave telescope.
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Out-of-print science fiction writer Kilgore Trout in Cohoes, New York, in 1975, having learned of the death of his estranged son, Leon, in a Swedish shipyard, having given his parakeet, “Cyclone Bill”, his freedom, and about to become a vagabond.